MA232A: Euclidean and non-Euclidean Geometry
Michaelmas Term 2015
Notes on Vector Algebra and Spherical Trigometry

David R. Wilkins

Contents

5 Vector Algebra and Spherical Trigonometry

5.1 Vectors in Three-Dimensional Euclidean Space

A 3-dimensional *vector* $\mathbf{v}$ in the vector space $\mathbb{R}^3$ can be represented as a triple (v_1, v_2, v_3) of real numbers. Vectors in $\mathbb{R}^3$ are added together, subtracted from one another, and multiplied by real numbers by the usual rules, so that

$$\begin{aligned}(u_1, u_2, u_3) + (v_1, v_2, v_3) &= (u_1 + v_1, u_2 + v_2, u_3 + v_3),\\ (u_1, u_2, u_3) - (v_1, v_2, v_3) &= (u_1 - v_1, u_2 - v_2, u_3 - v_3),\\ t(u_1, u_2, u_3) &= (tu_1, tu_2, tu_3)\end{aligned}$$

for all vectors (u_1, u_2, u_3) and (v_1, v_2, v_3) in $\mathbb{R}^3$, and for all real numbers t. The operation of vector addition is commutative and associative. Also $\mathbf{0} + \mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$, where $\mathbf{0} = (0, 0, 0)$, and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^3$, where $-(v_1, v_2, v_3) = (-v_1, v_2, v_3)$ for all $(v_1, v_2, v_3) \in \mathbb{R}^3$. Moreover

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}), \quad t(\mathbf{u} + \mathbf{v}) = t\mathbf{u} + t\mathbf{v}, \quad (s + t)\mathbf{v} = s\mathbf{v} + t\mathbf{v},$$

$$s(t\mathbf{v}) = (st)\mathbf{v}, \quad 1\mathbf{v} = \mathbf{v}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{R}^3$ and $s, t \in \mathbb{R}$. The set of all vectors in three-dimensional space, with the usual operations of vector addition and of scalar multiplication constitute a three-dimensional real vector space.

The *Euclidean norm* $|\mathbf{v}|$ of a vector $\mathbf{v}$ is defined so that if $\mathbf{v} = (v_1, v_2, v_3)$ then

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

The *scalar product* $\mathbf{u} \,.\, \mathbf{v}$ and the *vector product* $\mathbf{u} \times \mathbf{v}$ of vectors $\mathbf{u}$ and $\mathbf{v}$ are defined such that

$$\begin{aligned}(u_1, u_2, u_3) \,.\, (v_1, v_2, v_3) &= u_1v_1 + u_2v_2 + u_3v_3,\\ (u_1, u_2, u_3) \times (v_1, v_2, v_3) &= (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\end{aligned}$$

for all vectors (u_1, u_2, u_3) and (v_1, v_2, v_3) in $\mathbb{R}^3$. Then

$$(\mathbf{u} + \mathbf{v}) \,.\, \mathbf{w} = \mathbf{u} \,.\, \mathbf{w} + \mathbf{v} \,.\, \mathbf{w}, \quad \mathbf{u} \,.\, (\mathbf{v} + \mathbf{w}) = \mathbf{u} \,.\, \mathbf{v} + \mathbf{u} \,.\, \mathbf{w},$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w},$$

$$(t\mathbf{u}) \,.\, \mathbf{v} = \mathbf{u} \,.\, (t\mathbf{v}) = t(\mathbf{u} \,.\, \mathbf{v}), \quad (t\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (t\mathbf{v}) = t(\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \,.\, \mathbf{v} = \mathbf{v} \,.\, \mathbf{u}, \quad \mathbf{u} \,.\, \mathbf{u} = |\mathbf{u}|^2, \quad \mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

The unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ of the *standard basis* of $\mathbb{R}^3$ are defined so that

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Then

$$\mathbf{i} \,.\, \mathbf{i} = \mathbf{j} \,.\, \mathbf{j} = \mathbf{k} \,.\, \mathbf{k} = 1,$$

$$\mathbf{i} \,.\, \mathbf{j} = \mathbf{j} \,.\, \mathbf{i} = \mathbf{j} \,.\, \mathbf{k} = \mathbf{k} \,.\, \mathbf{j} = \mathbf{k} \,.\, \mathbf{i} = \mathbf{i} \,.\, \mathbf{k} = 0,$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}, \quad \mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}.$$

5.2 Displacement Vectors

Let points of three-dimensional Euclidean space be represented in Cartesian coordinates in the usual fashion, so that the line segments joining the origin to the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ are orthogonal and of unit lengths.

Let A and B be points in three-dimensional Euclidean space be represented in Cartesian coordinates so that

$$A = (a_1, a_2, a_3), \quad B = (b_1, b_2, b_3).$$

The *displacement vector* $\overrightarrow{AB}$ from A to B is defined such that

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3).$$

If A, B and C are points in three-dimensional Euclidean space then

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

Points A, B, C and D of three-dimensional Euclidean space are the vertices of a parallelogram (labelled in clockwise or anticlockwise) order if and only if $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$.

Let the origin O be the point with Cartesian coordinates. The *position vector* of a point A (with respect to the chosen origin) is defined to be the displacement vector $\overrightarrow{OA}$.

5.3 Geometrical Interpretation of the Scalar Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in three-dimensional space, represented in some Cartesian coordinate system by the ordered triples (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. The scalar product $\mathbf{u} \,.\, \mathbf{v}$ of the vectors $\mathbf{u}$ and $\mathbf{v}$ is then given by the formula

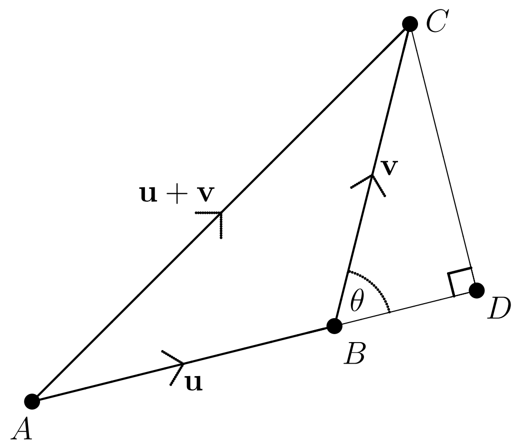
$$\mathbf{u} \,.\, \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Proposition 5.1 *The scalar product* $\mathbf{u} \,.\, \mathbf{v}$ *of non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in three-dimensional space satisfies*

$$\mathbf{u} \,.\, \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta,$$

where θ *denotes the angle between the vectors* $\mathbf{u}$ *and* $\mathbf{v}$*.*

Proof Suppose first that the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ is an acute angle, so that $0 < \theta < \frac{1}{2}\pi$. Let us consider a triangle ABC, where $\overrightarrow{AB} = \mathbf{u}$ and $\overrightarrow{BC} = \mathbf{v}$, and thus $\overrightarrow{AC} = \mathbf{u} + \mathbf{v}$. Let ADC be the right-angled triangle constructed as depicted in the figure below, so that the line AD extends AB and the angle at D is a right angle.



Then the lengths of the line segments AB, BC, AC, BD and CD may be expressed in terms of the lengths $|\mathbf{u}|$, $|\mathbf{v}|$ and $|\mathbf{u} + \mathbf{v}|$ of the displacement vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ and the angle θ between the vectors $\mathbf{u}$ and $\mathbf{v}$ by means of the following equations:

$$AB = |\mathbf{u}|, \quad BC = |\mathbf{v}|, \quad AC = |\mathbf{u} + \mathbf{v}|,$$

$$BD = |\mathbf{v}|\,\cos\theta \quad \text{and} \quad DC = |\mathbf{v}|\,\sin\theta.$$

Then

$$AD = AB + BD = |\mathbf{u}| + |\mathbf{v}|\cos\theta.$$

The triangle ADC is a right-angled triangle with hypotenuse AC. It follows from Pythagoras' Theorem that

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 &= AC^2 = AD^2 + DC^2 = (|\mathbf{u}| + |\mathbf{v}|\,\cos\theta)^2 + |\mathbf{v}|\,\sin^2\theta \\
&= |\mathbf{u}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\,\cos\theta + |\mathbf{v}|\,\cos^2\theta + |\mathbf{v}|\,\sin^2\theta \\
&= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2|\mathbf{u}|\,|\mathbf{v}|\,\cos\theta,
\end{aligned}$$

because $\cos^2\theta + \sin^2\theta = 1$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2, u_3+v_3),$$

and therefore

$$\begin{aligned}
|\mathbf{u}+\mathbf{v}|^2 &= (u_1+v_1)^2 + (u_2+v_2)^2 + (u_3+v_3)^2 \\
&= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 + 2u_2v_2 + v_2^2 + u_3^2 + 2u_3v_3 + v_3^2 \\
&= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2(u_1v_1 + u_2v_2 + u_3v_3) \\
&= |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u}\,.\,\mathbf{v}.
\end{aligned}$$

On comparing the expressions for $|\mathbf{u}+\mathbf{v}|^2$ given by the above equations, we see that $\mathbf{u}\,.\,\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ when $0 < \theta < \frac{1}{2}\pi$.

The identity $\mathbf{u}\,.\,\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ clearly holds when $\theta = 0$ and $\theta = \pi$. Pythagoras' Theorem ensures that it also holds when the angle θ is a right angle (so that $\theta = \frac{1}{2}\pi$. Suppose that $\frac{1}{2}\pi < \theta < \pi$, so that the angle θ is obtuse. Then the angle between the vectors $\mathbf{u}$ and $-\mathbf{v}$ is acute, and is equal to $\pi - \theta$. Moreover $\cos(\pi - \theta) = -\cos\theta$ for all angles θ. It follows that

$$\mathbf{u}\,.\,\mathbf{v} = -\mathbf{u}\,.\,(-\mathbf{v}) = -|\mathbf{u}|\,|\mathbf{v}|\,\cos(\pi-\theta) = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$$

when $\frac{1}{2}\pi < \theta < \pi$. We have therefore verified that the identity $\mathbf{u}\,.\,\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta$ holds for all non-zero vectors $\mathbf{u}$ and $\mathbf{v}$, as required. ∎

Corollary 5.2 *Two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in three-dimensional space are perpendicular if and only if* $\mathbf{u}\,.\,\mathbf{v} = 0$.

Proof It follows directly from Proposition 5.1 that $\mathbf{u}\,.\,\mathbf{v} = 0$ if and only if $\cos\theta = 0$, where θ denotes the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$. This is the case if and only if the vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

Example We can use the scalar product to calculate the angle θ between the vectors $(2, 2, 0)$ and $(0, 3, 3)$ in three-dimensional space. Let $\mathbf{u} = (2, 2, 0)$ and $\mathbf{v} = (3, 3, 0)$. Then $|\mathbf{u}|^2 = 2^2 + 2^2 = 8$ and $|\mathbf{v}|^2 = 3^2 + 3^2 = 18$. It follows that $(|\mathbf{u}|\,|\mathbf{v}|)^2 = 8 \times 18 = 144$, and thus $|\mathbf{u}|\,|\mathbf{v}| = 12$. Now $\mathbf{u} \,.\, \mathbf{v} = 6$. It follows that

$$6 = |\mathbf{u}|\,|\mathbf{v}|\,\cos\theta = 12\cos\theta.$$

Therefore $\cos\theta = \frac{1}{2}$, and thus $\theta = \frac{1}{3}\pi$.

We can use the scalar product to find the distance between points on a sphere. Now the Cartesian coordinates of a point P on the unit sphere about the origin O in three-dimensional space may be expressed in terms of angles θ and φ as follows:

$$P = (\sin\theta\,\cos\varphi,\,\sin\theta\,\sin\varphi,\,\cos\theta).$$

The angle θ is that between the displacement vector $\overrightarrow{OP}$ and the vectical vector $(0, 0, 1)$. Thus the angle $\frac{1}{2}\pi - \theta$ represents the 'latitude' of the point P, when we regard the point $(0, 0, 1)$ as the 'north pole' of the sphere. The angle φ measures the 'longitude' of the point P.

Now let P_1 and P_2 be points on the unit sphere, where

$$\begin{aligned} P_1 &= (\sin\theta_1\,\cos\varphi_1,\,\sin\theta_1\,\sin\varphi_1,\,\cos\theta_1), \\ P_2 &= (\sin\theta_2\,\cos\varphi_2,\,\sin\theta_2\,\sin\varphi_2,\,\cos\theta_2). \end{aligned}$$

We wish to find the angle ψ between the displacement vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ of the points P_1 and P_2 from the origin. Now $|\overrightarrow{OP_1}| = 1$ and $|\overrightarrow{OP_2}| = 1$. On applying Proposition 5.1, we see that

$$\begin{aligned} \cos\psi &= \overrightarrow{OP_1}\,.\,\overrightarrow{OP_2} \\ &= \sin\theta_1\,\sin\theta_2\,\cos\varphi_1\,\cos\varphi_2 + \sin\theta_1\,\sin\theta_2\,\sin\varphi_1\,\sin\varphi_2 \\ &\qquad + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,(\cos\varphi_1\,\cos\varphi_2 + \sin\varphi_1\,\sin\varphi_2) + \cos\theta_1\,\cos\theta_2 \\ &= \sin\theta_1\,\sin\theta_2\,\cos(\varphi_1 - \varphi_2) + \cos\theta_1\,\cos\theta_2. \end{aligned}$$

5.4 Geometrical Interpretation of the Vector Product

Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in three-dimensional space, with Cartesian components given by the formulae $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$. The vector product $\mathbf{a} \times \mathbf{b}$ is then determined by the formula

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1).$$

Proposition 5.3 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in three-dimensional space* $\mathbb{R}^3$*. Then their vector product* $\mathbf{a} \times \mathbf{b}$ *is a vector of length* $|\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta|$*, where* θ *denotes the angle between the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*. Moreover the vector* $\mathbf{a} \times \mathbf{b}$ *is perpendicular to the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*.*

Proof Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, and let l denote the length $|\mathbf{a} \times \mathbf{b}|$ of the vector $\mathbf{a} \times \mathbf{b}$. Then

$$\begin{aligned}
l^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 \\
&\quad + a_3^2b_1^2 + a_1^2b_3^2 - 2a_3a_1b_3b_1 \\
&\quad + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2 \\
&= a_1^2(b_2^2 + b_3^2) + a_2^2(b_1^2 + b_3^2) + a_3^2(b_1^2 + b_2^2) \\
&\quad - 2a_2a_3b_2b_3 - 2a_3a_1b_3b_1 - 2a_1a_2b_1b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - a_1^2b_1^2 - a_2^2b_2^2 - a_3^2b_3^2 - 2a_2b_2a_3b_3 - 2a_3b_3a_1b_1 - 2a_1b_1a_2b_2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \,.\, \mathbf{b})^2
\end{aligned}$$

since

$$|\mathbf{a}|^2 = a_1^2 + a_2^2 + a_3^2, \quad |\mathbf{b}|^2 = b_1^2 + b_2^2 + b_3^2, \quad \mathbf{a} \,.\, \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

But $\mathbf{a} \,.\, \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$ (Proposition 5.1). Therefore

$$l^2 = |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta$$

(since $\sin^2\theta + \cos^2\theta = 1$ for all angles θ) and thus $l = |\mathbf{a}|\,|\mathbf{b}|\,|\sin\theta|$. Also

$$\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{b}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0$$

and

$$\mathbf{b} \,.\, (\mathbf{a} \times \mathbf{b}) = b_1(a_2b_3 - a_3b_2) + b_2(a_3b_1 - a_1b_3) + b_3(a_1b_2 - a_2b_1) = 0$$

and therefore the vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (Corollary 5.2), as required. ∎

Using elementary geometry, and the formula for the length of the vector product $\mathbf{a} \times \mathbf{b}$ given by Proposition 5.3 it is not difficult to show that the length of this vector product is equal to the area of a parallelogram in three-dimensional space whose sides are represented, in length and direction, by the vectors $\mathbf{a}$ and $\mathbf{b}$.

Remark Let $\mathbf{a}$ and $\mathbf{b}$ be non-zero vectors that are not colinear (i.e., so that they do not point in the same direction, or in opposite directions). The direction of $\mathbf{a} \times \mathbf{b}$ may be determined, using the thumb and first two fingers of your right hand, as follows. Orient your right hand such that the thumb points in the direction of the vector $\mathbf{a}$ and the first finger points in the direction of the vector $\mathbf{b}$, and let your second finger point outwards from the palm of your hand so that it is perpendicular to both the thumb and the first finger. Then the second finger points in the direction of the vector product $\mathbf{a} \times \mathbf{b}$.

Indeed it is customary to describe points of three-dimensional space by Cartesian coordinates (x, y, z) oriented so that if the positive x-axis and positive y-axis are pointed in the directions of the thumb and first finger respectively of your right hand, then the positive z-axis is pointed in the direction of the second finger of that hand, when the thumb and first two fingers are mutually perpendicular. For example, if the positive x-axis points towards the East, and the positive y-axis points towards the North, then the positive z-axis is chosen so that it points upwards. Moreover if $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ then these vectors $\mathbf{i}$ and $\mathbf{j}$ are unit vectors pointed in the direction of the positive x-axis and positive y-axis respectively, and $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, where $\mathbf{k} = (0, 0, 1)$, and the vector $\mathbf{k}$ points in the direction of the positive z-axis. Thus the 'right-hand' rule for determining the direction of the vector product $\mathbf{a} \times \mathbf{b}$ using the fingers of your right hand is valid when $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$.

If the directions of the vectors $\mathbf{a}$ and $\mathbf{b}$ are allowed to vary continuously, in such a way that these vectors never point either in the same direction or in opposite directions, then their vector product $\mathbf{a} \times \mathbf{b}$ will always be a non-zero vector, whose direction will vary continuously with the directions of $\mathbf{a}$ and $\mathbf{b}$. It follows from this that if the 'right-hand rule' for determining the direction of $\mathbf{a} \times \mathbf{b}$ applies when $\mathbf{a} = \mathbf{i}$ and $\mathbf{b} = \mathbf{j}$, then it will also apply whatever the directions of $\mathbf{a}$ and $\mathbf{b}$, since, if your right hand is moved around in such a way that the thumb and first finger never point in the same direction, and if the second finger is always perpendicular to the thumb and first finger, then the direction of the second finger will vary continuously, and will therefore always point in the direction of the vector product of two vectors pointed in the direction of the thumb and first finger respectively.

Example We shall find the area of the parallelogram $OACB$ in three-dimensional space, where

$$O = (0, 0, 0), \quad A = (1, 2, 0), \qquad B = (-4, 2, -5), \quad C = (-3, 4, -5).$$

Note that $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$. Let $\mathbf{a} = \overrightarrow{OA} = (1, 2, 0)$ and $\mathbf{b} = \overrightarrow{OB} =$

$(-4, 2, -5)$. Then $\mathbf{a} \times \mathbf{b} = (-10, 5, 10)$. Now $(-10, 5, 10) = 5(-2, 1, 2)$, and $|(-2, 1, 2)| = \sqrt{9} = 3$. It follows that

$$\text{area}\, OACB = |\mathbf{a} \times \mathbf{b}| = 15.$$

Note also that the vector $(-2, 1, 2)$ is perpendicular to the parallelogram $OACB$.

Example We shall find the equation of the plane containing the points A, B and C where $A = (3, 4, 1)$, $B = (4, 6, 1)$ and $C = (3, 5, 3)$. Now if $\mathbf{u} = \overrightarrow{AB} = (1, 2, 0)$ and $\mathbf{v} = \overrightarrow{AC} = (0, 1, 2)$ then the vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel to the plane. It follows that the vector $\mathbf{u} \times \mathbf{v}$ is perpendicular to this plane. Now $\mathbf{u} \times \mathbf{v} = (4, -2, 1)$, and therefore the displacement vector between any two points of the plane must be perpendicular to the vector $(4, -2, 1)$. It follows that the function mapping the point (x, y, z) to the quantity $4x - 2y + z$ must be constant throughout the plane. Thus the equation of the plane takes the form

$$4x - 2y + z = k,$$

for some constant k. We can calculate the value of k by substituting for x, y and z the coordinates of any chosen point of the plane. On taking this chosen point to be the point A, we find that $k = 4 \times 3 - 2 \times 4 + 1 = 5$. Thus the equation of the plane is the following:

$$4x - 2y + z = 5.$$

(We can check our result by verifying that the coordinates of the points A, B and C do indeed satisfy this equation.)

5.5 Scalar Triple Products

Given three vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in three-dimensional space, we can form the *scalar triple product* $\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w})$. This quantity can be expressed as the determinant of a 3×3 matrix whose rows contain the Cartesian components of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. Indeed

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1),$$

and thus

$$\mathbf{u} \,.\, (\mathbf{v} \times \mathbf{w}) = u_1 (v_2 w_3 - v_3 w_2) + u_2 (v_3 w_1 - v_1 w_3) + u_3 (v_1 w_2 - v_2 w_1).$$

The quantity on the right hand side of this equality defines the determinant of the 3×3 matrix

$$\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

We have therefore obtained the following result.

Proposition 5.4 *Let* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u}\,.\,(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Corollary 5.5 *Let* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\begin{aligned} \mathbf{u}\,.\,(\mathbf{v} \times \mathbf{w}) &= \mathbf{v}\,.\,(\mathbf{w} \times \mathbf{u}) = \mathbf{w}\,.\,(\mathbf{u} \times \mathbf{v}) \\ &= -\mathbf{u}\,.\,(\mathbf{w} \times \mathbf{v}) = -\mathbf{v}\,.\,(\mathbf{u} \times \mathbf{w}) = -\mathbf{w}\,.\,(\mathbf{v} \times \mathbf{u}). \end{aligned}$$

Proof The basic theory of determinants ensures that 3×3 determinants are unchanged under cyclic permutations of their rows by change sign under transpositions of their rows. These identities therefore follow directly from Proposition 5.4. ∎

One can show that the absolute value of the scalar triple product $\mathbf{u}.(\mathbf{v}\times\mathbf{w})$ is the volume of the parallelepiped in three-dimensional space whose vertices are the points whose displacement vectors from some fixed point O are $\mathbf{0}$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}+\mathbf{v}$, $\mathbf{u}+\mathbf{w}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{u}+\mathbf{v}+\mathbf{w}$. (A *parallelepiped* is a solid like a brick, but whereas the faces of a brick are rectangles, the faces of the parallelepiped are parallelograms.)

Example We shall find the volume of the parallelepiped in 3-dimensional space with vertices at $(0,0,0)$, $(1,2,0)$, $(-4,2,-5)$, $(0,1,1)$, $(-3,4,-5)$, $(1,3,1)$, $(-4,3,-4)$ and $(-3,5,-4)$. The volume of this parallelepiped is the absolute value of the scalar triple product $\mathbf{u}\,.\,(\mathbf{v} \times \mathbf{w})$, where

$$\mathbf{u} = (1,2,0), \quad \mathbf{v} = (-4,2,-5), \quad \mathbf{w} = (0,1,1).$$

Now

$$\begin{aligned} \mathbf{u}\,.\,(\mathbf{v} \times \mathbf{w}) &= (1,2,0)\,.\,(\,(-4,2,-5) \times (0,1,1)\,) \\ &= (1,2,0)\,.\,(7,4,-4) = 7 + 2 \times 4 = 15. \end{aligned}$$

Thus the volume of the paralellepiped is 15 units.

5.6 The Vector Triple Product Identity

Proposition 5.6 (Vector Triple Product Identity) *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in three-dimensional space. Then*

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w}$$

and

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\,\mathbf{u}.$$

Proof Let $\mathbf{q} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{w} = (w_1, w_2, w_3)$, and $\mathbf{q} = (q_1, q_2, q_3)$. Then

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1).$$

and hence $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (q_1, q_2, q_3)$, where

$$\begin{aligned} q_1 &= u_2(v_1 w_2 - v_2 w_1) - u_3(v_3 w_1 - v_1 w_3) \\ &= (u_2 w_2 + u_3 w_3) v_1 - (u_2 v_2 + u_3 v_3) w_1 \\ &= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (u_1 v_1 + u_2 v_2 + u_3 v_3) w_1 \\ &= (\mathbf{u} \,.\, \mathbf{w}) v_1 - (\mathbf{u} \,.\, \mathbf{v}) w_1 \end{aligned}$$

Similarly

$$q_2 = (\mathbf{u} \,.\, \mathbf{w}) v_2 - (\mathbf{u} \,.\, \mathbf{v}) w_2$$

and

$$q_3 = (\mathbf{u} \,.\, \mathbf{w}) v_3 - (\mathbf{u} \,.\, \mathbf{v}) w_3$$

(In order to verify the formula for q_2 with an minimum of calculation, take the formulae above involving q_1, and cyclicly permute the subcripts 1, 2 and 3, replacing 1 by 2, 2 by 3, and 3 by 1. A further cyclic permutation of these subscripts yields the formula for q_3.) It follows that

$$\mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w},$$

as required, since we have shown that the Cartesian components of the vectors on either side of this identity are equal. Thus

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\,\mathbf{w}.$$

On replacing $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ by $\mathbf{w}$, $\mathbf{u}$ and $\mathbf{v}$ respectively, we find that

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w} \,.\, \mathbf{v})\,\mathbf{u} - (\mathbf{w} \,.\, \mathbf{u})\,\mathbf{v}.$$

It follows that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = -\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \,.\, \mathbf{w})\,\mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\,\mathbf{u},$$

as required. ∎

Remark When recalling these identities for use in applications, it is often helpful to check that the summands on the right hand side have the correct sign by substituting, for example, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{i}$ for $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

Thus, for example, $(\mathbf{i} \times \mathbf{j}) \times \mathbf{i} = \mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $(\mathbf{i}.\mathbf{i})\mathbf{j} - (\mathbf{j}.\mathbf{i})\mathbf{i} = \mathbf{j}$. This helps check that the summands on the right hand side of the identity $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{v} \,.\, \mathbf{w})\, \mathbf{u}$ have been chosen with the correct sign (assuming that these summands have opposite signs).

We present below a second proof making use of the following standard identity.

Proposition 5.7 *Let $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ be defined for $i, j, k \in \{1, 2, 3\}$ such that*

$$\varepsilon_{i,j,k} = \begin{cases} 1 & \textit{if } (i,j,k) \in \{(1,2,3),\ (2,3,1),\ (3,1,2)\}; \\ -1 & \textit{if } (i,j,k) \in \{(1,3,2),\ (2,1,3),\ (3,2,1)\}; \\ 0 & \textit{otherwise.} \end{cases}$$

and

$$\delta_{i,j} = \begin{cases} 1 & \textit{if } i = j; \\ 0 & \textit{otherwise.} \end{cases}$$

Then

$$\sum_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}$$

for all $i, j, m \in \{1, 2, 3\}$.

Proof Suppose that $j = k$. Then $\varepsilon_{i,j,k} = 0$ for $i = 1, 2, 3$ and thus the left hand side is zero. The right hand side is also zero in this case, because

$$\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = \delta_{j,m}\delta_{k,n} - \delta_{k,n}\delta_{j,m} = 0$$

when $j = k$. Thus $\sum\limits_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ when $j = k$. Similarly $\sum\limits_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ when $m = n$. Next suppose that $j \neq k$ and $m \neq n$ but $\{j, k\} \neq \{m, n\}$. In this case the single value of i in $\{1, 2, 3\}$ for which $\varepsilon_{i,j,k} \neq 0$ does not coincide with the single value of i for which $\varepsilon_{i,m,n} \neq 0$, and therefore $\sum\limits_{i=1}^{3} \varepsilon_{i,j,k}\, \varepsilon_{i,m,n} = 0$. Moreover either $j \notin \{m, n\}$, in which case $\delta_{j,m} = \delta_{j,n} = 0$ and thus $\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$, or else $k \notin \{m, n\}$, in which case $\delta_{k,m} = \delta_{k,n} = 0$ and thus $\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$. It follows from all the cases considered above that $\sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m} = 0$ unless both $j \neq k$ and $\{j, k\} = \{m, n\}$. Suppose then that $j \neq k$ and $\{j, k\} = \{m, n\}$. Then there is a single value of i for which $\varepsilon_{i,j,k} \neq 0$. For this particular value of i we find that

$$\varepsilon_{i,j,k}\,\varepsilon_{i,m,n} = \begin{cases} 1 & \text{if } j \neq k,\ j = m \text{ and } k = n; \\ -1 & \text{if } j \neq k,\ j = n \text{ and } k = m. \end{cases}$$

It follows that, in the cases where $j \neq k$ and $\{j, k\} = \{m, n\}$,

$$\begin{aligned} \sum_{i=1}^{3} \varepsilon_{i,j,k}\,\varepsilon_{i,m,n} &= \begin{cases} 1 & \text{if } j \neq k,\ j = m \text{ and } k = n, \\ -1 & \text{if } j \neq k,\ j = n \text{ and } k = m, \\ 0 & \text{otherwise,} \end{cases} \\ &= \delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}, \end{aligned}$$

as required. ∎

Second Proof of Proposition 5.6 Let $\mathbf{p} = \mathbf{v} \times \mathbf{w}$ and $\mathbf{q} = \mathbf{u} \times \mathbf{p} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$, and let

$$\mathbf{u} = (u_1, u_2, u_3), \quad \mathbf{v} = (v_1, v_2, v_3), \quad \mathbf{w} = (w_1, w_2, w_3),$$

$$\mathbf{p} = (p_1, p_2, p_3) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, q_3).$$

The definition of the vector product ensures that $p_i = \sum_{j,k=1}^{3} \varepsilon_{i,j,k} v_j w_k$ for $i = 1, 2, 3$, where $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ are defined for $i, j, k \in \{1, 2, 3\}$ as described in the statement of Proposition 5.7. It follows that

$$\begin{aligned} q_m &= \sum_{n,i=1}^{3} \varepsilon_{m,n,i}\, u_n p_i = \sum_{n,i,j,k=1}^{3} \varepsilon_{m,n,i}\varepsilon_{i,j,k}\, u_n v_j w_k \\ &= \sum_{n,j,k=1}^{3} \sum_{i=1}^{3} \varepsilon_{i,m,n}\varepsilon_{i,j,k}\, u_n v_j w_k \\ &= \sum_{n,j,k=1}^{3} (\delta_{j,m}\delta_{k,n} - \delta_{j,n}\delta_{k,m}) u_n v_j w_k \\ &= \sum_{n,k=1}^{3} \delta_{k,n}\, v_m u_n w_k - \sum_{n,j=1}^{3} \delta_{j,n}\, u_n v_j w_m = v_m \sum_{k=1}^{3} u_k w_k - w_m \sum_{j=1}^{3} u_j v_j \\ &= (\mathbf{u} \,.\, \mathbf{w}) v_m - (\mathbf{u} \,.\, \mathbf{v}) w_m \end{aligned}$$

for $m = 1, 2, 3$, and therefore

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{q} = (\mathbf{u} \,.\, \mathbf{w})\, \mathbf{v} - (\mathbf{u} \,.\, \mathbf{v})\, \mathbf{w},$$

as required. ∎

Remark The identity

$$\alpha \mathrm{S} \,.\, \alpha'\alpha'' - \alpha'\mathrm{S}.\alpha''\alpha = \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \alpha'')$$

occurs as equation (12) in article 22 of William Rowan Hamilton's *On Quaternions, or on a new System of Imaginaries in Algebra*, published in the *Philosophical Magazine* in August 1846. Hamilton noted in that paper that this identity "will be found to have extensive applications."

In Hamilton's quaternion algebra, vectors in three-dimensional space are represented as pure imaginary quaternions and are denoted by Greek letters. Thus α, α' and α'' denote (in Hamilton's notation) three arbitrary vectors. The product of two vectors α' and α'' in Hamilton's system is a quaternion which is the sum of a *scalar part* $\mathrm{S} \,.\, \alpha\alpha'$ and a *vector part* $\mathrm{V}.\alpha\alpha'$. (The scalar and vector parts of a quaternion are the analogues, in Hamilton's quaternion algebra, of the real and imaginary parts of a complex number.) Now a quaternion can be represented in the form $s + u_1 i + u_2 j + u_3 k$ where s, u_1, u_2, u_3 are real numbers. The operations of quaternion addition, quaternion subtraction and scalar multiplication by real numbers are defined so that the space $\mathbb{H}$ of quaternions is a four-dimensional vector space over the real numbers with basis 1, i, j, k. The operation of quaternion multiplication is defined so that quaternion multiplication is distributive over addition and is determined by the identities

$$i^2 = j^2 = k^2 = -1,$$

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$$

that Hamilton formulated in 1843. It then transpires that the operation of quaternion multiplication is associative. Hamilton described his discovery of the quaternion algebra in a letter to P.G. Tait dated October 15, 1858 as follows:—

> ... P.S.—To-morrow will be the 15th birthday of the Quaternions. They started into life, or light, full grown, on [Monday] the 16th of October, 1843, as I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, which my boys have since called the Quaternion Bridge. That is to say, I then and

> there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations between i, j, k; exactly such as I have used them ever since. I pulled out on the spot a pocket-book, which still exists, and made an entry, on which, at the very moment, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a problem to have been at that moment solved—an intellectual want relieved—which had haunted me for at least fifteen years before.

Let quaternions q and r be defined such that $q = s + u_1 i + u_2 j + u_3 k$ and $r = t + v_1 i + v_2 j + v_3 k$, where $s, t, u_1, u_2, u_3, v_1, v_2, v_3$ are real numbers. We can then write $q = s + \alpha$ and $r = t + \beta$, where

$$\alpha = u_1 i + u_2 j + u_3 k, \quad \beta = v_1 i + v_2 j + v_3 k.$$

Hamilton then defined the *scalar part* of the quaternion q to be the real number s, and the *vector part* of the quaternion q to be the quaternion α determined as described above. The Distributive Law for quaternion multiplication and the identities for the products of i, j and k then ensure that

$$qr = st + \mathrm{S}\,.\,\alpha\beta + s\beta + t\alpha + \mathrm{V}\,.\,\alpha\beta,$$

where

$$\mathrm{S}\,.\,\alpha\beta = -(u_1 v_1 + u_2 v_2 + u_3 v_3)$$

and

$$\mathrm{V}\,.\,\alpha\beta = (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k.$$

Thus the *scalar part* $\mathrm{S}\,.\,\alpha'\alpha''$ of the quaternion product $\alpha'\alpha''$ represents the negative of the scalar product of the vectors α' and α'', and the *vector part* $\mathrm{V}.\alpha'\alpha''$ represents the vector product of the quaternion $\alpha\alpha'$. Thus Hamilton's identity can be represented, using the now customary notation for the scalar and vector products, as follows:—

$$-\alpha(\alpha'\,.\,\alpha'') + \alpha'(\alpha''\,.\,\alpha) = (\alpha \times \alpha') \times \alpha''.$$

Hamilton's identity of 1846 (i.e., equation (12) in article 22 of *On quaternions*) is thus the Vector Triple Product Identity stated in Proposition 5.6.

Corollary 5.8 *Let* $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbb{R}^3$*. Then*

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = (\mathbf{u}.(\mathbf{v} \times \mathbf{w}))\mathbf{u}.$$

Proof Using the Vector Triple Product Identity (Proposition 5.6) and basic properties of the scalar triple product Corollary 5.5, we find that

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\times(\mathbf{u}\times\mathbf{w}) &= (\mathbf{u}.(\mathbf{u}\times\mathbf{w}))\mathbf{v}-(\mathbf{v}.(\mathbf{u}\times\mathbf{w}))\mathbf{u}\\ &= (\mathbf{u}.(\mathbf{v}\times\mathbf{w}))\mathbf{u},\end{aligned}$$

as required. ∎

5.7 Lagrange's Quadruple Product Identity

Proposition 5.9 (Lagrange's Quadruple Product Identity) *Let* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *and* $\mathbf{z}$ *be vectors in* $\mathbb{R}^3$*. Then*

$$(\mathbf{u}\times\mathbf{v})\,.\,(\mathbf{w}\times\mathbf{z}) = (\mathbf{u}\,.\,\mathbf{w})(\mathbf{v}\,.\,\mathbf{z}) - (\mathbf{u}\,.\,\mathbf{z})(\mathbf{v}\,.\,\mathbf{w}).$$

Proof Using the Vector Triple Product Identity (Proposition 5.6) and basic properties of the scalar triple product Corollary 5.5, we find that

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\,.\,(\mathbf{w}\times\mathbf{z}) &= \mathbf{z}.((\mathbf{u}\times\mathbf{v})\times\mathbf{w})\\ &= \mathbf{z}.((\mathbf{u}\,.\,\mathbf{w})\mathbf{v}-(\mathbf{v}\,.\,\mathbf{w})\mathbf{u})\\ &= (\mathbf{u}\,.\,\mathbf{w})(\mathbf{v}\,.\,\mathbf{z}) - (\mathbf{u}\,.\,\mathbf{z})(\mathbf{v}\,.\,\mathbf{w}),\end{aligned}$$

as required. ∎

Remark Substituting $\mathbf{i}$, $\mathbf{j}$, $\mathbf{i}$ and $\mathbf{j}$ for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$ respectively, where

$$\mathbf{i}=(1,0,0),\quad \mathbf{j}=(0,1,0),\quad \mathbf{k}=(0,0,1),$$

we find that $(\mathbf{i}\times\mathbf{j}).(\mathbf{i}\times\mathbf{j})=\mathbf{k}.\mathbf{k}=1$ and $(\mathbf{i}.\mathbf{i})(\mathbf{j}.\mathbf{j})-(\mathbf{i}.\mathbf{j})(\mathbf{j}.\mathbf{i})=1-0=1$. This helps check that the summands on the right hand side have been allocated the correct sign.

Second Proof of Proposition 5.9 Let

$$\mathbf{u}=(u_1,u_2,u_3),\quad \mathbf{v}=(v_1,v_2,v_3),\quad \mathbf{w}=(w_1,w_2,w_3),\quad \mathbf{z}=(z_1,z_2,z_3),$$

and let $\varepsilon_{i,j,k}$ and $\delta_{i,j}$ be defined for $i,j,k\in\{1,2,3\}$ as described in the statement of Proposition 5.7. Then the components of $\mathbf{u}\times\mathbf{v}$ are the values of $\sum\limits_{j,k=1}^{3}\varepsilon_{i,j,k}v_jw_k$ for $i=1,2,3$. It follows from Proposition 5.7 that

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\,.\,(\mathbf{w}\times\mathbf{z}) &= \sum_{i,j,k,m,n}\varepsilon_{i,j,k}\,\varepsilon_{i,m,n}\,u_jv_kw_mz_n\\ &= \sum_{j,k,m,n}(\delta_{j,m}\delta_{k,n}-\delta_{j,n}\delta_{k,m})\,u_jv_kw_mz_n\\ &= \sum_{j,k}(u_jv_kw_jz_k - u_jv_kw_kz_j)\\ &= (\mathbf{u}\,.\,\mathbf{w})(\mathbf{v}\,.\,\mathbf{z}) - (\mathbf{u}\,.\,\mathbf{z})(\mathbf{v}\,.\,\mathbf{w}),\end{aligned}$$

as required. ∎

5.8 Orthonormal Triads of Unit Vectors

Let $\mathbf{e}$ and $\mathbf{f}$ be unit vectors (i.e., vectors of length one) that are perpendicular to each other, and let $\mathbf{g} = \mathbf{e} \times \mathbf{f}$. It follows immediately from Proposition 5.3 that $|\mathbf{g}| = |\mathbf{e}|\,|\mathbf{f}| = 1$, and that this unit vector $\mathbf{g}$ is perpendicular to both $\mathbf{e}$ and $\mathbf{f}$. Then

$$\mathbf{e} \,.\, \mathbf{e} = \mathbf{f} \,.\, \mathbf{f} = \mathbf{g} \,.\, \mathbf{g} = 1$$

and

$$\mathbf{e} \,.\, \mathbf{f} = \mathbf{f} \,.\, \mathbf{g} = \mathbf{g} \,.\, \mathbf{e} = 0.$$

On applying the Vector Triple Product Identity (Proposition 5.6) we find that

$$\mathbf{f} \times \mathbf{g} = \mathbf{f} \times (\mathbf{e} \times \mathbf{f}) = (\mathbf{f} \,.\, \mathbf{f})\,\mathbf{e} - (\mathbf{f} \,.\, \mathbf{e})\,\mathbf{f} = \mathbf{e},$$

and

$$\mathbf{g} \times \mathbf{e} = -\mathbf{e} \times \mathbf{g} = -\mathbf{e} \times (\mathbf{e} \times \mathbf{f}) = -(\mathbf{e} \,.\, \mathbf{f})\,\mathbf{e} + (\mathbf{e} \,.\, \mathbf{e})\,\mathbf{f} = \mathbf{f},$$

Therefore

$$\mathbf{e} \times \mathbf{f} = -\mathbf{f} \times \mathbf{e} = \mathbf{g}, \quad \mathbf{f} \times \mathbf{g} = -\mathbf{g} \times \mathbf{f} = \mathbf{e}, \quad \mathbf{g} \times \mathbf{e} = -\mathbf{e} \times \mathbf{g} = \mathbf{f},$$

Three unit vectors, such as the vectors $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ above, that are mutually perpendicular, are referred to as an *orthonormal triad* of vectors in three-dimensional space. The vectors $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ in any orthonormal triad are linearly independent. It follows from the theory of bases and dimension in finite-dimensional vector spaces that that any vector in three-dimensional space may be expressed, uniquely, as a linear combination of the form

$$p\mathbf{e} + q\mathbf{f} + r\mathbf{g}.$$

Any Cartesian coordinate system on three-dimensional space determines an orthonormal triad $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, where

$$\mathbf{i} = (1, 0, 0), \quad \mathbf{j} = (0, 1, 0), \quad \mathbf{k} = (0, 0, 1).$$

The scalar and vector products of these vectors satisfy the same relations as the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ above. A vector represented in these Cartesian components by an ordered triple (x, y, z) then satisfies the identity

$$(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

5.9 Some Applications of Vector Algebra to Spherical Trigometry

Let S^2 be the unit sphere

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

in three-dimensional Euclidean space $\mathbb{R}^3$. Each point of S^2 may be represented in the form

$$(\sin\theta\, \cos\varphi, \sin\theta\, \sin\varphi, \cos\theta).$$

Let I, J and K denote the points of S^2 defined such that

$$I = (1, 0, 0), \quad J = (0, 1, 0), \quad K = (0, 0, 1).$$

We take the origin O of Cartesian coordinates to be located at the centre of the sphere. The position vectors of the points I, J and K are then the standard unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

It may be helpful to regard the point K as representing the "north pole" of the sphere. The "equator" is then the great circle consisting of those points (x, y, z) of S^2 for which $z = 0$. Every point P of S^2 is the pole of a great circle on S^2 consisting of those points of S^2 whose position vectors are orthogonal to the position vector $\mathbf{p}$ of the point P.

Let L and L' be distinct points of S^2 with position vectors $\mathbf{r}$ and $\mathbf{r}'$ respectively. We denote by $\sin LL'$ and $\cos LL'$ the sine and cosine of the angles between the lines joining the centre of the sphere to the points L and L'.

Lemma 5.10 *Let L and L' be points on the unit sphere S^2 in $\mathbb{R}^3$, and let $\mathbf{r}$ and $\mathbf{r}'$ denote the displacement vectors of those points from the centre of the sphere. Then*

$$\mathbf{r} \,.\, \mathbf{r}' = \cos LL'$$

and

$$\mathbf{r} \times \mathbf{r}' = \sin LL'\, \mathbf{n}_{L,L'},$$

where $\mathbf{n}_{L,L'}$ is a unit vector orthogonal to the plane through the centre of the sphere that contains the points L and L'.

Proof The displacement vectors $\mathbf{r}$ and $\mathbf{r}'$ of the points L and L' from the centre of the sphere satisfy $|\mathbf{r}| = 1$ and $|\mathbf{r}'| = 1$ (because the sphere has unit radius). The required identities therefore follows from basic properties of the scalar and vector products stated in Proposition 5.1 and Proposition 5.3. ∎

Lemma 5.11 *Let V and W be planes in $\mathbb{R}^3$ that are not parallel, and let $\mathbf{n}_V$ and $\mathbf{v}_W$ be the unit vectors orthogonal to the planes V and W, and let α be the angle between those planes. Then*

$$\mathbf{n}_V \,.\, \mathbf{n}_W = \cos\alpha,$$

and

$$\mathbf{n}_V \times \mathbf{n}_W = \sin\alpha\, \mathbf{u},$$

where $\mathbf{u}$ is a unit vector in the direction of the line of intersection of the planes V and W.

Proof The vectors $\mathbf{n}_V$ and $\mathbf{n}_W$ are not parallel, because the planes are not parallel, and therefore $\mathbf{n}_V \times \mathbf{n}_W$ is a non-zero vector. Let $t = |\mathbf{n}_V \times \mathbf{n}_W|$. Then $\mathbf{n}_V \times \mathbf{n}_W = t\mathbf{u}$, where $\mathbf{u}$ is a unit vector orthogonal to both $\mathbf{n}_V$ and $\mathbf{n}_W$. This vector $\mathbf{u}$ must be parallel to both V and W, and must therefore be parallel to the line of intersection of these two planes. Let $\mathbf{v} = \mathbf{u} \times \mathbf{n}_V$ and $\mathbf{w} = \mathbf{u} \times \mathbf{n}_W$. Then the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel to the planes V and W respectively, and both vectors are orthogonal to the line of intersection of these planes. It follows that angle between the vectors $\mathbf{v}$ and $\mathbf{w}$ is the angle α between the planes V and W.

Now the vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{n}_V$ and $\mathbf{n}_W$ are all parallel to the plane that is orthogonal to $\mathbf{u}$, the angle between the vectors $\mathbf{v}$ and $\mathbf{n}_V$ is a right angle, and the angle between the vectors $\mathbf{w}$ and $\mathbf{n}_W$ is also a right angle. It follows that the angle between the vectors $\mathbf{n}_V$ and $\mathbf{n}_W$ is equal to the angle α between the vectors $\mathbf{v}$ and $\mathbf{w}$, and therefore

$$\begin{aligned} \mathbf{n}_V \,.\, \mathbf{n}_W &= \mathbf{v} \,.\, \mathbf{w} = \cos\alpha, \\ \mathbf{n}_V \times \mathbf{n}_W &= \mathbf{v} \times \mathbf{w} = \sin\alpha\, \mathbf{u}. \end{aligned}$$

These identities can also be verified by vector algebra. Indeed, using Lagrange's Quadruple Product Identity, we see that

$$\begin{aligned} \mathbf{v} \,.\, \mathbf{w} &= (\mathbf{n}_V \times \mathbf{u}) \,.\, (\mathbf{n}_W \times \mathbf{u}) \\ &= (\mathbf{n}_V \,.\, \mathbf{n}_W)(\mathbf{u} \,.\, \mathbf{u}) - (\mathbf{n}_V \,.\, \mathbf{u})(\mathbf{u} \,.\, \mathbf{n}_W) \\ &= \mathbf{n}_V \,.\, \mathbf{n}_W, \end{aligned}$$

because $\mathbf{u} \,.\, \mathbf{u} = |\mathbf{u}|^2 = 1$ $\mathbf{n}_V \,.\, \mathbf{u} = 0$ and $\mathbf{n}_W \,.\, \mathbf{u} = 0$. Thus $\mathbf{n}_V \,.\, \mathbf{n}_W = \cos\alpha$. Also $\mathbf{n}_V \times \mathbf{n}_W$ is parallel to the unit vector $\mathbf{u}$, and therefore

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &= (\mathbf{n}_V \times \mathbf{u}) \times (\mathbf{n}_W \times \mathbf{u}) = (\mathbf{u} \times \mathbf{n}_V) \times (\mathbf{u} \times \mathbf{n}_W) \\ &= (\mathbf{u}.(\mathbf{n}_V \times \mathbf{n}_W))\mathbf{u} = \mathbf{n}_V \times \mathbf{n}_W. \end{aligned}$$

(see Corollary 5.8). It follows that

$$|\mathbf{n}_V \times \mathbf{n}_W| = |\mathbf{v} \times \mathbf{w}| = \sin\alpha,$$

and therefore

$$\mathbf{n}_V \times \mathbf{n}_W = \sin\alpha\,\mathbf{u},$$

as required. ∎

Proposition 5.12 (Cosine Rule of Spherical Trigonometty) *Let L, L' and L'' be distinct points on the unit sphere in $\mathbb{R}^3$, let α be the angle at L between the great circle through L and L' and the great circle through L and L''. Then*

$$\cos L'L'' = \cos LL' \cdot \cos LL'' + \sin LL' \cdot \sin LL'' \cdot \cos\alpha.$$

Proof The angle α at L between the great circle LL' and the great circle LL'' is equal to the angle between the planes through the origin that intersect the unit sphere in those great circles, and this angle is in turn equal to the angle between the normal vectors $\mathbf{n}_{L,L'}$ and $\mathbf{n}_{L,L''}$ to those planes, and therefore $\mathbf{n}_{L,L'} \,.\, \mathbf{n}_{L,L''} = \cos\alpha$ (see Lemma 5.11). Let $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ denote the displacement vectors of the points L, L' and L'' respectively from the centre of the sphere. Then

$$\mathbf{r} \times \mathbf{r}' = \sin LL'\,\mathbf{n}_{L,L'}, \quad \mathbf{r} \times \mathbf{r}'' = \sin LL''\,\mathbf{n}_{L,L''}.$$

It follows that

$$(\mathbf{r} \times \mathbf{r}').(\mathbf{r} \times \mathbf{r}'') = \sin LL' \cdot \sin LL'' \cdot \cos\alpha.$$

But it follows from Lagrange's Quadruple Product Identity that Proposition 5.9 that

$$(\mathbf{r} \times \mathbf{r}') \,.\, (\mathbf{r} \times \mathbf{r}'') = (\mathbf{r} \,.\, \mathbf{r})(\mathbf{r}' \,.\, \mathbf{r}'') - (\mathbf{r} \,.\, \mathbf{r}'')(\mathbf{r}' \,.\, \mathbf{r}).$$

But $\mathbf{r}.\mathbf{r} = |\mathbf{r}|^2 = 1$, because the point $\mathbf{r}$ lies on the unit sphere. Therefore

$$(\mathbf{r} \times \mathbf{r}') \,.\, (\mathbf{r} \times \mathbf{r}'') = (\mathbf{r}' \,.\, \mathbf{r}'') - (\mathbf{r} \,.\, \mathbf{r}')(\mathbf{r} \,.\, \mathbf{r}'') = \cos L'L'' - \cos LL' \cos LL''.$$

Equating the two formulae for $(\mathbf{r} \times \mathbf{r}') \,.\, (\mathbf{r} \times \mathbf{r}'')$, we find that

$$\cos L'L'' = \cos LL' \cdot \cos LL'' + \sin LL' \cdot \sin LL'' \cdot \cos\alpha,$$

as required. ∎

Second Proof Let $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ denote the displacement vectors of the points L, L' and L'' respectively from the centre O of the sphere. Without loss of generality, we may assume that the Cartesian coordinate system with origin at the centre O of the sphere has been oriented so that

$$\begin{aligned}
\mathbf{r} &= (0, 0, 1), \\
\mathbf{r}' &= (\sin LL', 0, \cos LL'), \\
\mathbf{r}'' &= (\sin LL'' \cos\alpha, \sin LL'' \sin\alpha, \cos LL'').
\end{aligned}$$

Then $|\mathbf{r}'| = 1$ and $|\mathbf{r}''| = 1$. It follows that

$$\cos L'L'' = \mathbf{r}' \,.\, \mathbf{r}'' = \cos LL' \cdot \cos LL'' + \sin LL' \cdot \sin LL'' \cdot \cos\alpha,$$

as required. ∎

Theorem 5.13 (Gauss) *If L, L', L'' and L''' denote four points on the sphere, and α the angle which the arcs LL', $L''L'''$ make at their point of intersection, then we shall have*

$$\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' = \sin LL' \cdot \sin L''L''' \cdot \cos\alpha.$$

Proof Let $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$ and $\mathbf{r}'''$ denote the displacement vectors of the points L, L', L'' and L''' from the centre of the sphere. It follows from Lagrange's Quadruple Product Identity (Proposition 5.9) that

$$(\mathbf{r} \,.\, \mathbf{r}'')(\mathbf{r}' \,.\, \mathbf{r}''') - (\mathbf{r} \,.\, \mathbf{r}''')(\mathbf{r}' \,.\, \mathbf{r}'') = (\mathbf{r} \times \mathbf{r}') \,.\, (\mathbf{r}'' \times \mathbf{r}''').$$

Now it follows from the standard properties of the scalar and vector products recorded in the statement of Lemma 5.10 that $\mathbf{r} \,.\, \mathbf{r}'' = \cos LL''$ etc., $\mathbf{r} \times \mathbf{r}' = \sin LL' \,\mathbf{n}_{L,L'}$ and $\mathbf{r}'' \times \mathbf{r}''' = \sin L''L''' \,\mathbf{n}_{L'',L'''}$, where $\mathbf{n}_{L,L'}$ is a unit vector orthogonal to the plane through the origin containing the points L and L', and $\mathbf{n}_{L'',L'''}$ is a unit vector orthogonal to the plane through the origin containing the points L'' and L'''. Now $\mathbf{n}_{L,L'} \,.\, \mathbf{n}_{L'',L'''} = \cos\alpha$, where $\cos\alpha$ is the cosine of the angle α between these two planes (see Lemma 5.11). This angle is also the angle, at the points of intersection, between the great circles on the sphere that represent the intersection of those planes with the sphere. It follows that

$$\begin{aligned}
&\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' \\
&\quad= (\mathbf{r} \,.\, \mathbf{r}'')(\mathbf{r}' \,.\, \mathbf{r}''') - (\mathbf{r} \,.\, \mathbf{r}''')(\mathbf{r}' \,.\, \mathbf{r}'') \\
&\quad= (\mathbf{r} \times \mathbf{r}') \,.\, (\mathbf{r}'' \times \mathbf{r}''') \\
&\quad= \sin LL' \cdot \sin L''L''' \cdot (\mathbf{n}_{L,L'} \,.\, \mathbf{n}_{L'',L'''}) \\
&\quad= \sin LL' \cdot \sin L''L''' \cdot \cos\alpha,
\end{aligned}$$

as required. ∎

Second Proof (This proof follows fairly closely the proof given by Gauss in the *Disquisitiones Generales circa Superficies Curvas*, published in 1828.) Let the point O be the centre of the sphere, and let P be the point where the great circle passing through LL' intersects the great circle passing through $L''L'''$. The angle α is then the angle between these great circles at the point P. Let the angles between the line OP and the lines OL, OL', OL'' and OL''' be denoted by θ, θ', θ'', θ''' respectively (so that $\cos PL = \cos\theta$ etc.). It then follows from the Cosine Rule of Spherical Trigonometry (Proposition 5.12) that

$$\begin{aligned}
\cos LL'' &= \cos\theta\,\cos\theta'' + \sin\theta\,\sin\theta''\,\cos\alpha,\\
\cos LL''' &= \cos\theta\,\cos\theta''' + \sin\theta\,\sin\theta'''\,\cos\alpha,\\
\cos L'L'' &= \cos\theta'\,\cos\theta'' + \sin\theta'\,\sin\theta''\,\cos\alpha,\\
\cos LL''' &= \cos\theta'\,\cos\theta''' + \sin\theta'\,\sin\theta'''\,\cos\alpha
\end{aligned}$$

(see Lemma 5.10). From these equations it follows that

$$\begin{aligned}
&\cos LL''\cdot\cos L'L''' - \cos LL'''\cdot\cos L'L''\\
&\quad= \cos\alpha(\cos\theta\,\cos\theta''\,\sin\theta'\,\sin\theta''' + \cos\theta'\,\cos\theta'''\,\sin\theta\,\sin\theta''\\
&\qquad\qquad - \cos\theta\,\cos\theta'''\,\sin\theta'\,\sin\theta'' - \cos\theta'\,\cos\theta''\,\sin\theta\,\sin\theta''')\\
&\quad= \cos\alpha(\cos\theta\,\sin\theta' - \sin\theta\cos\theta')(\cos\theta''\,\sin\theta''' - \sin\theta''\cos\theta''')\\
&\quad= \cos\alpha\cdot\sin(\theta'-\theta)\cdot\sin(\theta'''-\theta'')\\
&\quad= \cos\alpha\cdot\sin LL'\cdot L''L''',
\end{aligned}$$

as required. ∎

Remark In his *Disquisitiones Generales circa Superficies Curvas*, published in 1828, Gauss proved Theorem 5.13, using the method of the second of the proofs of that theorem given above, and used it to deduce that if L, L' and L'' are three points on the unit sphere in $\mathbb{R}^3$ with Cartesian coordinates

$$L = (x,y,z),\quad L' = (x',y',z'),\quad L'' = (x'',y'',z''),$$

and if

$$\Delta = xy'z'' + x'y''z + x''yz' - xy''z' - x'yz'' - x''y'z,$$

then $\Delta = \cos NL''\cdot\sin LL'$, where N is a pole of the great circle passing through L and L' (i.e., a point on the surface whose displacement vector from the sphere is orthogonal to the plane through the centre O of the sphere that contains the points L and L'). Now if the displacement vectors of the points

L, L' and L'' from the centre of the sphere are $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ respectively then $\mathbf{r} \times \mathbf{r}' = \sin LL' \, \mathbf{n}_{L,L'}$, where the vector $\mathbf{n}_{L,L'}$ is orthogonal to the vectors $\mathbf{r}$ and $\mathbf{r}'$ and has unit length. We let N be the point on the surface of the sphere whose displacement vector from the centre of the sphere is $\mathbf{n}_{L,L'}$. Then N is a pole of the great circle passing through L and L'. It follows from this that $\cos NL'' = \pm \sin p$, where p is the angle between the line OL'' joining the centre O of the sphere to L'' and the plane through the origin that contains L and L'. It follows that

$$\Delta = \mathbf{r}'' \, . \, (\mathbf{r} \times \mathbf{r}') = \mathbf{r} \, . \, (\cos LL' \, \mathbf{n}_{L,L'}) = \cos NL'' \cdot \cos LL' = \pm \sin p \, \cos LL'.$$

Now Gauss's paper was published nearly two decades before William Rowan Hamilton started publishing papers concerning vectors, using a form of vector notation that he developed in his theory of quaternions, that included standard vector identities such as those satisfied by the scalar triple product, the Vector Triple Product identity and Lagrange's Quadruple Product Identity.

Gauss deduced the identity $\Delta = \cos NL'' \cdot \sin LL'$ in the *Disquisitiones Generales super Superficies Curvas* using the following method. Let I, J and K be the points on the surface of the sphere where the coordinate axes cut the sphere, so that, taking the origin of Cartesian coordinates at the centre of the sphere,

$$I = (1, 0, 0), \quad J = (0, 1, 0) \quad \text{and} \quad K = (0, 0, 1).$$

It then follows from an earlier theorem (Theorem 5.13 above) proved by Gauss in the *Disquisitiones Generales* that

$$\cos LI \cdot \cos L'J - \cos LJ \cdot \cos L'I = \sin LL' \cdot \sin IJ \cdot \cos \alpha = \sin LL' \cdot \cos \alpha,$$

where α is the angle between the equatorial great circle passing through I and J and the great circle containing L and L' at the points of intersection of these two circles. Now the points K and N are the poles of these two circles, and the angle between the great circles is equal to the angle between the poles of those great circles. It follows that $\cos \alpha = \cos NK$. Also

$$\cos LI = x, \quad \cos LJ, \quad y, \quad L'I = x', \quad L'J = y'.$$

It follows that $xy' - yx' = \sin LL' \cdot n_z$, where $n_z = \cos NK$. Similarly $yz' - zy' = \sin LL' \cdot n_x$ and $xy' - yz' = \sin LL' \cdot n_z$, where $n_x = \cos NI$ and $n_y = \cos NJ$.

$$\Delta = (n_x x'' + n_y y'' + n_z z'') \cdot \sin LL' = \cos NL'' \cdot \sin LL',$$

which is the identity to be proved.

Proposition 5.14 (Gauss) *Let L, L' and L'' be three distinct points on the unit sphere that do not all lie on any one great circle of the sphere, and let p be the angle which the line from the centre of the sphere to the point L'' makes with the plane through the centre of the sphere that contains the points L and L'. Then*

$$\sin p = \sin L \cdot \sin LL'' = \sin L' \cdot \sin L'L'',$$

where $\sin L$ *denotes the sine of the angle between the arcs LL' and LL'' at L and* $\sin L'$ *denotes the sine of the angle between the arcs $L'L''$ and $L'L$ at L'.*

Proof Let $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$ denote the diplacement vectors of the points L, L' and L'' from the centre of the sphere. A straightforward application of the Vector Triple Product Identity shows that

$$(\mathbf{r} \times \mathbf{r}') \times (\mathbf{r} \times \mathbf{r}'') = (\mathbf{r}.(\mathbf{r}' \times \mathbf{r}''))\mathbf{r}.$$

(see Corollary 5.8). Now $\mathbf{r} \times \mathbf{r}' = \sin LL'\, \mathbf{n}_{L,L'}$, where $\mathbf{n}_{L,L'}$ is a unit vector orthogonal to the plane spanned by L and L'. Similarly $\mathbf{r}\times\mathbf{r}'' = \sin LL''\, \mathbf{n}_{L,L''}$, where $\mathbf{n}_{L,L''}$ is a unit vector orthogonal to the plane spanned by L and L'. Moreover the vector $\mathbf{n}_{L,L'} \times \mathbf{n}_{L,L''}$ is orthogonal to the vectors $\mathbf{n}_{L,L'}$ and $\mathbf{n}_{L,L''}$, and therefore is parallel to the line of intersection of the plane through the centre of the sphere containing L and L' and the plane through the centre of the sphere containing L and L''. Moreover the magnitude of this vector is the sine of the angle between them. It follows that $\mathbf{n}_{L,L'} \times \mathbf{n}_{L,L''} = \pm \sin L\, \mathbf{r}$. We note also that $\mathbf{r}.(\mathbf{r}' \times \mathbf{r}'') = \mathbf{r}''.(\mathbf{r} \times \mathbf{r}')$. (see Corollary 5.5.) Putting these identities together, we see that we see that

$$\sin LL' \cdot \sin LL'' \cdot \sin L = \pm \mathbf{r} \,.\, (\mathbf{r}' \times \mathbf{r}'') = \pm \mathbf{r}'' \,.\, (\mathbf{r} \times \mathbf{r}') = \pm \sin LL' \cdot \mathbf{r}'' \,.\, \mathbf{n}_{L,L'}.$$

Now the cosine of the angle between the unit vector $\mathbf{r}''$ and the unit vector $\mathbf{n}_{L,L'}$ is the sine $\sin p$ of the angle between the vector $\mathbf{r}''$ and the plane through the centre of the sphere that contains the points L and L'. It follows that $\mathbf{r}'' \,.\, \mathbf{n}_{L,L'} = \sin p$, and therefore

$$\sin LL' \cdot \sin LL'' \cdot \sin L = \pm \sin LL' \cdot \sin p.$$

Now the angles concerned are all between 0 and π, and therefore their sines are non-negative. Also $\sin LL' \neq 0$, because L and L' are distinct and are not antipodal points on opposite sides of the sphere. Dividing by $\sin LL'$, we find that

$$\sin L \cdot \sin LL'' = \sin p.$$

Interchanging L and L', we find that

$$\sin L' \cdot \sin L'L'' = \sin p,$$

as required. ∎

Corollary 5.15 (Sine Rule of Spherical Trigonometry) *Let L, L' and L'' be three distinct points on the unit sphere that do not all lie on any one great circle of the sphere. Then*

$$\frac{\sin L'L''}{\sin L} = \frac{\sin LL''}{\sin L'},$$

where $\sin L$ denotes the sine of the angle between the arcs LL' and LL'' at L and $\sin L'$ denotes the sine of the angle between the arcs $L'L''$ and $L'L$ at L'.

Proposition 5.16 (Gauss) *Let L, L', L'' be points on the unit sphere in $\mathbb{R}^3$, and let the point O be at the centre of that sphere. Then the volume V of the tetrahedron with apex O and base $LL'L''$ satisfies*

$$\begin{aligned} V &= \tfrac{1}{6}\sin L \cdot \sin LL' \cdot \sin LL'' \\ &= \tfrac{1}{6}\sin L' \cdot \sin LL' \cdot \sin L'L'' \\ &= \tfrac{1}{6}\sin L'' \cdot \sin LL'' \cdot \sin L'L'' \end{aligned}$$

where $\sin LL'$, $\sin LL''$ and $\sin L'L''$ are the sines of the angles between the lines joining the indicated points to the centre of the sphere, and where $\sin L$, $\sin L'$ and $\sin L''$ are the sines of angles of the geodesic triangle $LL'L''$ whose vertices are L and L' and L'' and whose sides are the arcs of great circles joining its vertices.

Proof This tetrahedron may be described as the tetrahedron with base OLL' and apex L''. Now the area of the base of the tetrahedron is $\sin LL'$, and the height is $\sin p$, where p is the perpendicular distance from the point L'' to the plane passing through the centre of the sphere that contains the points L and L'. The volume V of the tetrahedron is one sixth of the area of the base of the tetrahedron multiplied by the height of the tetrahedron. On applying Proposition 5.14 we see that

$$V = \tfrac{1}{6}\sin p \cdot \sin LL' = \tfrac{1}{6}\sin L \cdot \sin LL' \cdot \sin LL''.$$

The remaining equalities can be derived by permuting the order of the vertices L, L' and L''. ∎