



Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin

Ollscoil Átha Cliath | The University of Dublin

Faculty of Engineering, Mathematics and Science
School of Mathematics

Sample Paper

Trinity Term 2016

MA232A: Euclidean and non-Euclidean Geometry

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Instructions to Candidates:

Credit will be given for the best 3 questions answered.

Materials Permitted for this Examination:

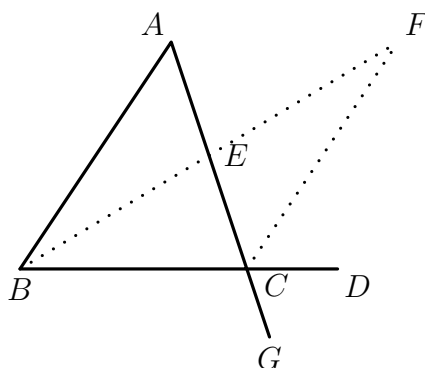
Formulae and Tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

1. The *SAS Congruence Rule* and the *Isosceles Triangle Theorem (Pons Asinorum)* included in the *Selected Propositions from Euclid* at the end of this examination paper may be used without proof throughout this question.

(a) (*Euclid*, Book I, Proposition 16, John Casey's Edition)



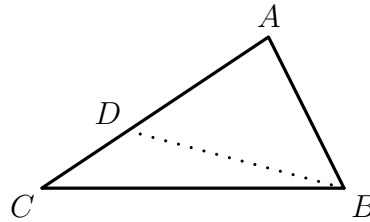
Prove that if any side (BC) of a triangle (ABC) be produced, the exterior angle (ACD) is greater than either of the two interior non-adjacent angles. (Thus the task is to prove that the external angle $\angle ACD$ is greater than the interior angles of the triangle ABC at A and B .) The proof must not rely on later propositions in Euclid. In particular the proof must not make use of the results of Book I, Proposition 32 that assert that the interior angles of the triangle sum to two right angles and the exterior angle $\angle ACD$ is the sum of the two interior non-adjacent angles at A and B .

Note that the construction underlying the above diagram ensures that the point E bisects both AC and BF .

(6 points)

(Question continues on the next page)

(b) (*Euclid*, Book I, Proposition 18, John Casey's Edition)



Prove that, if in any triangle (ABC) one side (AC) be greater than the other (AB), the angle opposite to the greater side is greater than the angle opposite to the less.

Note that, in the above figure, AC is greater than AB and AD is equal to AB . It must be shown that the angle of ABC at B is greater than that at C .

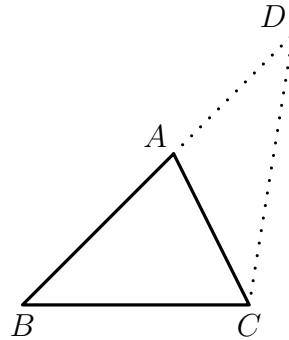
(4 points)

(c) Prove that, if one angle (B) of a triangle (ABC) be greater than another angle (C), the side (AC) which is opposite to the greater angle is greater than the side (AB) which is opposite to the less.

(2 points)

(Question continues on the next page)

(d) (*Euclid*, Book I, Proposition 20, John Casey's Edition)

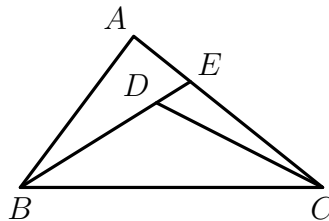


Prove that, *the sum of any two sides (BA , AC) of a triangle (ABC) is greater than the third.*

Note that, in the above diagram, the points B , A and D are colinear, and the line segments AD and AE are equal.

(4 points)

(e) (*Euclid*, Book I, Proposition 21, John Casey's Edition)



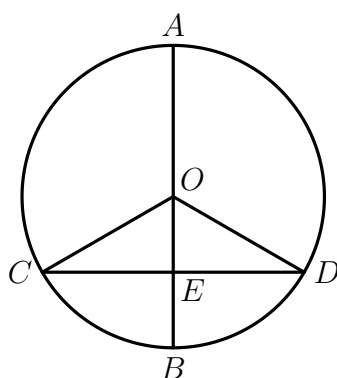
Prove that, *if two lines (BD , CD) be drawn to a point (D) within a triangle from the extremities of its base (BC), their sum is less than the sum of the remaining sides (BA , CA), but they contain a greater angle.*

(4 points)

(End of Question)

2. The *Selected Propositions from Euclid* at the end of this examination paper may be used without proof throughout this question. In addition you may use, without proof, the result of Euclid, Book I, Proposition 24, which ensures that if in two triangles ABC and DEF the side AB and AC are equal to sides DE and DF respectively, and if angle $\angle BAC$ is greater than angle $\angle EDF$ then BC is greater than EF . You may also use, without proof, any of the results stated in Question 1.

(a) (*Euclid*, Book III, Proposition 3, John Casey's Edition)

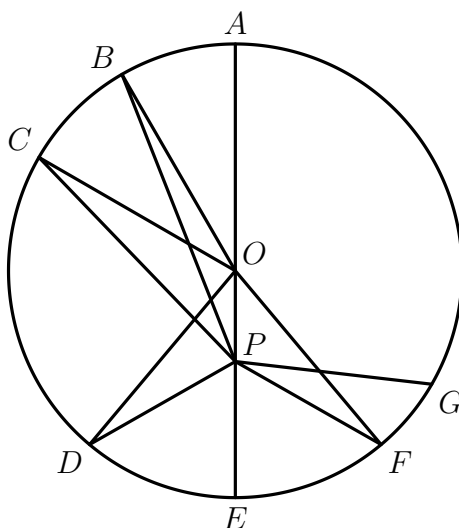


- (i) Prove that if a line (AB) passing through the centre of a circle bisects a chord (CD) , which does not pass through the centre, then it cuts it at right angles.
- (ii) Prove that if a line (AB) passing through the centre of a circle cuts at right angles a chord (CD) , which does not pass through the centre, then it bisects it.

(8 points)

(Question continues on the next page)

(b) (*Euclid*, Book III, Proposition 7, adapted from John Casey's Edition)



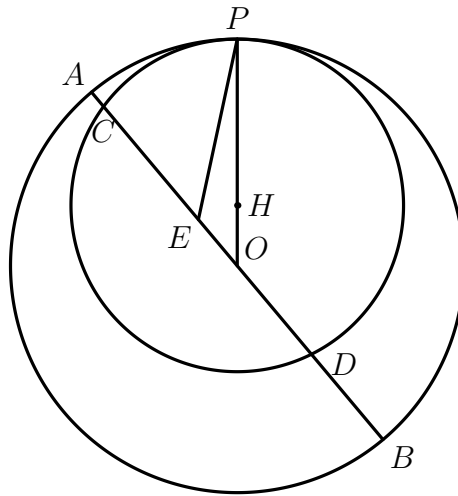
From any point P within a circle, which is not the centre O , let AE be the diameter of the circle that passes through the points O and P , so that the points A , O , P and E occur in that order along the diameter and the points A and E lie on the circumference of the circle. Let B , C and D be points around the circumference of the circle between A and E located so that the points A , B , C , D and E are distinct and occur in that order around the circumference of the circle, as depicted in the above figure.

- (i) Prove that PD is greater than PE , and that PB is greater than PC .
- (ii) Explain why there exists exactly one point F on the circumference of the circle distinct from the point D for which the line segments PD and PF are equal.

(6 points)

(Question continues on the next page)

(c) (*Euclid*, Book III, Proposition 11, John Casey's Edition)



Prove that if one circle (CPD) touch another circle (APB) internally at any point P , the line joining the centres must pass through that point.

(6 points)

(End of Question)

3. (a) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Prove the following identity (known as *Lagrange's Quadruple Product Identity*):

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w}).$$

(5 points)

- (b) Let $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ be vectors in $\mathbb{R}^3$. Prove that

$$(\mathbf{r} \times \mathbf{r}') \times (\mathbf{r} \times \mathbf{r}'') = (\mathbf{r} \cdot (\mathbf{r}' \times \mathbf{r}''))\mathbf{r}.$$

(5 points)

- (c) Let L , L' and L'' be three distinct points on the unit sphere that do not all lie on any one great circle of the sphere, and let p be the angle which the line from the centre of the sphere to the point L'' makes with the plane through the centre of the sphere that contains the points L and L' . Prove that

$$\sin p = \sin L \cdot \sin LL'' = \sin L' \cdot \sin L'L'',$$

where $\sin L$ denotes the sine of the angle between the arcs LL' and LL'' at L and $\sin L'$ denotes the sine of the angle between the arcs $L'L''$ and $L'L$ at L' .

(10 points)

[Note: throughout the above question, you may use, without proof, the *Vector Triple Product Identity*, which asserts that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

for all vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$. You may also use, without proof, standard properties of the Scalar Triple Product.]

(End of Question)

4. Let Σ be a smooth surface in $\mathbb{R}^3$, and let p and q be smooth local coordinates defined throughout Σ . Let the partial derivatives of the Cartesian coordinate functions with respect to these smooth local coordinates p and q be denoted by a, b, c, a', b' and c' , so that

$$\begin{aligned}\frac{\partial x}{\partial p} &= a, & \frac{\partial y}{\partial p} &= b, & \frac{\partial z}{\partial p} &= c, \\ \frac{\partial x}{\partial q} &= a', & \frac{\partial y}{\partial q} &= b', & \frac{\partial z}{\partial q} &= c' .\end{aligned}$$

- (a) Show that the unit normal vector field (X, Y, Z) along the surface Σ satisfies

$$(X, Y, Z) = \frac{\pm 1}{\Delta} (bc' - cb', ca' - ac', ab' - ba'),$$

where

$$\Delta = \sqrt{(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}.$$

(5 points)

- (b) Using the notation of (a), prove that

$$\Delta(p, q) = \frac{\pm 1}{Z(p, q)} \left(\frac{\partial x}{\partial p} \frac{\partial y}{\partial q} - \frac{\partial x}{\partial q} \frac{\partial y}{\partial p} \right).$$

(5 points)

(Question continues on next page)

(c) Let $\nu: \Sigma \rightarrow S^2$ denote the Gauss map of the smooth surface Σ , defined such that

$$\nu(p, q) = (X(p, q), Y(p, q), Z(p, q))$$

where $X(p, q)$, $Y(p, q)$ and $Z(p, q)$ are the Cartesian coordinates of the unit normal vector field to the surface at a point with local coordinates p and q . Prove that

$$\text{area}(\nu(R)) = \int_R k \, d\sigma,$$

where R is a region with regular boundary contained within the domain of a smooth local coordinate system (p, q) on the surface, the integral is taken with respect to surface area, and

$$k(p, q) = \frac{\frac{\partial X}{\partial p} \frac{\partial Y}{\partial q} - \frac{\partial X}{\partial q} \frac{\partial Y}{\partial p}}{\frac{\partial x}{\partial p} \frac{\partial y}{\partial q} - \frac{\partial x}{\partial q} \frac{\partial y}{\partial p}}.$$

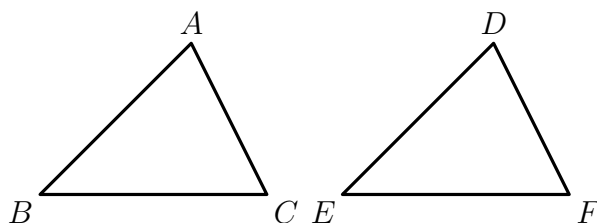
(You may use without proof the result that $d\sigma = \sqrt{\Delta(p, q)} \, dp \, dq$, where Δ is defined as specified in (a).)

(10 points)

(End of Question)

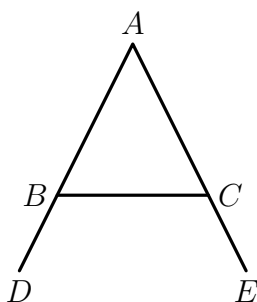
Selected Propositions from Euclid

SAS Congruence Rule (see *Euclid*, Book I, Proposition 4)



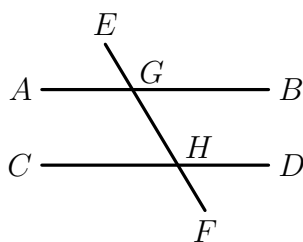
If two triangles (BAC , EDF) have two sides (BA , AC) of one equal respectively to two sides (ED , DF) of the other, and also have the angles ($\angle A$, $\angle D$) included by those sides equal, the triangles shall be equal in every respect—that is, their bases or third sides (BC , EF) shall be equal, and the angles ($\angle B$, $\angle C$) at the base of one shall be respectively equal to the angles ($\angle E$, $\angle F$) at the base of the other; namely, those shall be equal to which the equal sides are opposite.

Isosceles Triangle Theorem (Pons Asinorum) (see *Euclid*, Book I, Proposition 5)



The angles ($\angle ABC$, $\angle ACB$) at the base (BC) of an isosceles triangle are equal to one another, and if the equal sides (AB , AC) be produced, the external angles ($\angle DBC$, $\angle ECB$) below the base shall be equal.

Consider the following configuration:—



Suppose that the line EF cuts the lines AB and CD as shown in the figure, intersecting AB at G and CD at H . The angles $\angle AGH$ and $\angle EGB$ are said to be *vertically opposite angles*, the angles $\angle AGH$ and $\angle GHD$ are said to be *alternate angles*, and the angles $\angle EGB$ and $\angle GHD$ are said to be *corresponding angles*.

Vertically Opposite Angles Theorem (see *Euclid*, Book I, Proposition 15)

Let AB and EH be lines in the Euclidean plane that intersect at the point G .

Then $\angle AGH = \angle EGB$.

(Thus vertical angles are equal.)

Alternate Angles Theorem (see *Euclid*, Book I, Propositions 27 and 29)

Let AB , CD and EF be lines in the Euclidean plane, where A and C lie on the same side of EF , and let the line EF cut the lines AB and CD at G and H respectively (as depicted above). Then the lines AB and CD are parallel if and only if $\angle AGH = \angle GHD$.

(Thus AB and CD are parallel if and only if alternate angles $\angle AGH$ and $\angle GHD$ are equal.)

Corresponding Angles Theorem (see *Euclid*, Book I, Proposition 29)

Let AB , CD and EF be lines in the Euclidean plane, where A and C lie on the same side of EF , and let the line EF cut the lines AB and CD at G and H respectively (as depicted above). Then the lines AB and CD are parallel if and only if $\angle EGB = \angle GHD$.

(Thus AB and CD are parallel if and only if corresponding angles $\angle EGB$ and $\angle GHD$ are equal.)