MA232A: Euclidean and non-Euclidean Geometry
Michaelmas Term 2015
Curvature of Smooth Surfaces in Three-Dimensional Space

David R. Wilkins

Contents

7 The Gaussian Curvature of a Smooth Surface in Three-Dimensional Space

7.1 The Unit Normal Vector Field to a Smooth Surface

We consider here the properties of the normal vector field to a smooth surface discussed in Section 4 of Gauss's General Investigations on Curved Surfaces

The *auxiliary sphere* is the unit sphere about the origin in $\mathbb{R}^3$ that contains the endpoints of all unit vectors (vectors of unit length). The quantities that Gauss denotes by "$\cos(1)L$", "$\cos(2)L$" and "$\cos(3)L$" are the cosines of the angles which the unit normal to a curved surface makes with the directions of the coordinate axes. This unit normal is thus to be considered as a vector with components (X, Y, Z), where

$$X^2 + Y^2 + Z^2 = 1.$$

Gauss considers a displacement from a point A on the curved surface to another point A' that is infinitesimally close to A. To understand the situation, avoiding explicit use of "infinitesimals", it may be helpful to consider a smooth curve $\theta\colon I \to \Sigma$ in the smooth surface Σ, parameterized by arclength s, where the values of s range over an open interval I in the real line. The Cartesian coordinates of a point $\theta(s)$ on the curve are then $(x(s), y(s), z(s))$. Now standard principles of calculus ensure that arclength along the curve between $s = s_0$ and $s = s_1$ is given by the integral

$$\int_{s_0}^{s_1} \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}\, ds,$$

and this arclengh must be equal to $s_1 - s_0$. It follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} 1 &= \frac{d}{ds}\int_{s_0}^{s} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \\ &= \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2}. \end{aligned}$$

Therefore

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

The quantities $\dfrac{dx}{ds}\,\dfrac{dy}{ds}$ and $\dfrac{dz}{ds}$ are thus the components of a unit vector $\theta'(s)$ that is the velocity vector of the smooth curve at the point $\theta(s)$. Gauss denotes by λ the point on the unit sphere that represents the direction of this unit tangent vector. Accordingly

$$\lambda = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right).$$

Thus if, following Gauss, the angles that this tangent vector makes with the directions of the three coordinate axes are denoted by $\cos(1)\lambda$, $\cos(2)\lambda$ and $\cos(3)\lambda$ respectively, then

$$\frac{dx}{ds} = \cos(1)\lambda, \quad \frac{dy}{ds} = \cos(2)\lambda \quad \text{and} \quad \frac{dz}{ds} = \cos(3)\lambda.$$

Now the vector $\theta'(s)$ is in the tangent space to the smooth surface at the point $\theta(s)$, and is therefore orthogonal to the normal vector (X, Y, Z) to the surface. It follows that

$$X \frac{dx}{ds} + Y \frac{dy}{ds} + Z \frac{dx}{ds} = 0.$$

Gauss expresses this identity in the language of "differentials" as follows:

$$X\,dx + Y\,dy + Z\,dz = 0.$$

7.2 Orthogonality and Differentials on Tangent Spaces

The "differentials" dx, dy and dz may be interpreted as linear functionals on the tangent space to the surface Σ at the point $\mathbf{p}$. Let $T_{\mathbf{p}}\Sigma$ denote the tangent space to the surface Σ at the point $\mathbf{p}$. This tangent space is a two-dimensional vector subspace of $\mathbb{R}^3$. Moreover a vector (b_x, b_y, b_z) belongs to $T_{\mathbf{p}}\Sigma$ if and only if there exists a smooth curve $\gamma\colon I \to \Sigma$ in the surface, parameterized by a real variable t ranging over some open interval I in $\mathbb{R}$ that contains 0, such that

$$b_x = \left.\frac{dx(\gamma(t))}{dt}\right|_{t=0}, \quad b_y = \left.\frac{dy(\gamma(t))}{dt}\right|_{t=0}, \quad \text{and} \quad b_z = \left.\frac{dz(\gamma(t))}{dt}\right|_{t=0}.$$

A *linear functional* $\theta\colon T_{\mathbf{p}}\Sigma \to \mathbb{R}$ on the tangent space $T_{\mathbf{p}}\Sigma$ to the surface Σ at the point $\mathbf{p}$ is a linear transformation from the real vector space $T_{\mathbf{p}}\Sigma$ to the field $\mathbb{R}$ of real numbers. Linear functions can be added together and can be multiplied by real scalars, and the set of all linear functionals on the tangent space $T_{\mathbf{p}}\Sigma$, with these operations of addition and multiplication-by-scalars is itself a two-dimensional real vector space.

The space of linear functionals from $T_{\mathbf{p}}\Sigma$ to $\mathbb{R}$ is spanned by the differentials $(dx)_{\mathbf{p}}$, $(dy)_{\mathbf{p}}$ and $(dz)_{\mathbf{p}}$, where

$$(dx)_{\mathbf{p}}(b_x, b_y, b_z) = b_x, \quad (dy)_{\mathbf{p}}(b_x, b_y, b_z) = b_y, \quad (dz)_{\mathbf{p}}(b_x, b_y, b_z) = b_z$$

for all tangent vectors (b_x, b_y, b_z) to the smooth surface Σ at the point $\mathbf{p}$. Now the unit normal vector (X, Y, Z) has zero scalar product with all tangent vectors to the surface Σ at the point $\mathbf{p}$. Thus if $\mathbf{b}$ is a tangent vector to the surface Σ at the point $\mathbf{p}$, and if $\mathbf{b} = (b_x, b_y, b_z)$, then

$$X\,(dx)_{\mathbf{p}}(\mathbf{b}) + Y\,(dy)_{\mathbf{p}}(\mathbf{b}) + Z\,(dz)_{\mathbf{p}}(\mathbf{b}) = Xb_x + Yb_y + Zb_z = 0.$$

It follows that the linear functional

$$X\,(dx)_{\mathbf{p}} + Y\,(dy)_{\mathbf{p}} + Z\,(dz)_{\mathbf{p}}.$$

on the tangent space $T_{\mathbf{p}}\Sigma$ to Σ on $\mathbf{p}$ sends all tangent vectors to Σ at $\mathbf{p}$ to zero. It follows that

$$(X\,dx + Y\,dy + Z\,dz)_{\mathbf{p}}: T_{\mathbf{p}}\Sigma \to \mathbb{R}$$

is the zero linear transformation from $T_{\mathbf{p}}\Sigma$ to $\mathbb{R}$. The components of the normal vector field are thus the coefficients of a linear dependence relation satisfied by the linearly dependent linear functionals $(dx)_{\mathbf{p}}$, $(dy)_{\mathbf{p}}$ and $(dz)_{\mathbf{p}}$ on the tangent space $T_{\mathbf{p}}\Sigma$.

Thus the equation

$$X\,dx + Y\,dy + Z\,dz = 0$$

7.3 Line Integrals of the Normal Vector Field

It is also worth noting that, if line integrals along smooth curves in the surface Σ are to be calculated in accordance with the usual rules of multivariate calculus then

$$\int_\gamma (X\,dx + Y\,dy + Z\,dz) = 0$$

for all smooth curves $\gamma: [a, b] \to \Sigma$ in the surface Σ paramaterized by a real variable that ranges over some closed bounded interval $[a, b]$. Indeed

$$\begin{aligned} &\int_\gamma (X\,dx + Y\,dy + Z\,dz) \\ &\quad = \int_{t=a}^{b} \left(X(\gamma(t))\,\frac{dx(\gamma(t))}{dt} + Y(\gamma(t))\,\frac{dy(\gamma(t))}{dt} + Z(\gamma(t))\,\frac{dz(\gamma(t))}{dt} \right) dt \\ &\quad = 0, \end{aligned}$$

because the unit normal $(X(\gamma(t)), Y(\gamma(t), Z(\gamma(t)))$ to the surface at $\gamma(t)$ is orthogonal to the tangent vector

$$\left(\frac{dx(\gamma(t))}{dt}, \frac{dy(\gamma(t))}{dt}, \frac{dz(\gamma(t))}{dt}\right).$$

Thus the identity $X\,dx + Y\,dy + Z\,dz$ is the "differential" form of the result that establishes that $\int_\gamma (X\,dx + Y\,dy + Z\,dz) = 0$ for all smoooth curves $\gamma\colon [a, b] \to \Sigma$ within the surface Σ.

7.4 Smooth Surfaces that are the zero sets of smooth functions

Let W be a smooth real-valued function defined over some open set Ω in $\mathbb{R}^3$, and let $\Sigma = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}$. Suppose that the gradient

$$\left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right)$$

of W is non-zero at each point of Σ. Then Σ is a smooth surface in $\mathbb{R}^3$. This can be deduced as a consequence of the *Inverse Function Theorem* (or the *Implicit Function Theorem*, see the *Notes on Smooth Surfaces*).

If $\gamma\colon I \to \Sigma$ is a smooth curve in the surface Σ parameterized by some open interval I in the real line then $W(\gamma(t)) = 0$ for all $t \in I$. Differentiating with respect to t, using the Chain Rule for differentiating compositions of differentiable functions of several real variables, we see that

$$0 = \frac{dW(\gamma(t))}{dt} = \left.\frac{\partial W}{\partial x}\right|_{\gamma(t)} \frac{dx(\gamma(t))}{dt} + \left.\frac{\partial W}{\partial y}\right|_{\gamma(t)} \frac{dy(\gamma(t))}{dt} + \left.\frac{\partial W}{\partial z}\right|_{\gamma(t)} \frac{dz(\gamma(t))}{dt}.$$

For all $t \in I$. Thus if $\mathbf{p} = \gamma(t_0)$ for some $t_0 \in 1$, if $\mathbf{b} = \gamma'(t_0)$, and if $\mathbf{b} = (b_x, b_y, b_z)$, then

$$\left.\frac{dx(\gamma(t))}{dt}\right|_{t=t_0} = b_x, \qquad \left.\frac{dy(\gamma(t))}{dt}\right|_{t=t_0} = b_y, \qquad \left.\frac{dz(\gamma(t))}{dt}\right|_{t=t_0} = b_z.$$

It follows that

$$0 = \left.\frac{W(\gamma(t))}{dt}\right|_{t=0} = \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} b_x + \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} b_y + \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}} b_x.$$

Using the language of differential forms, we can write

$$\left.\frac{W(\gamma(t))}{dt}\right|_{t=0} = (dW)_{\mathbf{p}}(\mathbf{b}), \qquad \left.\frac{x(\gamma(t))}{dt}\right|_{t=0} = (dx)_{\mathbf{p}}(\mathbf{b}),$$

$$\left.\frac{y(\gamma(t))}{dt}\right|_{t=0} = (dy)_{\mathbf{p}}(\mathbf{b}), \qquad \left.\frac{z(\gamma(t))}{dt}\right|_{t=0} = (dz)_{\mathbf{p}}(\mathbf{b}).$$

It then follows that

$$\begin{aligned} 0 &= (dW)_{\mathbf{p}}(\mathbf{b}) \\ &= \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} (dx)_{\mathbf{p}}(\mathbf{b}), \quad + \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} (dy)_{\mathbf{p}}(\mathbf{b}), \quad + \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}} (dz)_{\mathbf{p}}(\mathbf{b}) \\ &= \left(\frac{\partial W}{\partial x}\, dx + \frac{\partial W}{\partial y}\, dy + \frac{\partial W}{\partial z}\, dz\right)_{\mathbf{p}} (\mathbf{b}). \end{aligned}$$

for all vectors $\mathbf{b}$ that are tangent to the surface $W = 0$ at the point $\mathbf{P}$.

This relation can be re-expressed as an equation

$$\frac{\partial W}{\partial x}\, dx + \frac{\partial W}{\partial y}\, dy + \frac{\partial W}{\partial z}\, dz = 0.$$

satisfied by the differentials dx, dy and dz of the coordinate functions x, y, z, on restricting these coordinate functions to the surface Σ, so that the differentials $(dx)_{\mathbf{p}}$, $(dy)_{\mathbf{p}}$, and $(dz)_{\mathbf{p}}$ at a point $\mathbf{p}$ are considered to be linear transformations from the tangent space $T_{\mathbf{p}}\Sigma$ to the surface at $\mathbf{p}$ to the field $\mathbb{R}$ of real numbers that send each tangent vector $\mathbf{b}$ at the point $\mathbf{p}$ to the directional derivatives

$$(dx)_{\mathbf{p}}(\mathbf{b}), \quad (dy)_{\mathbf{p}}(\mathbf{b}), \quad (dz)_{\mathbf{p}}(\mathbf{b})$$

of the coordinate functions in the direction of the tangent vector $\mathbf{b}$.

Moreover the gradient vector

$$\left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right)$$

is orthogonal to the tangent spaces to the surface.

Gauss uses the notation

$$P = \frac{\partial W}{\partial x}, \quad Q = \frac{\partial W}{\partial y}, \quad R = \frac{\partial W}{\partial z},$$

so that

$$dW = P\, dx + Q\, dy + R\, dz$$

on the surface Σ, where $\Sigma = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}$. Then the vector (P, Q, R) is orthogonal to the tangent space to the surface Σ at each point of this surface. Now, in Gauss's notation, the unit normal (X, Y, Z) to the surface has unit length. This vector is parallel to the vector (P, Q, R). Therefore either

$$X = \frac{P}{\sqrt{P^2+Q^2+R^2}}, \quad Y = \frac{Q}{\sqrt{P^2+Q^2+R^2}}, \quad Z = \frac{R}{\sqrt{P^2+Q^2+R^2}},$$

or

$$X = \frac{-P}{\sqrt{P^2+Q^2+R^2}}, \quad Y = \frac{-Q}{\sqrt{P^2+Q^2+R^2}}, \quad Z = \frac{-R}{\sqrt{P^2+Q^2+R^2}}.$$

7.5 Smoothly Parameterized Surfaces

Gauss also considers surfaces that are expressible as functions of two variables. Such a surface can be represented as the image of a function $\chi\colon D \to \mathbb{R}^3$, where D is some open set in $\mathbb{R}^2$. The Cartesian coordinates x, y, z may then be expressed as functions of u and v, where u and v denote Cartesian coordinates on D, so that

$$\chi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

An application of the Inverse Function Theorem guarantees that the image $\chi(D)$ is a smooth surface in $\mathbb{R}^3$ provided that the vectors $\dfrac{\partial \mathbf{r}}{\partial u}$ are $\dfrac{\partial \mathbf{r}}{\partial v}$ linearly independent for all $(u, v) \in D$, where

$$\frac{\partial \mathbf{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

at all $(u, v) \in D$ (see the *Notes on Smooth Surfaces*). We suppose that these vectors are indeed linearly independent for all $(u, v) \in D$, and denote the resulting smooth surface by Σ, so that $\Sigma = \chi(D)$. The parameterization of this smooth surface determines smooth real-valued functions p and q on Σ defined such that $p(\chi(u, v)) = u$ and $q(\chi(u, v)) = v$ for all $(u, v) \in D$. We then define the partial derivatives of x, y and z with respect to the parameterizing variables p and q on the surface so that

$$\begin{aligned}
\left(\left.\frac{\partial x}{\partial p}\right|_{\chi(u,v)}, \left.\frac{\partial y}{\partial p}\right|_{\chi(u,v)}, \left.\frac{\partial z}{\partial p}\right|_{\chi(u,v)}\right) &= \frac{\partial \chi(u, v)}{\partial u}, \\
\left(\left.\frac{\partial x}{\partial q}\right|_{\chi(u,v)}, \left.\frac{\partial y}{\partial q}\right|_{\chi(u,v)}, \left.\frac{\partial z}{\partial q}\right|_{\chi(u,v)}\right) &= \frac{\partial \chi(u, v)}{\partial v}.
\end{aligned}$$

Gauss uses the notation

$$\frac{\partial x}{\partial p} = a, \quad \frac{\partial y}{\partial p} = b, \quad \frac{\partial z}{\partial p} = c,$$

$$\frac{\partial x}{\partial q} = a', \quad \frac{\partial y}{\partial q} = b', \quad \frac{\partial z}{\partial q} = c'.$$

Then

$$\frac{\partial \mathbf{r}}{\partial p} = (a, b, c), \quad \frac{\partial \mathbf{r}}{\partial q} = (a', b', c').$$

Moreover these vectors $\dfrac{\partial \mathbf{r}}{\partial p}$ and $\dfrac{\partial \mathbf{r}}{\partial q}$ constitute a basis of the tangent space to the surface at each point of Σ.

Let (X, Y, Z) denote the unit normal vector field along the surface that is orthogonal to the tangent spaces at each point of the surface. Then the vectors (X, Y, Z) and

$$\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}$$

must be parallel. Let

$$\Delta = \left| \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} \right|.$$

Then

$$\begin{aligned}
(X, Y, Z) &= \frac{\pm 1}{\Delta} \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}. \\
&= \frac{\pm 1}{\Delta} (a, b, c) \times (a', b', c') \\
&= \frac{\pm 1}{\Delta} (bc' - cb', ca' - ac', ab' - ba'),
\end{aligned}$$

where

$$\Delta = \sqrt{(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}.$$

In particular these results can be applied in the special case where the surface is expressed by an equation of the form $z = f(x, y)$, where f is a smooth function of the first two Cartesian coordinates x and y. In that case $p = x$ and $q = y$, and therefore

$$(a, b, c) = (1, 0, t) \quad \text{and} \quad (a', b', c') = (0, 1, u).$$

$$t = \frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x} \quad \text{and} \quad u = \frac{\partial z}{\partial y} = \frac{\partial f(x, y)}{\partial y}.$$

Suppose the direction of the unit normal (X, Y, Z) is chosen such that $Z > 0$. Then the previous formulae yield

$$X = \frac{-t}{\sqrt{1 + t^2 + u^2}}, \quad Y = \frac{-u}{\sqrt{1 + t^2 + u^2}}, \quad Z = \frac{1}{\sqrt{1 + t^2 + u^2}}.$$

7.6 Surface Area

We consider integrals taken over regions of a smooth surface that lie within the domain of a local coordinate system (p, q) on a portion of the surface. We say that such a region R has *regular boundary* if the boundary of the region R within the surface Σ is such as to ensure that integral $\int_R f(p,q)\,dp\,dq$ of any continuous real-valued function f over the region R is well-defined. This would be the case, for example, if the boundary of R is a simple closed curve consisting of a finite number of smooth segments. But we do not enter here into questions as to precisely what conditions should be satisfied by the boundary of the region R to ensure that such integrals are well-defined.

Let Σ be a smooth surface in $\mathbb{R}^3$, let (p, q) be a smooth local coordinate system around a given point $\mathbf{r}_0$ on that surface, and let the values of the coordinate functions p and q at that given point be denoted by p_0 and q_0 respectively. We suppose that the surface area of a region R of the surface contained within the domain of the local coordinate system (u, v) can be evaluated as an integral of the form

$$\int_R m(p,q)\,dp\,dq,$$

where $m(p, q)$ is a function of p and q, provided that the boundary of the region R is regular. We seek to determine an expression for $m(p, q)$ as a function of p and q.

Let $\mathbf{r}(p, q)$ denote the position vector of a point on the surface as a function of the local coordinates p and q. This point can be mapped to a corresponding point $\overline{\mathbf{r}}(p, q)$ on the tangent space to the surface Σ at the point $\mathbf{r}_0$, where

$$\overline{\mathbf{r}}(p,q) = \mathbf{r}(p_0,q_0) + \left.\frac{\partial \mathbf{r}}{\partial p}\right|_{(p_0,q_0)} (p - p_0) + \left.\frac{\partial \mathbf{r}}{\partial q}\right|_{(p_0,q_0)} (q - q_0).$$

Then the smoothness of the surface Σ ensures the existence of a real constant K_0 such that

$$|\mathbf{r}(p,q) - \overline{\mathbf{r}}(p,q)| \leq K_0((p-p_0)^2 + (q-q_0)^2).$$

for all points of the surface with local coordinates (p, q) sufficiently close to (p_0, q_0).

Any region R with regular boundary contained within a portion of the surface close to the given point with local coordinates (p_0, q_0) determines a corresponding region $\overline{R}$ of the tangent plane, consisting of those points of the tangent plane for which the values of the parameters p and q are local coordinates of a point in R. Now the area area($\overline{R}$) of $\overline{R}$ satisfies

$$\mathrm{area}(\overline{R}) = m_0 \int_{\overline{R}} dp\, dq,$$

where m_0 is the area of the parallelogram in the tangent plane consisting of those points $\overline{r}(p, q)$ for which $p_0 \leq p \leq p_0 + 1$ and $q_0 \leq q \leq q_0 + 1$. The area of this parallelogram is the length of the vector product of the displacement vectors generating the sides of the parallelogram. It follows that $m_0 = \Delta(p_0, q_0)$, where

$$\Delta(p, q) = \left| \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} \right|.$$

We suppose that the surface area of a region R in the surface Σ with regular boundary is defined so that the ratio of the area area(R) of R and the area area($\overline{R}$) of the corresponding region $\overline{R}$ in the tangent plane approaches the limiting value 1 as the region R shrinks down around the point with local coordinates (p_0, q_0). This can be formalized by the requirement that, given any positive real number ε, there exists a positive real number δ such that

$$1 - \varepsilon < \frac{\mathrm{area}(R)}{\mathrm{area}(\overline{R})} < 1 + \varepsilon$$

whenever the local coordinates (p, q) of all points within the region R satisfy $p_0 - \delta < p < p_0 + \delta$ and $q_0 - \delta < q < q_0 + \delta$. This requirement can only be satisfied if

$$m(p_0, q_0) = m_0 = \Delta(p_0, q_0).$$

Applying this result at all points within the coordinate patch, we obtain the result stated in the following lemma.

Lemma 7.1 *Let Σ be a smooth surface in $\mathbb{R}^3$, let (p, q) be smooth local coordinates defined over a coordinate patch on the surface Σ, and let R be a region of the surface Σ with regular boundary (ensuring that the integral of any continuous real-valued function of p and q over R is well-defined). Then*

$$\mathrm{area}(R) = \int_R \Delta(p, q)\, dp\, dq,$$

where

$$\Delta(p, q) = \left| \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} \right|.$$

The definition of Δ ensures that if

$$\frac{\partial \mathbf{r}}{\partial p} = (a, b, c) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial q} = (a', b', c'),$$

then

$$\Delta = \sqrt{(bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2}.$$

This is the formula for Δ presented in Section 4 of Gauss's *General Investigations of Curved Surfaces*.

7.7 Transformation of Areas under the Gauss Map

Definition Let Σ be a smooth orientable surface in three-dimensional Euclidean space $\mathbb{R}^3$. The *Gauss map* $\nu\colon \Sigma \to S^2$ of the surface Σ is the smooth map from Σ to the unit sphere S^2 in $\mathbb{R}^3$ defined such that $\nu(\mathbf{r})$ is a vector of unit length normal to the surface Σ at each point $\mathbf{r}$ of Σ.

The definition of the Gauss map requires that the direction of the unit normal vector field $\mathbf{r} \mapsto \nu(\mathbf{r})$ be chosen so that it varies continuously (and thus smoothly) throughout the surface. The requirement that the surface be orientable ensures that this can be achieved. The unit normal vector field is then determined up to multiplication by the scalar -1, reversing the direction of all normal vectors to the surface. There are thus two possible choices for the Gauss map of a smooth orientable surface in $\mathbb{R}^3$, corresponding to the two possible orientations of that surface.

Let $\nu\colon \Sigma \to S^2$ be the Gauss map of a smooth surface Σ in $\mathbb{R}^3$. Gauss denotes the components of the Gauss map $\nu(\mathbf{r})$ by X, Y, Z, so that

$$\nu(\mathbf{r}) = \Bigl(X(\mathbf{r}), Y(\mathbf{r}), Z(\mathbf{r})\Bigr)$$

for all $\mathbf{r} \in \Sigma$.

Lemma 7.2 *Let Σ be a smooth oriented surface, and let X, Y, Z be smooth functions on Σ whose values at each point of the surface Σ are the Cartesian components of the unit normal vector field on that surface. Let (p, q) be a smooth local coordinate system on Σ, and let*

$$\Delta(p, q) = \left| \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} \right|,$$

where $\mathbf{r}(p, q)$ denotes the position vector of the point on the surface determined by local coordinates p and q. Then

$$\Delta(p, q) = \frac{\pm 1}{Z(p, q)} \left(\frac{\partial x}{\partial p}\frac{\partial y}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial y}{\partial p} \right)$$

at points where $Z(p, q) \neq 0$.

Proof Reversing the direction of the unit normal vector field, if necessary, we may assume, without loss of generality, that $Z(p,q) > 0$. Now

$$\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p}\right) \times \left(\frac{\partial x}{\partial q}, \frac{\partial y}{\partial q}, \frac{\partial z}{\partial q}\right) = (\Delta X, \Delta Y, \Delta Z),$$

where

$$\Delta = \left|\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p}\right) \times \left(\frac{\partial x}{\partial q}, \frac{\partial y}{\partial q}, \frac{\partial z}{\partial q}\right)\right|,$$

and therefore

$$\Delta Z = \frac{\partial x}{\partial p}\frac{\partial y}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial y}{\partial p}.$$

The result follows. ∎

Consider the parallelogram in three-dimensional space whose sides are generated by the vectors $\dfrac{\partial \mathbf{r}}{\partial p}$ and $\dfrac{\partial \mathbf{r}}{\partial q}$ and the projection of that parallelogram onto the plane $z = 0$. Now

$$\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \frac{\partial z}{\partial p}\right) \times \left(\frac{\partial x}{\partial q}, \frac{\partial y}{\partial q}, \frac{\partial z}{\partial q}\right) = (\Delta X, \Delta Y, \Delta Z),$$

where Δ is the length of the area of this parallelogram and the vector (X, Y, Z) is normal to the parallelogram. If we project this parallelogram onto the plane $z = 0$, the resultant parallelogram has sides generated by the vectors

$$\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, 0\right) \quad \text{and} \quad \left(\frac{\partial x}{\partial q}, \frac{\partial y}{\partial q}, 0\right).$$

The vector product of these two vectors is $(0, 0, \Delta Z)$, and therefore the projected parallelogram has area ΔZ, Thus the ratio of the projected parallelogram and the original parallelogram parallel to the tangent plane of the surface is equal to the third component Z of the normal vector to the surface.

This result is interpreted, in Section 7 of Gauss's *General Investigations of Curved Surfaces* as follows: if an (infinitesimal) element of a smooth surface with area $d\sigma$ is projected onto the plane of the coordinates x and y, then the area of the projection of that element is $Z d\sigma$. Gauss then observes that, because the tangent plane to the auxiliary sphere at (X, Y, Z) is parallel to the tangent space to the surface, the same relationship holds between the area of a corresponding element of the auxiliary sphere and its projection onto the plane of the coordinates x and y. (Gauss presents his arguments using triangles rather than parallelograms, but the areas of the parallelograms are double the areas of the corresponding triangles.)

Now the point on the auxiliary sphere corresponding to the point (x, y, z) of the surface has coordinates (X, Y, Z). It follows that formulae relating the area of an element of the auxiliary sphere to that of its projection into the plane of the coordinates x and y can be obtained from the corresponding formulae for the smooth surface by replacing x, y and z in those formulae by X, Y and Z respectively. It then follows that the ratio k of an "element" of the auxiliary sphere to the corresponding "element" of the given smooth surface is equal to the ratio between the areas of their projections onto the plane $z = 0$, and is therefore given by the formula

$$k = \frac{\dfrac{\partial X}{\partial p}\dfrac{\partial Y}{\partial q} - \dfrac{\partial X}{\partial q}\dfrac{\partial Y}{\partial p}}{\dfrac{\partial x}{\partial p}\dfrac{\partial y}{\partial q} - \dfrac{\partial x}{\partial q}\dfrac{\partial y}{\partial p}}.$$

Denoting "infinitesimal" variations of X with respect to p and q by dX and δX respectively, and denoting the corresponding "infinitesimal" variations of Y by dY and δY, so that

$$dX = \frac{\partial X}{\partial p}\, dp, \quad \delta X = \frac{\partial X}{\partial q}\, \delta q, \quad dY = \frac{\partial Y}{\partial p}\, dp, \quad \delta Y = \frac{\partial Y}{\partial q}\, \delta q,$$

we arrive at the formula

$$k = \frac{dX \cdot \delta Y - dY \cdot \delta X}{dx \cdot \delta y - dy \cdot \delta x}.$$

presented in Section 7 of Gauss's *General Investigations of Curved Surfaces*

We now derive this result through appropriate applications of Lemma 7.1.

Proposition 7.3 *Let $\nu: \Sigma \to S^2$ be the Gauss map of a smooth surface Σ, and let R be a region in Σ with regular boundary contained within the domain of a smooth local coordinate system (p, q) on the surface. Then*

$$\operatorname{area}(\nu(R)) = \int_R k\, d\sigma,$$

where the integral over R is taken with respect to surface area, and where $k(p, q)$ is expressed in terms of the partial derivatives of the the Cartesian components X, Y, Z of the Gauss map of the surface by means of the formula

$$k(p, q) = \frac{\dfrac{\partial X}{\partial p}\dfrac{\partial Y}{\partial q} - \dfrac{\partial X}{\partial q}\dfrac{\partial Y}{\partial p}}{\dfrac{\partial x}{\partial p}\dfrac{\partial y}{\partial q} - \dfrac{\partial x}{\partial q}\dfrac{\partial y}{\partial p}}.$$

Proof Let $\mathbf{r}(p,q)$ denote the position vector of a point on the surface Σ determined by the local coordinates p and q, and let $\mathbf{n}(p,q) = \nu(\mathbf{r}(p,q))$. Then $\mathbf{n}(p,q)$ is a unit vector normal to the surface Σ at the point $\mathbf{r}(p,q)$, and

$$\mathbf{n}(p,q) = \Bigl(X(p,q),\, Y(p,q),\, Z(p,q)\Bigr).$$

In this way we can consider the local coordinates (p,q) as parameterizing points on the auxiliary sphere S^2. It follows from Lemm 7.1, applied to the auxiliary sphere, that

$$\mathrm{area}(\nu(R)) = \int_R \theta(p,q)\, dp\, dq,$$

where

$$\theta(p,q) = \left|\frac{\partial \mathbf{n}}{\partial p} \times \frac{\partial \mathbf{n}}{\partial q}\right|.$$

Moreover the normal vector to the auxiliary sphere at $\mathbf{n}(p,q)$ is equal to $\mathbf{n}(p,q)$ itself, and its components are therefore equal to $X(p,q)$, $Y(p,q)$ and $Z(p,q)$. It follows from Lemma 7.2 that

$$\begin{aligned}\theta(p,q) &= \frac{\pm 1}{Z(p,q)}\left(\frac{\partial X}{\partial p}\frac{\partial Y}{\partial q} - \frac{\partial X}{\partial q}\frac{\partial Y}{\partial p}\right)\\ &= \frac{\pm k(p,q)}{Z(p,q)}\left(\frac{\partial x}{\partial p}\frac{\partial y}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial y}{\partial p}\right)\\ &= k(p,q)\Delta(p,q)\end{aligned}$$

where $\Delta(p,q)$ is the area of the parallelogram in the tangent space to the surface generated by $\dfrac{\partial \mathbf{r}}{\partial p}$ and $\dfrac{\partial \mathbf{r}}{\partial q}$, and $k(p,q)$ is determined by the formula given in the statement of the proposition. The result then follows on applying Lemma 7.1. ∎

Definition Let Σ be a smooth surface in $\mathbb{R}^3$, and let $\nu\colon \Sigma \to S^2$ denote the Gauss map of the surface. The *Gaussian curvature* of Σ at a point $\mathbf{r}$ of Σ is the limit of the ratio

$$\frac{\mathrm{area}(\nu(R))}{\mathrm{area}(R)}$$

as the region R shrinks down to the point $\mathbf{r}$.

The limiting process involved in the above definition can be made more explicit as follows. Let $k(\mathbf{r})$ denote the Gaussian curvature at a point r of the smooth surface Σ. Then, given any strictly positive real number ϵ, there exists a strictly positive real number δ such that

$$k(\mathbf{r}) - \delta < \frac{\text{area}(\nu(R))}{\text{area}(R)} < k(\mathbf{r}) + \delta$$

for all regions R of the surface with regular boundary whose points all lie within a distance δ of the point $\mathbf{r}$.

The Gaussian curvature of a smooth surface at a point of that surface has a value that is independent of any choice of coordinate system on that surface. But, given any smooth local coordinate system (p, q) on a portion of that surface, the Gaussian curvature can be calculated according to the formula given in the statement of Proposition 7.3.

Corollary 7.4 *Let $f: D \to \mathbf{R}$ be a smooth real-valued function defined over an open set D in $\mathbb{R}^2$, let Σ be the smooth surface in $\mathbb{R}^3$ defined such that*

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } z = f(x, y)\},$$

and let (X, Y, Z) be the triple of smooth functions on Σ whose values at each point of Σ are the components of the unit normal vector field there. Then the Gaussian curvature $k(x, y)$ of the surface Σ at a point $(x, y, f(x, y))$ of the surface satisfies

$$k(x, y) = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}.$$

Proof This follows directly from Proposition 7.3 on applying that result to the smooth local coordinate system (p, q) with $p = x$ and $q = y$. ∎

7.8 Calculation of the Gaussian Curvature

Let Σ be a surface of the form

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D \text{ and } z = f(x, y)\},$$

where D is an open set in $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ is a smooth real-valued function on D. Let

$$t = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} \quad \text{and} \quad u = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}.$$

Suppose the direction of the unit normal (X, Y, Z) is chosen such that $Z > 0$. Then

$$X = \frac{-t}{\sqrt{1 + t^2 + u^2}}, \quad Y = \frac{-u}{\sqrt{1 + t^2 + u^2}}, \quad Z = \frac{1}{\sqrt{1 + t^2 + u^2}}.$$

(These formula were obtained by Gauss at the end of Section 4 of *General Investigations of Curved Surface.*)

Let

$$\frac{\partial^2 z}{\partial x^2} = T, \quad \frac{\partial^2 z}{\partial x\, \partial y} = U, \quad \frac{\partial^2 z}{\partial y^2} = V,$$

so that

$$\frac{\partial t}{\partial x} = T, \quad \frac{\partial t}{\partial y} = \frac{\partial u}{\partial x} = U, \quad \frac{\partial u}{\partial u} = V.$$

Now

$$X = -tZ, \quad Y = -uZ, \quad (1 + t^2 + u^2)Z^2 = 1.$$

It follows that

$$\begin{aligned}
\frac{\partial X}{\partial x} &= -Z\frac{\partial t}{\partial x} - t\frac{\partial Z}{\partial x} = -ZT - t\frac{\partial Z}{\partial x}, \\
\frac{\partial X}{\partial y} &= -Z\frac{\partial t}{\partial y} - t\frac{\partial Z}{\partial y} = -ZU - t\frac{\partial Z}{\partial y}, \\
\frac{\partial Y}{\partial x} &= -Z\frac{\partial u}{\partial x} - u\frac{\partial Z}{\partial x} = -ZU - u\frac{\partial Z}{\partial x}, \\
\frac{\partial Y}{\partial y} &= -Z\frac{\partial u}{\partial y} - u\frac{\partial Z}{\partial y} = -ZV - u\frac{\partial Z}{\partial y}, \\
0 &= 2(1 + t^2 + u^2)Z\frac{\partial Z}{\partial x} + 2Z^2\left(t\frac{\partial t}{\partial x} + u\frac{\partial u}{\partial x}\right) \\
&= 2Z\left((1 + t^2 + u^2)\frac{\partial Z}{\partial x} + Z(tT + uU)\right), \\
0 &= 2(1 + t^2 + u^2)Z\frac{\partial Z}{\partial y} + 2Z^2\left(t\frac{\partial t}{\partial y} + u\frac{\partial u}{\partial y}\right) \\
&= 2Z\left((1 + t^2 + u^2)\frac{\partial Z}{\partial y} + Z(tU + uV)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial Z}{\partial x} &= \frac{-Z}{1 + t^2 + u^2}(tT + uU) = -Z^3(tT + uU), \\
\frac{\partial Z}{\partial y} &= \frac{-Z}{1 + t^2 + u^2}(tU + uV) = -Z^3(tU + uV).
\end{aligned}$$

Therefore

$$\frac{\partial X}{\partial x} = -ZT + tZ^3(tT + uU)$$

$$
\begin{aligned}
&= Z^3(-(1+t^2+u^2)T + t^2 T + tuU)\\
&= Z^3(-(1+u^2)T + tuU),\\
\frac{\partial X}{\partial y} &= -ZU + tZ^3(tU + uV)\\
&= Z^3(-(1+t^2+u^2)U + t^2 U + tuV)\\
&= Z^3(-(1+u^2)U + tuV),\\
\frac{\partial Y}{\partial x} &= -ZU + uZ^3(tT + uU)\\
&= Z^3(-(1+t^2+u^2)U + tuT + u^2 U)\\
&= Z^3(-(1+t^2)U + tuT),\\
\frac{\partial Y}{\partial y} &= -ZV + uZ^3(tU + uV)\\
&= Z^3(-(1+t^2+u^2)V + tuU + u^2 V)\\
&= Z^3(-(1+t^2)V + tuU).
\end{aligned}
$$

The Gaussian curvature k of the surface satisfies

$$
k(x,y) = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x}
$$

(see Lemma 7.4). It follows that

$$
\begin{aligned}
Z^{-6}k &= (-(1+u^2)T + tuU)(-(1+t^2)V + tuU)\\
&\qquad - (-(1+u^2)U + tuV)(-(1+t^2)U + tuT)\\
&= = (1+u^2)(1+t^2)TV - tu(1+u^2)TU\\
&\qquad - tu(1+t^2)UV + t^2u^2U^2\\
&\qquad - (1+u^2)(1+t^2)U^2 + tu(1+u^2)TU\\
&\qquad + tu(1+t^2)UV - t^2u^2TV\\
&= \Big((1+t^2)(1+u^2) - t^2u^2\Big)(TV - U^2)\\
&= (1+t^2+u^2)(TV - U^2) = Z^{-2}(TV - U^2).
\end{aligned}
$$

Thus

$$
k = Z^4(TV - U^2) = \frac{TV - U^2}{(1+t^2+u^2)^2}.
$$

We summarize the result just obtained in the following proposition.

Proposition 7.5 *Let $f\colon \Omega_0 \to \mathbb{R}$ be a smooth real-valued function defined over some open set Ω_0 in $\mathbb{R}^2$, and let*

$$
\Sigma = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in \Omega_0 \text{ and } z = f(x,y)\}.
$$

Then the Gaussian curvature k of the surface Σ satisfies

$$k = \frac{TV - U^2}{(1 + t^2 + u^2)^2},$$

where

$$t = \frac{\partial z}{\partial x}, \quad u = \frac{\partial z}{\partial y}, \quad T = \frac{\partial^2 z}{\partial x^2}, \quad U = \frac{\partial^2 z}{\partial x\, \partial y}, \quad V = \frac{\partial^2 z}{\partial y^2}.$$

This completes the calculation of the Gaussian curvature of a surface of the form $z = f(x, y)$ that concludes Section 7 of Gauss's *General Investigations of Curved Surfaces.*

7.9 The Sectional Curvatures of a Smooth Surface

Let Σ be a smooth surface in $\mathbb{R}^3$. In Section 8 of *General Investigations of Curved Surfaces* Gauss observes that a Cartesian coordinate system can be rotated if necessary in order to ensure that the coordinate functions x, y and z are all zero at some given point of the smooth surface, the tangent plane to the surface at that point is the plane $z = 0$. Let T°, U° and V° denote the values of the functions T, U and V respectively at the given point. Then the equation of the surface with respect to the Cartesian coordinate system chosen as described above takes the form

$$z = \tfrac{1}{2}T^\circ x^2 + U^\circ xy + \tfrac{1}{2}V^\circ y^2 + \Omega(x, y),$$

where $\Omega(x, y)$ is a smooth real-valued function of x and y which tends to zero as (x, y) tends to $(0, 0)$ fast enough to ensure the existence of positive constants L and δ such that

$$|\Omega(x, y)| \leq L(x^2 + y^2)^{\frac{3}{2}}$$

whenever $0 < \sqrt{x^2 + y^2} < \delta$. The paraboloid

$$z = \frac{1}{2}T^\circ x^2 + U^\circ xy + V^\circ y^2$$

is the *osculating paraboloid* to the surface at the origin.

Now one can rotate the Cartesian coordinate system about the z-axis, if necessary, to ensure that $U^\circ = 0$. The equation of the surface then becomes

$$z = \frac{1}{2}(T^\circ x^2 + V^\circ y^2) + \Omega(x, y),$$

Given a real number φ, the plane

$$\{(x,y,z) \in \mathbb{R}^3 : x \sin\varphi = y \cos\varphi\}$$

intersects the surface along the curve η_φ, where

$$\eta_\varphi(t) = \tfrac{1}{2}(T^\circ \cos^2\varphi + V^\circ \cos^2\varphi)t^2 + \Omega(t\cos\varphi, t\sin\varphi)$$

for all values of the real parameter t sufficiently close to zero. Moreover there exists a constant L such that

$$|\Omega(t\cos\varphi, t\sin\varphi)| \leq L|t|^3$$

for all real numbers φ, and for all values of the real number t that are sufficiently close to zero. The *curvature* κ_φ of the smooth curve η_φ at $t = 0$ is defined to be the rate of change of its unit tangent vector $|\eta'_\varphi(t)|^{-1}\eta'_\varphi(t)$. Now

$$|\eta'_\varphi(0)| = 1 \quad \text{and} \quad \left.\frac{d}{dt}|\eta'_\varphi(t)|\right|_{t=0} = 0.$$

It follows that

$$\kappa_\varphi = |\eta''_\varphi(0)| = T^\circ \cos^2\varphi + V^\circ \cos^2\varphi.$$

This quantity κ_φ is the *sectional curvature* of the surface in the direction of the tangent vector $(\cos\varphi, \sin\varphi, 0)$.

The sectional curvature κ_φ of the surface at the origin in the direction of the tangent vector $(\cos\varphi, \sin\varphi, 0)$. is thus equal to that of the parabola

$$z = \tfrac{1}{2}(T^\circ x^2 + V^\circ y^2)$$

at the origin in the same direction $(\cos\varphi, \sin\varphi)$. It is also equal to the curvature of a circle passing through the origin in the given plane whose centre is located at the point

$$\left(0, 0, \frac{1}{T^\circ \cos^2\varphi + V^\circ \sin^2\varphi}\right).$$

The radius of this circle is the *radius of curvature* of the smooth curve η_φ at $t - 0$. This radius of curvature is thus equal to the reciprocal $1/\kappa_\varphi$ of the sectional curvature κ_φ.

If $T^\circ = V^\circ$ then all sectional curvatures of the surface Σ at the origin are equal. The origin is then said to be an *umbilic point* of the surface. Otherwise one of the quantities T° and V° is the minimum value of the sectional curvatures at the origin, and the other is the maximum of those sectional curvatures. The quantities T° and V° are referred to as the *principal curvatures* of the surface at the origin, and the corresponding directions $(1,0,0)$ and $(0,1,0)$ are referred to as the *principal directions* of curvature at the origin. The Gaussian curvature k at the origin then satisfies the equation $k = T^\circ V^\circ$, and is thus the product of the principal curvatures at the origin.

7.10 The Gaussian Curvature of a Parameterized Surface

[Discussion of Section 9 of Gauss's *General Investigations of Curved Surfaces* is omitted from these notes.]

Let Σ be a smooth surface in $\mathbb{R}^3$, and let (p, q) be a smooth local coordinate system on a portion of the surface Σ. We discuss below the calculation in Section 10 of Gauss's *General Investations of Curved Surfaces* for determining the Gaussian curvature over the coordinate patch.

We denote by $\mathbf{r}(p, q)$ the position vector of a point on the surface determined by local coordinates p and q. The normal vector (X, Y, Z) then satisfies

$$(X, Y, Z) = \frac{1}{\Delta}\left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right),$$

where

$$\Delta = \left|\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right|.$$

Let

$$\frac{\partial x}{\partial p} = a, \quad \frac{\partial y}{\partial p} = b, \quad \frac{\partial z}{\partial p} = c,$$

$$\frac{\partial x}{\partial q} = a', \quad \frac{\partial y}{\partial q} = b', \quad \frac{\partial z}{\partial q} = c',$$

$$\frac{\partial^2 x}{\partial p^2} = \alpha, \quad \frac{\partial^2 y}{\partial p^2} = \beta, \quad \frac{\partial^2 z}{\partial p^2} = \gamma,$$

$$\frac{\partial^2 x}{\partial p\,\partial q} = \alpha', \quad \frac{\partial^2 y}{\partial p\,\partial q} = \beta', \quad \frac{\partial^2 z}{\partial p\,\partial q} = \gamma',$$

$$\frac{\partial^2 x}{\partial q^2} = \alpha'', \quad \frac{\partial^2 y}{\partial q^2} = \beta'', \quad \frac{\partial^2 z}{\partial q^2} = \gamma'',$$

(Here the notation a', b', c', α', β', γ', α'', β'', γ'' is used to be consistent with the notation in Gauss's *General Investigations*: these notation is *not* intended to indicate that those functions that are first or second derivatives in any sense of the functions a, b, c, α, β or γ.) Then

$$\frac{\partial \mathbf{r}}{\partial p} = (a, b, c), \quad \frac{\partial \mathbf{r}}{\partial q} = (a', b', c'),$$

$$X = \frac{A}{\Delta}, \quad Y = \frac{B}{\Delta}, \quad Z = \frac{C}{\Delta},$$

where

$$A = bc' - cb', \quad B = ca' - ac', \quad C = ab' - ba'$$

and

$$\Delta^2 = A^2 + B^2 + C^2 = (bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2.$$

The vector (A, B, C) is then orthogonal to the tangent space to the surface at a given point of the surface, and therefore has zero scalar product with the vectors (a, b, c) and (a', b', c') that span the tangent space at that point. It follows that

$$Aa + Bb + Cc = 0 \quad \text{and} \quad Aa' + Bb' + Cc' = 0.$$

Now one of the Cartesian coordinate functions may be expressed on the surface as a smooth function of the other two around any given point $\mathbf{r}_0$ of the surface. We may there suppose, without loss of generality, that there exists a smooth real-valued function f defined over some open set in $\mathbb{R}^2$ such that the surface takes the form $z = f(x, y)$ around the point $\mathbf{r}_0$. Then

$$\begin{aligned} c &= \frac{\partial z}{\partial p} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial p} = ta + ub, \\ c' &= \frac{\partial z}{\partial q} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial q} = ta' + ub', \end{aligned}$$

where

$$t = \frac{\partial z}{\partial x}, \quad u = \frac{\partial z}{\partial y}.$$

It follows that

$$(A + Ct)a + (B + Cu)b = 0, \quad (A + Ct)a' + (B + Cu)b' = 0.$$

Moreover the projection onto the plane $z = 0$ sending $(x, y, z) \in \mathbb{R}^3$ to $(x, y, 0)$ maps the tangent plane to the surface at $\mathbf{r_0}$ surjectively onto the plane $z = 0$. It follows that the vectors (a, b) and (a', b') span the vector space $\mathbb{R}^2$ and are therefore linearly independent. The vector $(A + Ct, B + Cu)$ therefore has zero scalar product with all vectors in $\mathbb{R}^2$, and therefore $A + Ct = 0$ and $B + Cu = 0$. Thus

$$\frac{\partial z}{\partial x} = t = -\frac{A}{C} \quad \text{and} \quad \frac{\partial z}{\partial y} = u = -\frac{B}{C}.$$

Next we note that it follows from the Chain Rule for computing derivatives of compositions of differentiable functions of several real variables that

$$\begin{pmatrix} \dfrac{\partial p}{\partial x} & \dfrac{\partial p}{\partial y} \\ \dfrac{\partial q}{\partial x} & \dfrac{\partial q}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial p} & \dfrac{\partial x}{\partial q} \\ \dfrac{\partial y}{\partial p} & \dfrac{\partial y}{\partial q} \end{pmatrix}^{-1} = \begin{pmatrix} a & a' \\ b & b' \end{pmatrix}^{-1}$$

$$= \frac{1}{ab' - a'b} \begin{pmatrix} b' & -a' \\ -b & a \end{pmatrix} = \frac{1}{C} \begin{pmatrix} b' & -a' \\ -b & a \end{pmatrix},$$

and thus

$$C\,\frac{\partial p}{\partial x} = b', \quad C\,\frac{\partial p}{\partial y} = -a', \quad C\,\frac{\partial q}{\partial x} = -b, \quad C\,\frac{\partial q}{\partial y} = a.$$

Differentiating the identities $t = -A/C$ and $u = -B/C$ with respect to p and q, we find that

$$\begin{aligned}
\frac{\partial t}{\partial p} &= \frac{1}{C^2}\left(A\,\frac{\partial C}{\partial p} - C\frac{\partial A}{\partial p}\right), \\
\frac{\partial t}{\partial q} &= \frac{1}{C^2}\left(A\,\frac{\partial C}{\partial q} - C\frac{\partial A}{\partial q}\right), \\
\frac{\partial u}{\partial p} &= \frac{1}{C^2}\left(B\,\frac{\partial C}{\partial p} - C\frac{\partial B}{\partial p}\right), \\
\frac{\partial u}{\partial q} &= \frac{1}{C^2}\left(B\,\frac{\partial C}{\partial b} - C\frac{\partial B}{\partial q}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
C^3\frac{\partial t}{\partial x} &= C^3\frac{\partial t}{\partial p}\,\frac{\partial p}{\partial x} + C^3\frac{\partial t}{\partial q}\,\frac{\partial q}{\partial x} \\
&= \left(A\,\frac{\partial C}{\partial p} - C\frac{\partial A}{\partial p}\right) b' - \left(A\,\frac{\partial C}{\partial q} - C\frac{\partial A}{\partial q}\right) b \\
C^3\frac{\partial t}{\partial y} &= C^3\frac{\partial t}{\partial p}\,\frac{\partial p}{\partial y} + C^3\frac{\partial t}{\partial q}\,\frac{\partial q}{\partial y} \\
&= -\left(A\,\frac{\partial C}{\partial p} - C\frac{\partial A}{\partial p}\right) a' + \left(A\,\frac{\partial C}{\partial q} - C\frac{\partial A}{\partial q}\right) a \\
C^3\frac{\partial u}{\partial x} &= C^3\frac{\partial t}{\partial p}\,\frac{\partial p}{\partial x} + C^3\frac{\partial t}{\partial q}\,\frac{\partial q}{\partial x} \\
&= \left(B\,\frac{\partial C}{\partial p} - C\frac{\partial B}{\partial p}\right) b' - \left(B\,\frac{\partial C}{\partial q} - C\frac{\partial B}{\partial q}\right) b \\
C^3\frac{\partial u}{\partial y} &= C^3\frac{\partial t}{\partial p}\,\frac{\partial p}{\partial y} + C^3\frac{\partial t}{\partial q}\,\frac{\partial q}{\partial y} \\
&= -\left(B\,\frac{\partial C}{\partial p} - C\frac{\partial B}{\partial p}\right) a' + \left(B\,\frac{\partial C}{\partial q} - C\frac{\partial B}{\partial q}\right) a
\end{aligned}$$

Now

$$\frac{\partial A}{\partial p} = c'\beta + b\gamma' - c\beta' - b'\gamma,$$

$$
\begin{aligned}
\frac{\partial A}{\partial q} &= c'\beta' + b\gamma'' - c\beta'' - b'\gamma', \\
\frac{\partial B}{\partial p} &= a'\gamma + c\alpha' - a\gamma' - c'\alpha, \\
\frac{\partial B}{\partial q} &= a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha', \\
\frac{\partial C}{\partial p} &= b'\alpha + a\beta' - b\alpha' - a'\beta, \\
\frac{\partial C}{\partial q} &= b'\alpha' + a\beta'' - b\alpha'' - a'\beta',
\end{aligned}
$$

and therefore

$$
\begin{aligned}
A\,\frac{\partial C}{\partial p} - C\frac{\partial A}{\partial p} &= \alpha Ab' + \beta' Aa - \alpha' Ab - \beta Aa' \\
&\qquad - \beta Cc' - \gamma' Cb + \beta' Cc + \gamma Cb', \\
A\,\frac{\partial C}{\partial q} - C\frac{\partial A}{\partial q} &= \alpha' Ab' + \beta'' Aa - \alpha'' Ab - \beta' Aa' \\
&\qquad - \beta' Cc' - \gamma'' Cb + \beta'' Cc + \gamma' Cb', \\
B\,\frac{\partial C}{\partial p} - C\frac{\partial B}{\partial p} &= \alpha Bb' + \beta' Ba - \alpha' Bb - \beta Ba' \\
&\qquad - \gamma Ca' - \alpha' Cc + \gamma' Ca + \alpha Cc', \\
B\,\frac{\partial C}{\partial q} - C\frac{\partial B}{\partial q} &= \alpha' Bb' + \beta'' Ba - \alpha'' Bb - \beta' Ba' \\
&\qquad - \gamma' Ca' - \alpha'' Cc + \gamma'' Ca + \alpha' Cc'.
\end{aligned}
$$

The quantities T, U and V have been defined such that

$$
T = \frac{\partial^2 z}{\partial x^2} = \frac{\partial t}{\partial x}, \quad U = \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial t}{\partial y} = \frac{\partial u}{\partial x}, \quad V = \frac{\partial^2 z}{\partial y^2} = \frac{\partial u}{\partial u}.
$$

On using the identities $Aa + Bb + Cc = 0$ and $Aa' + Bb' + Cc' = 0$, we find that

$$
\begin{aligned}
C^3 T &= \alpha Ab'^2 + \beta' Aab' - \alpha' Abb' - \beta Aa'b' \\
&\qquad - \beta Cc'b' - \gamma' Cbb' + \beta' Ccb' + \gamma Cb'^2, \\
&\qquad - \alpha' Abb' - \beta'' Aab + \alpha'' Ab^2 + \beta' Aa'b \\
&\qquad + \beta' Cbc' + \gamma'' Cb^2 - \beta'' Cbc - \gamma' Cbb', \\
&= \alpha Ab'^2 - \beta Aa'b' - \beta Cc'b' + \gamma Cb'^2 - 2\alpha' Abb' \\
&\qquad + \beta' Aab' + \beta' Aba' + \beta' Ccb' + \beta' Cbc'
\end{aligned}
$$

$$
\begin{aligned}
& \quad - 2\gamma' Cbb' + \alpha'' Ab^2 - \beta'' Aab - \beta'' Cbc + \gamma'' Cb^2 \\
= \; & \alpha A b'^2 + \beta B b'^2 + \gamma C b'^2 \\
& \quad - 2\alpha' Abb' - 2\beta' Bbb' - 2\gamma' Cbb' \\
& \quad + \alpha'' Ab^2 + \beta'' Bb^2 + \gamma'' Cb^2.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
C^3 U \; = \; & -\alpha A a' b' - \beta' A a a' + \alpha' A a' b + \beta A a'^2 \\
& \quad + \beta C a' c' + \gamma' C a' b - \beta' C a' c - \gamma C a' b', \\
& \quad + \alpha' A a b' + \beta'' A a^2 - \alpha'' A a b - \beta' A a a' \\
& \quad - \beta' C a c' - \gamma'' C a b + \beta'' C a c + \gamma' C a b', \\
= \; & -\alpha A a' b' + \beta A a'^2 + \beta C a' c' - \gamma C a' b' + \alpha' A b a' + \alpha' A a b' \\
& \quad - \beta' A a a' - \beta' C c a' - \beta' A a a' - \beta' C a c' \\
& \quad + \gamma' C a b' + \gamma' C b a' - \alpha'' A a b + \beta'' A a^2 + \beta'' C a c - \gamma'' C a b \\
= \; & -\alpha A a' b' - \beta B a' b' - \gamma C a' b' \\
& \quad + \alpha' A (a b' + b a') + \beta' B (a b' + b a') + \gamma' C (a b' + b a') \\
& \quad - \alpha'' A a b - \beta'' B a b - \gamma'' C a b,
\end{aligned}
$$

and

$$
\begin{aligned}
C^3 V \; = \; & -\alpha B a' b' - \beta' B a a' + \alpha' B b a' + \beta B a'^2 \\
& \quad + \gamma C a'^2 + \alpha' C a' c - \gamma' C a a' - \alpha C a' c', \\
& \quad + \alpha' B a b' + \beta'' B a^2 - \alpha'' B a b - \beta' B a a' \\
& \quad - \gamma' C a a' - \alpha'' C a c + \gamma'' C a^2 + \alpha' C a c' \\
= \; & -\alpha B a' b' - \alpha C a' c' + \beta B a'^2 + \gamma C a'^2 + \alpha' B b a' + \alpha' C a' c \\
& \quad + \alpha' B a b' + \alpha' C a c', -\beta' B a a' - \beta' B a a' - \gamma' C a a' - \gamma' C a a' \\
& \quad - \alpha'' B a b - \alpha'' C a c + \beta'' B a^2 + \gamma'' C a^2 \\
= \; & \alpha A a'^2 + \beta B a'^2 + \gamma C a'^2 \\
& \quad - 2\alpha' A a a' - 2\beta' B a a' - 2\gamma' C a a' \\
& \quad + \alpha'' A a^2 + \beta'' B a^2 + \gamma'' C a^2.
\end{aligned}
$$

Let

$$
\begin{aligned}
A\alpha + B\beta + C\gamma &= D, \\
A\alpha' + B\beta' + C\gamma' &= D', \\
A\alpha'' + B\beta'' + C\gamma'' &= D''.
\end{aligned}
$$

Then

$$\begin{aligned} C^3T &= Db'^2 - 2D'bb' + D''b^2, \\ C^3U &= -Da'b' + D'(ab' + ba') - D''ab, \\ C^3V &= Da'^2 - 2D'aa' + D''a^2. \end{aligned}$$

It follows that

$$\begin{aligned} C^6TV &= D^2a'^2b'^2 + 4D'^2aba'b' + D''^2a^2b^2 \\ &\quad - 2DD'a'b'(ab' + ba') - 2D'D''ab(ab' + ba') \\ &\quad + DD''(a^2b'^2 + b^2a'^2), \\ C^6U^2 &= D^2a'^2b'^2 + D'^2(a^2b'^2 + b^2a'^2 + 2aba'b') + D''^2a^2b^2 \\ &\quad - 2DD'a'b'(ab' + ba') - 2D'D''ab(ab' + ba') \\ &\quad + 2DD''aa'bb', \end{aligned}$$

and therefore

$$\begin{aligned} C^6(TV - U^2) &= (DD'' - D'^2)(a^2b'^2 + b^2a'^2 - 2aba'b') \\ &= (DD'' - D'^2)(ab' - ba')^2 \\ &= (DD'' - D'^2)C^2. \end{aligned}$$

Now the Gaussian curvature k of the surface satisfies

$$k = \frac{TV - U^2}{(1 + t^2 + u^2)^2}.$$

(see Proposition 7.5). Also

$$1 + t^2 + u^2 = 1 + \frac{A^2}{C^2} + \frac{B^2}{C^2} = \frac{1}{C^2}(A^2 + b^2 + C^2).$$

It follows that

$$k = \frac{C^4(TV - U^2)}{(A^2 + B^2 + C^2)^2} = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}.$$

Proposition 7.6 *Let Σ be a smooth surface in $\mathbb{R}^3$ and let (p, q) be a smooth local coordinate system on a portion of Σ. Let $\mathbf{r}(p, q)$ represent the position vector of a point of Σ as a smooth function of the local coordinates p and q, let*

$$\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} = (A, B, C),$$

where A, B and C are smooth functions on the surface Σ, and let

$$\begin{aligned} D &= \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial p^2}, \\ D' &= \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial p\, \partial q}, \\ D'' &= \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial q^2}. \end{aligned}$$

Then the Gaussian curvature k of the surface satisfies

$$k = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}.$$

This concludes the discussion of Section 11 of Gauss's *General Investigations of Curved Surfaces.*

7.11 Determining Gaussian Curvature from Metric Coefficients

We recall the definitions of the following real-valued functions that are determined by partial derivatives of the position vector $\mathbf{r}$ of a point on a smooth surface Σ with respect to smooth local coordinates (p, q) defined over a coordinate patch on that surface:

$$(a, b, c) = \frac{\partial \mathbf{r}}{\partial p}, \quad (a', b', c') = \frac{\partial \mathbf{r}}{\partial q},$$

$$(\alpha, \beta, \gamma) = \frac{\partial^2 \mathbf{r}}{\partial p^2}, \quad (\alpha', \beta', \gamma') = \frac{\partial^2 \mathbf{r}}{\partial p\, \partial q}, \quad (\alpha'', \beta'', \gamma'') = \frac{\partial^2 \mathbf{r}}{\partial q^2},$$

$$(A, B, C) = \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}, \quad \Delta = \sqrt{A^2 + B^2 + C^2},$$

$$X = \frac{A}{\Delta}, \quad Y = \frac{B}{\Delta}, \quad Z = \frac{C}{\Delta}.$$

$$\begin{aligned} D &= A\alpha + B\beta + C\gamma = \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial p^2}, \\ D' &= A\alpha' + B\beta' + C\gamma' = \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial p\, \partial q}, \\ D'' &= A\alpha'' + B\beta'' + C\gamma'' = \left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) \cdot \frac{\partial^2 \mathbf{r}}{\partial q^2}. \end{aligned}$$

Moreover

$$A = (bc' - cb'), \quad B = (ca' - ac'), \quad C = (ab' - ba').$$

Gauss, in Section 11 of *General Investigations of Curved Surfaces*, introduces further functions:

$$\begin{aligned}
E &= a^2 + b^2 + c^2 = \left|\frac{\partial \mathbf{r}}{\partial p}\right|^2, \\
F &= aa' + bb' + cc' = \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial \mathbf{r}}{\partial q}, \\
G &= a'^2 + b'^2 + c'^2 = \left|\frac{\partial \mathbf{r}}{\partial q}\right|^2, \\
m &= a\alpha + b\beta + c\gamma = \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial^2 \mathbf{r}}{\partial p^2}, \\
m' &= a\alpha' + b\beta' + c\gamma' = \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial^2 \mathbf{r}}{\partial p\,\partial q}, \\
m'' &= a\alpha + b\beta + c\gamma = \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^2}, \\
n &= a\alpha + b\beta + c\gamma = \frac{\partial \mathbf{r}}{\partial q} \cdot \frac{\partial^2 \mathbf{r}}{\partial p^2}, \\
n' &= a\alpha' + b\beta' + c\gamma' = \frac{\partial \mathbf{r}}{\partial q} \cdot \frac{\partial^2 \mathbf{r}}{\partial p\,\partial q}, \\
n'' &= a\alpha + b\beta + c\gamma = \frac{\partial \mathbf{r}}{\partial q} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^2}.
\end{aligned}$$

Now it follows from Lagrange's Quadruple Product Identity (Proposition 5.9) that

$$\left|\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right|^2 = \left|\frac{\partial \mathbf{r}}{\partial p}\right|^2 \left|\frac{\partial \mathbf{r}}{\partial q}\right|^2 - \left(\frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial \mathbf{r}}{\partial q}\right)^2.$$

It follows that

$$\Delta^2 = A^2 + B^2 + C^2 = EG - F^2.$$

(Gauss uses Δ in Section 11 of the *General Investigations* to denote the quantity whose square root was labelled Δ^2 in Section 4 of the *General Investigations*. In what follows, we continue to use the notation Δ^2 for the quantity that Gauss denotes by Δ in Section 11.) This identity can also be verified by direct computation (and indeed the relevant computation was carried through in the proof of Proposition 5.3).

Then

$$\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} D \\ m \\ n \end{pmatrix}.$$

Now

$$\begin{vmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{vmatrix} = A(bc' - cb') + B(ca' - ac') + C(ab' - ba') = A^2 + B^2 + C^2 = \Delta^2,$$

and the standard procedure for inverting a 3×3 matrix then establishes that

$$\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix}^{-1} = \frac{1}{\Delta^2} \begin{pmatrix} A & Cb' - Bc' & Bc - Cb \\ B & Ac' - Ca' & Ca - Ac \\ C & Ba' - Ab' & Ab - Ba \end{pmatrix}.$$

It follows that

$$\begin{aligned} \Delta^2\alpha &= AD + (Cb' - Bc')m + (Bc - Cb)n, \\ \Delta^2\beta &= BD + (Ac' - Ca')m + (Ca - Ac)n, \\ \Delta^2\gamma &= CD + (Ba' - Ab')m + (Ab - Ba)n. \end{aligned}$$

Now

$$\begin{aligned} Cb' - Bc' &= ab'^2 - ba'b' - ca'c' + ac'^2 \\ &= a(a'^2 + b'^2 + c'^2) - a'(aa' + bb' + cc') \\ &= aG - a'F, \\ Bc - Cb &= c^2a' - acc' - abb' + b^2a' \\ &= (a^2 + b^2 + c^2)a' - a(aa' + bb' + cc') \\ &= a'E - aF. \end{aligned}$$

It follows that

$$\begin{aligned} AD &= \Delta^2\alpha + (a'F - aG)m + (aF - a'E)n \\ &= \Delta^2\alpha + a(nF - mG) + a'(mF - nE). \end{aligned}$$

Similarly

$$\begin{aligned} BD &= \Delta^2\beta + b(nF - mG) + b'(mF - nE), \\ CD &= \Delta^2\gamma + c(nF - mG) + c'(mF - nE). \end{aligned}$$

These three identities combine to yield the following vector identity:

$$\left(\left(\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right) . \frac{\partial^2 \mathbf{r}}{\partial p^2}\right) \frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} = \left|\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q}\right|^2 \frac{\partial^2 \mathbf{r}}{\partial p^2} + (nF - mG)\frac{\partial \mathbf{r}}{\partial p} + (mF - nE)\frac{\partial \mathbf{r}}{\partial q}.$$

This identity can be expressed in the form

$$\frac{\partial^2 \mathbf{r}}{\partial p^2} - \left(\nu(\mathbf{r}) . \frac{\partial^2 \mathbf{r}}{\partial p^2}\right) \nu(\mathbf{r}) + \frac{nF - mG}{\Delta^2}\frac{\partial \mathbf{r}}{\partial p} + \frac{mF - nE}{\Delta^2}\frac{\partial \mathbf{r}}{\partial q} = 0,$$

where

$$\nu(\mathbf{r}) = \left(\frac{A}{\Delta}, \frac{B}{\Delta}, \frac{C}{\Delta}\right).$$

Moreover $\nu(\mathbf{r})$ is a unit normal vector to the surface, and

$$\frac{\partial \mathbf{r}}{\partial p} \times \frac{\partial \mathbf{r}}{\partial q} = \Delta\, \nu(\mathbf{r}).$$

We can therefore verify the identity obtained by Gauss using vector methods as described below.

Let

$$\mathbf{S} = \frac{\partial^2 \mathbf{r}}{\partial p^2} - \left(\nu(\mathbf{r}) . \frac{\partial^2 \mathbf{r}}{\partial p^2}\right) \nu(\mathbf{r}) + \frac{nF - mG}{\Delta^2}\frac{\partial \mathbf{r}}{\partial p} + \frac{mF - nE}{\Delta^2}\frac{\partial \mathbf{r}}{\partial q}.$$

Now

$$\nu(\mathbf{r}) . \frac{\partial \mathbf{r}}{\partial p} = 0 \quad \text{and} \quad \nu(\mathbf{r}) . \frac{\partial \mathbf{r}}{\partial q} = 0,$$

because the vector $\nu(\mathbf{r})$ is orthogonal to the tangent space to the surface at a given point, whilst $\dfrac{\partial \mathbf{r}}{\partial p}$ and $\dfrac{\partial \mathbf{r}}{\partial q}$ are parallel to that tangent space. It follows that

$$\nu(\mathbf{r}) . \mathbf{S} = \nu(\mathbf{r}) . \frac{\partial^2 \mathbf{r}}{\partial p^2} - \left(\nu(\mathbf{r}) . \frac{\partial^2 \mathbf{r}}{\partial p^2}\right) \nu(\mathbf{r}) . \nu(\mathbf{r}) = 0.$$

Also

$$\begin{aligned} \frac{\partial r}{\partial p} . \mathbf{S} &= \frac{\partial r}{\partial p} . \frac{\partial^2 \mathbf{r}}{\partial p^2} + \frac{nF - mG}{\Delta^2}\left|\frac{\partial \mathbf{r}}{\partial p}\right|^2 + \frac{mF - nE}{\Delta^2}\frac{\partial \mathbf{r}}{\partial p} . \frac{\partial \mathbf{r}}{\partial q} \\ &= m + \frac{nF - mG}{\Delta^2} E + + \frac{mF - nE}{\Delta^2} F \\ &= m - \frac{EG - F^2}{\Delta^2} m = 0, \end{aligned}$$

$$\begin{aligned}
\frac{\partial r}{\partial q} \cdot \mathbf{S} &= \frac{\partial r}{\partial q} \cdot \frac{\partial^2 \mathbf{r}}{\partial p^2} + \frac{nF - mG}{\Delta^2} \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial \mathbf{r}}{\partial q} + \frac{mF - nE}{\Delta^2} \left| \frac{\partial \mathbf{r}}{\partial q} \right|^2 \\
&= n + \frac{nF - mG}{\Delta^2} F + + \frac{mF - nE}{\Delta^2} G \\
&= n - \frac{EG - F^2}{\Delta^2} n = 0.
\end{aligned}$$

It follows that $\mathbf{S} = 0$. This completes the verification of Gauss's identities

$$\begin{aligned}
AD &= \Delta^2 \alpha + a(nF - mG) + a'(mF - nE), \\
BD &= \Delta^2 \beta + b(nF - mG) + b'(mF - nE), \\
CD &= \Delta^2 \gamma + c(nF - mG) + c'(mF - nE).
\end{aligned}$$

using vector methods.

The vector $(\alpha', \beta', \gamma')$ satisfies the matrix identity

$$\begin{pmatrix} A & B & C \\ a & b & c \\ a' & b' & c' \end{pmatrix} \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} D' \\ m' \\ n' \end{pmatrix}.$$

The solution of this matrix equation leads to identities

$$\begin{aligned}
AD' &= \Delta^2 \alpha + a(n'F - m'G) + a'(m'F - n'E), \\
BD' &= \Delta^2 \beta + b(n'F - m'G) + b'(m'F - n'E), \\
CD' &= \Delta^2 \gamma + c(n'F - m'G) + c'(m'F - n'E).
\end{aligned}$$

These identities correspond to the vector identity

$$\frac{\partial^2 \mathbf{r}}{\partial p \, \partial q} - \left(\nu(\mathbf{r}) \, . \, \frac{\partial^2 \mathbf{r}}{\partial p \, \partial q} \right) \nu(\mathbf{r}) + \frac{n'F - m'G}{\Delta^2} \frac{\partial \mathbf{r}}{\partial p} + \frac{m'F - n'E}{\Delta^2} \frac{\partial \mathbf{r}}{\partial q} = 0.$$

Moreover this vector identity can be verified by establishing that the scalar product of the left hand side with each of the vectors $\nu(\mathbf{r})$, $\dfrac{\partial \mathbf{r}}{\partial p}$ and $\dfrac{\partial \mathbf{r}}{\partial q}$ is equal to zero.

Now $A\alpha'' + B\beta'' + C\gamma'' = D''$. It follows that

$$\begin{aligned}
DD'' &= \Delta^2 (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'') \\
&\qquad + (a\alpha'' + b\beta'' + c\gamma'')(nF - mG) \\
&\qquad + (a'\alpha'' + b'\beta'' + c'\gamma'')(mF - nE) \\
&= \Delta^2 (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'') + m''(nF - mG) + n''(mF - nE).
\end{aligned}$$

Similarly $A\alpha' + B\beta' + C\gamma' = D'$, and therefore

$$\begin{aligned} D'^2 &= \Delta^2(\alpha'^2 + \beta'^2 + \gamma'^2) \\ &\qquad + (a\alpha' + b\beta' + c\gamma')(n'F - m'G) \\ &\qquad + (a'\alpha' + b'\beta' + c'\gamma')(m'F - n'E) \\ &= \Delta^2(\alpha'^2 + \beta'^2 + \gamma'^2) + m'(n'F - m'G) + n'(m'F - n'E). \end{aligned}$$

It follows that

$$\begin{aligned} DD'' - D'^2 &= \Delta^2(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2) \\ &\qquad + m''(nF - mG) + n''(mF - nE) \\ &\qquad - m'(n'F - m'G) - n'(m'F - n'E) \\ &= \Delta^2(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2) \\ &\qquad + (n'^2 - nn'')E + (mn'' - 2m'n' + nm'')F \\ &\qquad + (m'^2 - mm'')G. \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial E}{\partial p} &= 2a\frac{\partial a}{\partial p} + 2b\frac{\partial b}{\partial p} + 2c\frac{\partial c}{\partial p} \\ &= 2(a\alpha + b\beta + c\gamma) = 2m, \\ \frac{\partial E}{\partial q} &= 2a\frac{\partial a}{\partial q} + 2b\frac{\partial b}{\partial q} + 2c\frac{\partial c}{\partial q} \\ &= 2(a\alpha' + b\beta' + c\gamma') = 2m', \\ \frac{\partial F}{\partial p} &= a\frac{\partial a'}{\partial p} + a'\frac{\partial a}{\partial p} + b\frac{\partial b'}{\partial p} + b'\frac{\partial b}{\partial p} + c\frac{\partial c'}{\partial p} + c'\frac{\partial c}{\partial p} \\ &= a\alpha' + b\beta' + c\gamma' + a'\alpha + b'\beta + c'\gamma = m' + n, \\ \frac{\partial F}{\partial q} &= a\frac{\partial a'}{\partial q} + a'\frac{\partial a}{\partial q} + b\frac{\partial b'}{\partial q} + b'\frac{\partial b}{\partial q} + c\frac{\partial c'}{\partial q} + c'\frac{\partial c}{\partial q} \\ &= a\alpha'' + b\beta'' + c\gamma'' + a'\alpha' + b'\beta' + c'\gamma' = m'' + n', \\ \frac{\partial G}{\partial p} &= 2a'\frac{\partial a'}{\partial p} + 2b'\frac{\partial b'}{\partial p} + 2c'\frac{\partial c'}{\partial p} \\ &= 2(a'\alpha' + b'\beta' + c\gamma') = 2n', \\ \frac{\partial G}{\partial q} &= 2a'\frac{\partial a'}{\partial q} + 2b'\frac{\partial b'}{\partial q} + 2c'\frac{\partial c'}{\partial q} \\ &= 2(a'\alpha'' + b'\beta'' + c'\gamma'') = 2n''. \end{aligned}$$

It follows, as noted by Gauss in the *General Investigations* that

$$m = \tfrac{1}{2}\frac{\partial E}{\partial p}, \quad m' = \tfrac{1}{2}\frac{\partial E}{\partial q}, \quad m'' = \frac{\partial F}{\partial q} - \tfrac{1}{2}\frac{\partial G}{\partial p},$$

$$n = \frac{\partial F}{\partial p} - \tfrac{1}{2}\frac{\partial E}{\partial q}, \quad n' = \tfrac{1}{2}\frac{\partial G}{\partial p}, \quad n'' = \tfrac{1}{2}\frac{\partial G}{\partial q}.$$

Also

$$\frac{\partial \alpha'}{\partial p} = \frac{\partial \alpha}{\partial q}, \quad \frac{\partial \beta'}{\partial p} = \frac{\partial \beta}{\partial q}, \quad \frac{\partial \gamma'}{\partial p} = \frac{\partial \gamma}{\partial q}$$

and

$$\begin{gathered} \frac{\partial a}{\partial p} = \alpha, \quad \frac{\partial b}{\partial p} = \beta, \quad \frac{\partial c}{\partial p} = \gamma, \\ \frac{\partial a}{\partial q} = \alpha', \quad \frac{\partial b}{\partial q} = \beta', \quad \frac{\partial c}{\partial q} = \gamma', \\ \frac{\partial a'}{\partial p} = \alpha', \quad \frac{\partial b'}{\partial p} = \beta', \quad \frac{\partial c'}{\partial p} = \gamma', \\ \frac{\partial a'}{\partial q} = \alpha'', \quad \frac{\partial b'}{\partial q} = \beta'', \quad \frac{\partial c'}{\partial q} = \gamma'', \end{gathered}$$

and therefore

$$\begin{aligned} \frac{\partial n}{\partial q} - \frac{\partial n'}{\partial p} &= \frac{\partial}{\partial q}(a'\alpha + b'\beta + c'\gamma) - \frac{\partial}{\partial p}(a'\alpha' + b'\beta' + c'\gamma') \\ &= \alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2, \\ \frac{\partial m''}{\partial p} - \frac{\partial m'}{\partial q} &= \frac{\partial}{\partial p}(a\alpha'' + b\beta'' + c\gamma'') - \frac{\partial}{\partial q}(a\alpha' + b\beta' + c\gamma') \\ &= \alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2. \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2 \\ &\qquad = \frac{\partial n}{\partial q} - \frac{\partial n'}{\partial p} \\ &\qquad = \frac{\partial^2 F}{\partial p\,\partial q} - \tfrac{1}{2}\frac{\partial E}{\partial q^2} - \tfrac{1}{2}\frac{\partial G}{\partial p^2}. \end{aligned}$$

Now the Gaussian curvature k of the surface satisfies

$$k = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}$$

Proposition 7.6. Moreover $A^2 + B^2 + C^2 = EG - F^2 = \Delta^2$. It follows that

$$\begin{aligned} 4(EG - F^2)^2 k &= 4(DD'' - D'^2) \\ &= 4E(n'^2 - nn'') + 4F(mn'' - 2m'n' + nm'') + 4G(m'^2 - mm'') \end{aligned}$$

$$
\begin{aligned}
&\quad + 4(EG - F^2)(\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2) \\
= \;& E\left(\left(\frac{\partial G}{\partial p}\right)^2 - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} + \frac{\partial E}{\partial q}\frac{\partial G}{\partial q}\right) \\
&\quad + F\Big(\frac{\partial E}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial G}{\partial p} + 4\frac{\partial F}{\partial p}\frac{\partial F}{\partial q} \\
&\qquad - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial p} - 2\frac{\partial F}{\partial q}\frac{\partial E}{\partial q}\Big) \\
&\qquad + G\left(\left(\frac{\partial E}{\partial q}\right)^2 - 2\frac{\partial E}{\partial p}\frac{\partial F}{\partial q} + \frac{\partial E}{\partial p}\frac{\partial G}{\partial p}\right) \\
&\qquad - 2(EG - F^2)\left(\frac{\partial E}{\partial q^2} - 2\frac{\partial^2 F}{\partial p\,\partial q} + \frac{\partial G}{\partial p^2}\right).
\end{aligned}
$$

[Note that there is a typographical error in the Project Gutenberg edition of Gauss's *General Investigations of Curved Surfaces*, where the left hand side of the above inequality is given as $4(EG - F^2)k$. This error is not in the translation as published by Princeton University Press in 1902.]

Proposition 7.7 *Let Σ be a smooth surface in $\mathbb{R}^3$ and let (p, q) be a smooth local coordinate system on a portion of Σ. Let $\mathbf{r}(p, q)$ represent the position vector of a point of Σ as a smooth function of the local coordinates p and q, and let*

$$
\begin{aligned}
E &= \left|\frac{\partial \mathbf{r}}{\partial p}\right|^2, \\
F &= \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial \mathbf{r}}{\partial q}, \\
G &= \left|\frac{\partial \mathbf{r}}{\partial q}\right|^2.
\end{aligned}
$$

Then the Gaussian curvature k of the surface is expressible in terms of the functions E, F, G and their partial derivatives of first and second order with respect to the local coordinates p and q by means of the following formula:

$$
\begin{aligned}
4(EG - F^2)^2 k \;=\;& E\left(\frac{\partial E}{\partial q}\frac{\partial G}{\partial q} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p}\right)^2\right) \\
&\quad + F\Big(\frac{\partial E}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial q}\frac{\partial F}{\partial q} \\
&\qquad + 4\frac{\partial F}{\partial p}\frac{\partial F}{\partial q} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial p}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ G\left(\frac{\partial E}{\partial p}\,\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial p}\,\frac{\partial F}{\partial q} + \left(\frac{\partial E}{\partial q}\right)^2\right) \\
&- 2(EG - F^2)\left(\frac{\partial E}{\partial q^2} - 2\frac{\partial^2 F}{\partial p\,\partial q} + \frac{\partial G}{\partial p^2}\right).
\end{aligned}
$$