Commentaries on Propositions in Book I of Euclid's Elements
Module MA232A, Michaelmas Term 2017

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1 Commentaries on Propositions in Book I of Euclid's Elements

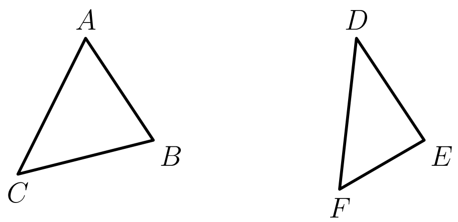
1.1 Commentary on Proposition 24 in Book I of Euclid's Elements

The statement of proposition 24 in Book I of Euclid is translated by Heath as follows:

> *If two triangles have the two sides equal to two sides respectively, but have the one of the angles contained by the equal straight lines greater than the other, they will also have the base greater than the base.*

This is followed by a restatement, assigning letters to the vertices of the triangles involved.

> *Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE, and AC to DF, and let the angle at A be greater than the angle at D; I say that the base BC is also greater than the base EF.*



Euclid begins the proof by constructing a triangle DEG that is congruent to ABC, where the triangle DEG shares a side DE with the triangle DEF and the points F and G lie on the same side of the line DE. In Book I of the *Elements*, this is justified as an application of the preceding proposition. (In the axiomatic scheme set out in Hilbert's *Grundlagen der Geometrie*, the existence of such a point G follows directly from one of the Axioms of Congruence, namely that consistently labelled III-4 in the various editions of Hilbert's treatise.)

The result of the 23rd proposition in Book I of the Elements therefore follows from the following special case:

> Let DEF and DEG be two triangles sharing a common side DE, and having the side DF equal to the side DG, let the point G lie on the same side of the side DE as the point F, and let the triangle DEG have an angle greater than that of the triangle DEF at the vertex D. Then the base EG of DEG is also greater than the base EF of DEF.

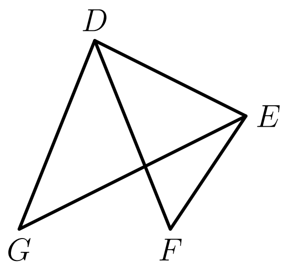
Now, in the configuration described, the point F lies in the interior of the angle EDG, because the points F and G lie on the same side of DE and the angle EDG is greater than EDF. A result, often referred to as the *Crossbar Theorem* in formal axiomatic treatments of the foundations of Euclidean geometry written during the the past one-and-a-half centuries, then guarantees that the ray (or half-line) starting at the vertex D of the triangle DEG and passing through the point F must intersect the edge EG of that triangle. There are then three possible cases to consider:

(i) the point F lies outside the triangle DEG;

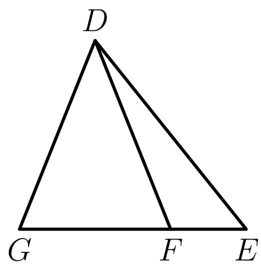
(ii) the point F lies on the side EG of the triangle DEG;

(iii) the point F lies inside the triangle DEG;

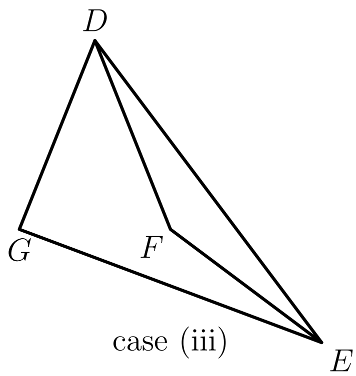
The configurations in these three cases are depicted below.



case (i)



case (ii)



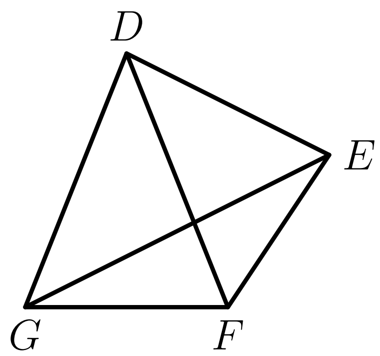
case (iii)

In case (ii), the point F lies in the interior of the line segment EG, and therefore EG is greater than EF. The required result therefore follows directly in case (ii).

In case (iii) it follows from the 21st proposition in Book I of the *Elements* that the sum of the sides DG and EG of the triangle DEG is greater than the sum of the sides DF and EF of the triangle DEF, because the point F lies inside the triangle DEG. But the sides DF and DG are equal by hypothesis. It follows that, in this case, the side EG is greater than the side EF. Thus the result follows in case (iii) also.

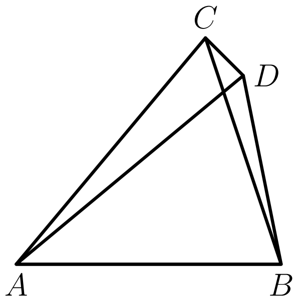
It remains to consider case (i), which is the only case explicitly covered in Euclid's proof of the 24th proposition of the *Elements*.

In case (i), we join the points F and G. The triangle DFG is isoceles, because DF is equal to DG, and therefore the angles DFG and DGF of this triangle at F and G are equal.



Now, in case (iii), the line segments DF and EG intersect, and therefore the point E lies in the interior of the angle DGF and the point D lies in the interior of EFG. It follows that FGE is less than FGD, which is equal to DFG, and thefore FGE is less than DFG, which in turn is less than EFG. Therefore the angle FGE of the triangle EFG opposite the side EF is less than the angle EFG of this triangle opposite the side EG. It then follows from 19th proposition in Book I of the *Elements* that the side EF of the triangle EFG is less than the side EG. The result follows.

The proof of the 24th proposition of the *Elements* is clearly related to that of the 7th proposition in Book I of the *Elements*. Indeed in the 7th proposition, Euclid explicitly considers only the situation depicted in the following figure, in which the point D lies outside the triangle ABC, on the same side of AC as the point B, on the same side of AB as the point C, but on the opposite side of BC to the point A. Euclid's proof amounts to showing that, in this configuration, if the sides AC and AD of the triangles ABC and ABD are equal, then the angle BCD is less than the angle BDC. Indeed angles ACD and ADC are equal by the Isosceles Triangle Theorem (the 5th proposition in the *Elements*). But the location of the point D ensures that angle BCD is less than ACD, whereas angle BDC is greater

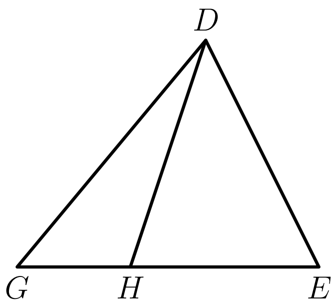


than ADC. Therefore angle BCD is less than BDC. In the context of the 7th proposition, one can conclude from this that, in this configuration, the sides BC and BD must be unequal, for if they were equal, then the angles BCD and BDC would be equal, which is not the case. It follows therefore that, in this configuration, the sides of the triangle ABC are not all equal to the corresponding sides of the triangle ABD.

But one can also combine the fact that, in this configuration, the angle BCD is less than BDC with the result of the 19th proposition in Book I of the *Elements* to conclude that the side BD of the triangle BCD is less than than the side BC of this triangle. Thus if triangles ABC and ABD share a common side, if the vertices C and D of these triangles both lie on the same side of AB, if the edges AC and AD are equal, and if the angle BAC is greater than BAD, then the edge BC is greater than BD. This is the result that proves the 24th proposition in the case explicitly considered by Euclid.

Most 19th century editions of Euclid's *Elements* published in Britain and Ireland tended to follow the variation in the proof of the 24th proposition of Book I introduced by Robert Simson, in his translations of Euclid into Latin and English first published in 1756. Simson noted that Euclid's proof was incomplete because not all cases were covered. Simson's version of the proof proceeds as follows. Let ABC and DEF be triangles for which the sides AB and DE are equal to one another, the sides AC and DF are also equal to one another, and the angle BAC is greater than the angle EDF. Then one of the two sides DE and DF must be greater than or equal to the other. We may therefore suppose, without loss of generality that DE is not greater than DF. The triangle ABC is then applied to the edge DE. The 23rd proposition in Book I of the *Elements* then ensures the existence of a point G for which the triangle DEG is congruent to ABC. Then DG is equal to AC, which is equal to DF, and thus the sides DG and DF are equal to one another. Simson's account of the proof then relies on the fact that, in this configuration, *the point F is guaranteed to lie outside the triangle DEG.*

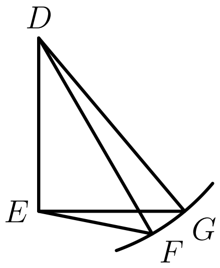
Indeed let H denote the point at which the ray (or half-line) starting at D and passing through the point F meets the edge EG of the triangle DEG. It follows from the 16th proposition in Book I of the *Elements* that



the external angle DHG of the triangle DEH is greater than the internal angle DEH of that triangle. Thus angle DHG is greater than angle DEG. Also the 5th and 18th propositions of Book I ensures that the angle DEG is greater than or equal to angle DGE, because the side DG is assumed greater than or equal to the side DE. Therefore the angle of the triangle DHG at H is greater than the angle of this triangle at D. It follows from the 19th proposition in Book I of the *Elements* that DG is greater than DH. Now F is the point on the ray starting from D and passing through H for which DF is equal to DG. It follows that the point F must lie outside the triangle DEG. Thus the proof explicitly given by Euclid proves the result of the 24th proposition in full generality.

Robert Simson's note on this proposition reads as follows:—

> *To this is added "of the two sides DE, DF, let DE be that "which is not greater than the other;" that is, take that side of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC. because without this restriction, there might be three different cases of the Proposition, as Campanus and others make.*
>
> *Mr. Thomas Simpson in p.262 of the second edition of his Elements of Geometry printed Ann. 1760. observes in his Notes that it ought to have been shewn that the point F falls below the line EG; this probably Euclid omitted, as it is very easy to perceive that DG being equal to DF, the point G is in the circumference of a circle described from the center D at the distance DF, and must be in that part of it which is above the straight line EF,*



because DG falls above DF, the angle EDG being greater than the angle EDF.