MA232A: Euclidean and non-Euclidean Geometry
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Notes on Smooth Surfaces

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6 Smooth Surfaces in Three-Dimensional Euclidean Space

6.1 Smooth Functions

Definition Let Ω be an open set in $\mathbb{R}^n$ for some positive integer n, and let $\varphi\colon \Omega \to \mathbb{R}^m$ be a vector-valued function mapping Ω into $\mathbb{R}^m$ for some positive integer m. The function φ is said to be *differentiable* at a point $\mathbf{p}$ of Ω, with *derivative* (or *total derivative*) $(D\varphi)_{\mathbf{p}}\colon \mathbb{R}^n \to \mathbb{R}^m$, where $(D\varphi)_{\mathbf{p}}$ is a linear transformation from the real vector space $\mathbb{R}^n$ to the real vector space $\mathbb{R}^m$, if and only if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}\Bigl(\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\mathbf{h}\Bigr) = \mathbf{0}.$$

(Here $|\mathbf{h}|$ denotes the Euclidean norm of the n-dimensional vector $\mathbf{h}$, defined such that

$$|\mathbf{h}|^2 = h_1^2 + h_2^2 + \cdots + h_n^2,$$

where $h_1, h_2, \ldots, h_n$ are the Cartesian components of the vector $\mathbf{h}$.)

The derivative $(D\varphi)_{\mathbf{p}}$ of a differentiable function $\varphi\colon \Omega \to \mathbb{R}^m$ at a point $\mathbf{p}$ can be represented with respect to the standard bases of the real vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ by an $m \times n$ matrix. The function $\varphi\colon \Omega \to \mathbb{R}^m$ is *continuously differentiable* on Ω if the components of the $m \times n$ matrix representing the derivative $(D\varphi)_{\mathbf{p}}$ are continuous functions of $\mathbf{p}$ as the point $\mathbf{p}$ varies over the open set Ω.

A theorem of real analysis in severable variables guarantees that a vector-valued function defined over an open set in $\mathbb{R}^n$ is continuously differentiable if and only if the partial derivatives of its components taken with respect to the Cartesian coordinates on Ω exist and are continuous throughout Ω.

One can regard the derivative of a continuously-differentiable vector-valued function $\varphi\colon \Omega \to \mathbb{R}^m$ mapping the open set Ω into $\mathbb{R}^m$ as being itself a continuous vector-valued function $D\varphi\colon \Omega \to M_{m,n}(\mathbb{R})$ mapping Ω into the space $M_{m,n}(\mathbb{R})$ of $m \times n$ matrices with real coefficients which sends each point $\mathbf{p}$ of Ω to the $m \times n$ matrix representing the derivative $(D\varphi)_{\mathbf{p}}$ of φ at the point $\mathbf{p}$. If this function is itself differentiable on Ω then its derivative represents the *second derivative* of $\varphi\colon \Omega \to \mathbb{R}^m$. The second derivative of φ exists and is continuous if and only if all second-order partial derivatives of the components of φ exist and are continuous throughout Ω. One can repeat the differentiation process to seek to construct derivatives of φ of all orders. A theorem of real analysis in several real variables ensures that the mth order derivative of $\varphi\colon \Omega \to \mathbb{R}^m$ exists and is continuous throughout Ω

if and only if all mth order partial derivatives of the components of φ exist and are continuous throughout Ω.

Definition Let Ω be an open set in $\mathbb{R}^n$. A vector-valued function $\varphi\colon \Omega \to \mathbb{R}^m$ mapping Ω into a Euclidean space $\mathbb{R}$ of dimension m is said to be *smooth* on the open set Ω if and only if the derivatives of the function φ of all orders exist throughout Ω.

Note that if the partial derivatives of a real-valued function of order $k+1$ are to exist for some positive integer k, the partial derivatives of order k must be continuous. The theorems of real analysis of several real variables described above therefore guarantee that a vector-valued function $\varphi\colon \Omega \to \mathbb{R}^m$ defined over an open set Ω in $\mathbb{R}^n$ is smooth on Ω if and only if the partial derivatives of the components of φ of all orders exist throughout the open set Ω.

Theorems of real analysis in several real variables guarantee that sums, differences, products, quotients and compositions of smooth vector-valued functions are smooth on open sets over which they are well-defined.

6.2 The Chain Rule for Differentiable Functions of Several Real Variables

Let Ω be an open set in $\mathbb{R}^n$ for some positive integer n, let $\varphi\colon \Omega \to \mathbb{R}^m$ be a differentiable vector-valued function mapping Ω into a Euclidean space $\mathbb{R}^m$ of dimension m, and let $f\colon \varphi(\Omega) \to \mathbb{R}$ be a differentiable real-valued function on $\varphi(\Omega)$. Let $x_1, x_2, \ldots x_n$ denote Cartesian coordinates on Ω, and let $u_1, u_2, \ldots, u_m$ denote Cartesian coordinates on $\varphi(\Omega)$. The *Chain Rule* for calculating partial derivatives of compositions of differentiable functions of several variables then ensures that the composition function $f \circ \varphi\colon \Omega \to \mathbb{R}$ is differentiable, and if $y = f(\varphi(x_1, x_2, \ldots, x_n))$ for all $x_1, x_2, \ldots, x_n$ then

$$\left.\frac{\partial y}{\partial x_i}\right|_{\mathbf{p}} = \sum_{j=1}^{m} \left.\frac{\partial y}{\partial u_j}\right|_{\varphi(\mathbf{p})} \left.\frac{\partial u_j}{\partial x_i}\right|_{\mathbf{p}}$$

for $i = 1, 2, \ldots, n$. Thus in order to determine the partial derivatives of y with respect to $x_1, x_2, \ldots, x_n$ at a point $\mathbf{p}$ of Ω, one applies the above identity using partial derivatives of y with respect to $u_1, u_2, \ldots, u_m$ evaluated at the point $\varphi(\mathbf{p})$ and partial derivatives of $u_1, u_2, \ldots, u_m$ evaluated at the point $\mathbf{p}$. Suppressing the specifications of the points at which the partial derivatives are to be evaluated yields the following more succinct equation:

$$\frac{\partial y}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial y}{\partial u_j} \frac{\partial u_j}{\partial x_i}.$$

Remark The Chain Rule will not be applicable in situations where the partial derivatives of the relevant functions exist throughout their domains but the functions themselves are not differentiable. The mere existence of partial derivatives throughout the domain of a function is not sufficient to ensure differentiability. But if those partial derivatives are continuous throughout the relevant domains then the functions themselves are guaranteed to be differentiable and therefore the Chain Rule for calculating the partial derivatives of a composition of such functions will be applicable.

6.3 Smooth Curvilinear Coordinate Systems on Three-Dimensional Space

Definition Let Ω be an open set in $\mathbb{R}^3$, let (U, V, W) be an ordered triple of smooth real-valued functions on Ω, and let $\varphi\colon \Omega \to \mathbb{R}^3$ be the smooth map defined such that

$$\varphi(\mathbf{r}) = \Bigl(U(\mathbf{r}),\, V(\mathbf{r}),\, W(\mathbf{r})\Bigr)$$

for all $\mathbf{r} \in \Omega$. Then the ordered triple (U, V, W) of smooth functions on Ω is said to constitute a *smooth curvilinear coordinate system* with *domain* Ω if $\varphi(\Omega)$ is an open set in $\mathbb{R}^3$ on which are defined smooth real-valued functions ξ, η and ζ that express the Cartesian coordinates (x, y, z) of each point $\mathbf{r}$ of Ω in terms of those of the corresponding point $\varphi(\mathbf{r})$ of $\varphi(\Omega)$ in accordance with the following equations:

$$\begin{aligned} x(\mathbf{r}) &= \xi(\varphi(\mathbf{r})) = \xi(U(\mathbf{r}), V(\mathbf{r}), W(\mathbf{r})), \\ y(\mathbf{r}) &= \eta(\varphi(\mathbf{r})) = \eta(U(\mathbf{r}), V(\mathbf{r}), W(\mathbf{r})), \\ z(\mathbf{r}) &= \zeta(\varphi(\mathbf{r})) = \zeta(U(\mathbf{r}), V(\mathbf{r}), W(\mathbf{r})). \end{aligned}$$

Example Let

$$\Omega = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 : x \leq 0 \text{ and } y = 0\},$$

and let smooth real-valued functions r, θ, φ be defined on Ω such that

$$r(x, y, z) > 0, \quad 0 < \theta(x, y, z) < \pi, \quad -\pi < \varphi(x, y, z) < \pi,$$

and

$$\begin{aligned} r(x, y, z) &= \sqrt{x^2 + y^2 + z^2}, \\ \theta(x, y, z) &= \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \end{aligned}$$

$$\varphi(x,y,z) = \begin{cases} \arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y \geq 0;, \\ -\arccos\left(\dfrac{x}{\sqrt{x^2+y^2}}\right) & \text{if } y < 0., \end{cases}$$

(Here $\arccos\colon [-1,1] \to [0,\pi]$ denotes the inverse of the restriction of the cosine function to the interval $[0,\pi]$.) Then r, θ and φ are smooth functions on the open set Ω. Moreover

$$\begin{aligned} x &= r\sin\theta\,\cos\varphi, \\ y &= r\sin\theta\,\sin\varphi, \\ z &= r\cos\theta \end{aligned}$$

where $r > 0$, $0 \leq \theta < \pi$ and $-\pi < \varphi < \pi$, and thus the Cartesian coordinates x, y, z are expressible as smooth functions of the values of r, θ and φ. It follows that (r, θ, φ) is a smooth curvilinear coordinate system with domain Ω. This is the *spherical polar coordinate system* over the open set Ω.

Lemma 6.1 *Let Ω be an open set in $\mathbb{R}^3$, and let (U, V, W) be a smooth curvilinear coordinate system with domain Ω. Let*

$$\varphi(\mathbf{r}) = \Big(U(\mathbf{r}), V(\mathbf{r}), W(\mathbf{r})\Big)$$

for all $\mathbf{r} \in \Omega$, and let ξ, η and ζ denote the smooth functions defined on $\varphi(\Omega)$ that satisfy the equations

$$\begin{aligned} x &= \xi(\varphi(x,y,z)), \\ y &= \eta(\varphi(x,y,z)), \\ z &= \zeta(\varphi(x,y,z)) \end{aligned}$$

for all $(x, y, z) \in \Omega$. Then

$$\begin{aligned} u &= U(\sigma(u,v,w)), \\ v &= V(\sigma(u,v,w)), \\ w &= W(\sigma(u,v,w)). \end{aligned}$$

for all $(u, v, w) \in \varphi(\Omega)$, where

$$\sigma(u,v,w) = (\xi(u,v,w),\ \eta(u,v,w),\ \zeta(u,v,w)).$$

Proof Let $(u,v,w) \in \varphi(\Omega)$. Then there exists $(x,y,z) \in \Omega$ such that

$$u = U(x,y,z), \quad v = V(x,y,z), \quad w = (x,y,z).$$

But then

$$x = \xi(u,v,w), \quad y = \eta(u,v,w), \quad z = \zeta(u,v,w),$$

and therefore

$$\begin{aligned} u &= U(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)) = U(\sigma(u,v,w)), \\ v &= V(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)) = V(\sigma(u,v,w)), \\ w &= W(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)) = W(\sigma(u,v,w)), \end{aligned}$$

as required. ▮

Lemma 6.2 *Let Ω be an open set in $\mathbb{R}^3$, and let (U,V,W) be a smooth curvilinear coordinate system with domain Ω. Let*

$$\varphi(\mathbf{r}) = \Big(U(\mathbf{r}), V(\mathbf{r}), W(\mathbf{r})\Big)$$

for all $\mathbf{r} \in \Omega$, and let ξ, η and ζ denote the smooth functions defined on $\varphi(\Omega)$ that satisfy the equations

$$\begin{aligned} x &= \xi(\varphi(x,y,z)), \\ y &= \eta(\varphi(x,y,z)), \\ z &= \zeta(\varphi(x,y,z)) \end{aligned}$$

for all $(x,y,z) \in \Omega$. Let $\partial_j\xi$, $\partial_j\eta$ and $\partial_j\zeta$ denote the partial derivatives of the functions ξ, η and ζ respectively with respect to the jth Cartesian coordinate on Ω for $j = 1,2,3$, and let

$$\frac{\partial\xi}{\partial U} = (\partial_1\xi)(U,V,W), \quad \frac{\partial\xi}{\partial V} = (\partial_2\xi)(U,V,W), \quad \frac{\partial\xi}{\partial W} = (\partial_3\xi)(U,V,W),$$

$$\frac{\partial\eta}{\partial U} = (\partial_1\eta)(U,V,W), \quad \frac{\partial\eta}{\partial V} = (\partial_2\eta)(U,V,W), \quad \frac{\partial\eta}{\partial W} = (\partial_3\eta)(U,V,W),$$

$$\frac{\partial\zeta}{\partial U} = (\partial_1\zeta)(U,V,W), \quad \frac{\partial\zeta}{\partial V} = (\partial_2\zeta)(U,V,W), \quad \frac{\partial\zeta}{\partial W} = (\partial_3\zeta)(U,V,W).$$

Then

$$\begin{pmatrix} \dfrac{\partial\xi}{\partial U} & \dfrac{\partial\xi}{\partial V} & \dfrac{\partial\xi}{\partial W} \\ \dfrac{\partial\eta}{\partial U} & \dfrac{\partial\eta}{\partial V} & \dfrac{\partial\eta}{\partial W} \\ \dfrac{\partial\zeta}{\partial U} & \dfrac{\partial\zeta}{\partial V} & \dfrac{\partial\zeta}{\partial V} \end{pmatrix} \begin{pmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof This follows on differentiating the equations

$$\begin{aligned} u &= U(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)), \\ v &= V(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)), \\ w &= W(\xi(u,v,w),\, \eta(u,v,w),\, \zeta(u,v,w)), \end{aligned}$$

applying the Chain Rule for calculating Partial Derivatives of compositions of continuously-differentiable functions of several real variables. ∎

6.4 Smooth Surfaces in Three-Dimensional Space

Definition Let Σ be a subset of three-dimensional Euclidean space $\mathbb{R}^3$. Then Σ is a *smooth surface* if, given any point $\mathbf{p}$ of Σ, there exists a smooth curvilinear coordinate system (U, V, W) with domain Ω, where Ω is an open set in $\mathbb{R}^3$ containing the point $\mathbf{p}$, such that

$$\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.$$

Example Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3\}.$$

Then S^2 is a smooth surface in $\mathbb{R}^3$, in accordance with the above definition. To verify this, let (r, θ, φ) be spherical polar coordinates on Ω_0, where

$$\Omega_0 = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0 \text{ or } x > 0\},$$

$$r > 0, \quad 0 < \theta < \pi, \quad -\pi < \varphi < \pi,$$

and

$$x = r \sin\theta\, \cos\varphi, \quad y = r \sin\theta\, \sin\varphi, \quad z = r\cos\theta$$

for all $(x, y, z) \in \Omega_0$. Then, given any point $\mathbf{p}$ of S^2, there exists a rotation T of $\mathbb{R}^3$ about the origin such that $T(\mathbf{p}) = (1, 0, 0)$. Let

$$\Omega = \{\mathbf{r} \in \mathbb{R}^3 : |\mathbf{r} - \mathbf{p}| < \sqrt{2}\},$$

and let

$$U(\mathbf{r}) = \theta(T(\mathbf{r})), \quad V(\mathbf{r}) = \varphi(T(\mathbf{r})), \quad w(\mathbf{r}) = r(T(\mathbf{r})) - 1$$

for all $\mathbf{r} \in \Omega$. Then (U, V, W) is a smooth curvilinear coordinate system with domain Ω, and

$$S^2 \cap \Omega = \{\mathbf{r} \in \Omega : W(\mathbf{r}) = 0\}.$$

6.5 Smooth Local Coordinate Systems on Smooth Surfaces

Definition Let Σ be a smooth surface in $\mathbb{R}^3$, let Ω be an open set in $\mathbb{R}^3$, and let (U, V, W) be a smooth curvilinear coordinate system with domain Ω. We say that this smooth curvilinear coordinate system is *adapted* to the surface Σ if

$$\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.$$

The definition of smooth surfaces ensures that, given any point of a smooth surface, that point is contained in the domain of some smooth curvilinear coordinate system adapted to the surface.

Let Σ be a smooth surface in $\mathbb{R}^3$, and let (U, V, W) be a smooth curvilinear coordinate system with domain Ω that is adapted to the surface Σ. Suppose that $\Sigma \cap \Omega \neq \emptyset$. Let $u: \Sigma \cap \Omega \to \mathbb{R}$ and $v: \Sigma \cap \Omega \to \mathbb{R}$ denote the restrictions of the functions U and V to the $\Sigma \cap \Omega$. Then the functions u and v parameterize the portion of the surface Σ represented by $\Sigma \cap \Omega$. The ordered pair (u, v) of real-valued functions on $\Sigma \cap \Omega$ is referred to as a *smooth local coordinate system* for the surface Σ whose *domain* $\Sigma \cap \Omega$ is a subset of Σ open in Σ.

Let $(\hat{U}, \hat{V}, \hat{W})$ be another smooth curvilinear coordinate system with domain $\hat{\Omega}$ which is also adapted to the surface Σ, where $\Sigma \cap \Omega \cap \hat{\Omega} \neq \emptyset$, and let $\hat{u}: \Sigma \cap \hat{\Omega} \to \mathbb{R}$ and $\hat{v}: \Sigma \cap \hat{\Omega} \to \mathbb{R}$ denote the restrictions of $\hat{U}$ and $\hat{V}$ respectively to $\Sigma \cap \hat{\Omega}$. Then $\hat{U}, \hat{V}, \hat{W}$ are expressible as smooth functions of U, V, W and vice versa over the open set $\Omega \cap \Omega'$. It follows that $\hat{u}, \hat{v}$ are expressible as smooth functions of u, v, and also u, v are expressible as smooth functions of $\hat{u}, \hat{v}$, over the intersection $\Sigma \cap \Omega \cap \hat{\Omega}$ of the relevant domains.

Definition A *smooth atlas* for a smooth surface Σ in three-dimensional Euclidean space is a collection of smooth local coordinate systems on Σ whose domains cover Σ. The smooth local coordinate systems belonging to a given atlas are called *charts*

Remark Let us regard the surface of the Earth as a smooth surface. This surface is of course curved. An atlas, such as one finds on bookshelves at home, or at the local library, is a collection of charts. Each chart represents some portion of the Earth's surface, determining a local coordinate system represented by horizontal and vertical displacements on the relevant printed page of the atlas. The charts within such an atlas should cover all areas of the Earth's surface, including the polar regions and the oceans.

6.6 Smooth Functions on Smooth Surfaces

Let Σ be a smooth surface, let (U, V, W) be a smooth curvilinear coordinate system with domain Ω adapted to the surface Σ, and let $(\hat{U}, \hat{V}, \hat{W})$ be another smooth curvilinear coordinate system with domain $\hat{\Omega}$ adapted to the surface Σ, where $\Sigma \cap \Omega \cap \hat{\Omega} \neq \emptyset$.

Let $f\colon \Sigma \to \mathbb{R}$ be a continuous real-valued function defined throughout the surface Σ. The function f is a differentiable function of smooth local coordinates u and v if and only if there exists a differentiable function F of two real variables, defined at $(u(\mathbf{p}), v(\mathbf{p}))$ for all $\mathbf{p} \in \Sigma \cap \Omega$, such that

$$f(\mathbf{p}) = F(u(\mathbf{p}), v(\mathbf{p}))$$

for all $\mathbf{p} \in \Sigma \cap \Omega$. Let $\partial_1 F$ and $\partial_2 F$ denote the partial derivatives of the function F with respect to its first and second arguments respectively. The partial derivatives of f with respect to the functions u and v constituting the smooth local coordinate system are then defined so that

$$\begin{aligned} \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} &= (\partial_1 F)(u(\mathbf{p}), v(\mathbf{p})), \\ \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} &= (\partial_2 F)(u(\mathbf{p}), v(\mathbf{p})) \end{aligned}$$

for all $\mathbf{p} \in \Sigma \cap \Omega$. We thus obtain real-valued functions $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ on $\Sigma \cap \Omega$ whose values at any point $\mathbf{p}$ of $\Sigma \cap \Omega$ are $\left.\dfrac{\partial f}{\partial u}\right|_{\mathbf{p}}$ and $\left.\dfrac{\partial f}{\partial v}\right|_{\mathbf{p}}$ respectively.

The smooth dependence of the local coordinates u, v on local coordinates $\hat{u}, \hat{v}$, and of $\hat{u}, \hat{v}$ on u, v, ensures that f is a differentiable function of u and v on $\Sigma \cap \Omega \cap \hat{\Omega}$ if and only if if and only if it is a differentiable function of $\hat{u}$ and $\hat{v}$, in which case it follows from the Chain Rule that

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial \hat{u}} \frac{\partial \hat{u}}{\partial u} + \frac{\partial f}{\partial \hat{v}} \frac{\partial \hat{v}}{\partial u}, \\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial \hat{u}} \frac{\partial \hat{u}}{\partial v} + \frac{\partial f}{\partial \hat{v}} \frac{\partial \hat{v}}{\partial v}, \end{aligned}$$

and

$$\begin{aligned}\frac{\partial f}{\partial \hat{u}} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial \hat{u}} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \hat{u}},\\ \frac{\partial f}{\partial \hat{v}} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial \hat{v}} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial \hat{v}}.\end{aligned}$$

Definition A continuous function $f\colon \Sigma \to \mathbb{R}$ on a smooth surface Σ is said to be k-times continuously differentiable (or C^k) if, given any smooth local coordinate system (u, v) defined over an open region of the surface, the restriction of the function f to that open region is expressible as a k-times continuously differentiable function of the smooth local coordinates u and v.

Definition A continuous function $f\colon \Sigma \to \mathbb{R}$ on a smooth surface Σ is said to be smooth if, given any smooth local coordinate system (u, v) defined over an open region of the surface, the restriction of the function f to that open region is expressible as a smooth function of the smooth local coordinates u and v.

In order to verify that a function is smooth around a point $\mathbf{p}$ of a smooth surface Σ, it suffices to verify that f is expressible as a smooth function of the coordinate functions u and v of at least one smooth local coordinate system (u, v) defined over an open region that contains the point $\mathbf{p}$. Indeed the results described above ensure that if f is expressible as a smooth function of the coordinate functions of at least one such smooth local coordinate system (u, v) around the point $\mathbf{p}$, then it is expressible as a smooth function of the coordinate functions of any other such smooth local coordinate system around the point $\mathbf{p}$.

Thus a continuous real-valued function $f\colon \Sigma \to \mathbb{R}$ on a surface Σ is smooth throughout Σ if and only if, for every chart in some smooth atlas for the surface, the function can be expressed as a smooth function of local coordinates determined by the chart throughout the domain of the chart.

6.7 Derivatives of Functions along Curves in Surfaces

Proposition 6.3 *Let Σ be a smooth surface in $\mathbb{R}^3$, let $f\colon \Sigma \to \mathbb{R}$ be a differentiable function on Σ, let $\gamma\colon I \to \Sigma$ be a smooth curve in the surface Σ parameterized by an open interval I, let $t_0 \in I$, and let $\mathbf{p} = \gamma(t_0)$ and $\mathbf{b} = \gamma'(t_0)$, where $\gamma'(t_0)$ denotes the velocity vector to the curve $t \mapsto \gamma(t)$ at $t = t_0$, and let $\mathbf{b} = (b_x, b_y, b_z)$. Let (U, V, W) be a smooth curvilinear coordinate system with domain Ω, where $\mathbf{p} \in \Omega$ and*

$$\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\},$$

let $\Omega_{(u,v)} = \Sigma \cap \Omega$, *and let* u *and* v *be the smooth real-valued functions on* $\Omega_{(u,v)}$ *that are the restrictions to the surface of the smooth functions* U *and* V. *Then*

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=t_0} = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} \left.\frac{du(\gamma(t))}{dt}\right|_{t=t_0} + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} \left.\frac{dv(\gamma(t))}{dt}\right|_{t=t_0},$$

where

$$\begin{aligned}
\left.\frac{du(\gamma(t))}{dt}\right|_{t=t_0} &= b_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}, \\
\left.\frac{dv(\gamma(t))}{dt}\right|_{t=t_0} &= b_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}.
\end{aligned}$$

Proof The differentiability of the function f ensures the existence of a differentiable function F of two real variables, defined at $(u(\mathbf{r}), v(\mathbf{r}))$ for all $\mathbf{r} \in \Omega_{(u,v)}$. Let $\partial_1 F$ and $\partial_2 F$ denote the partial derivatives of F with respect to its first and second arguments respectively. Then the definition of the partial derivative of f with respect to the local coordinates (u, v) ensures that

$$\left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} = (\partial_1 F)(u(\mathbf{p}), v(\mathbf{p})), \qquad \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} = (\partial_2 F)(u(\mathbf{p}), v(\mathbf{p})).$$

The Chain Rule for differentiating compositions of differentiable functions of several real variables ensures that

$$\begin{aligned}
\left.\frac{df(\gamma(t))}{dt}\right|_{t=t_0} &= (\partial_1 F)(u(\mathbf{p}), v(\mathbf{p}))(u \circ \gamma)'(t_0) \\
&\quad + (\partial_2 F)(u(\mathbf{p}), v(\mathbf{p}))(v \circ \gamma)'(t_0) \\
&= \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} \left.\frac{du(\gamma(t))}{dt}\right|_{t=t_0} + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} \left.\frac{dv(\gamma(t))}{dt}\right|_{t=t_0}.
\end{aligned}$$

The functions u and v are defined only over an open region on the surface, but they are the restrictions to that surface of the smooth functions U and V that are the first two components of a smooth curvilinear coordinate system (U, V, W) adapted to the surface Σ whose domain Ω is an open set in $\mathbb{R}^3$. It follows that

$$\begin{aligned}
\left.\frac{du(\gamma(t))}{dt}\right|_{t=t_0} &= \left.\frac{dU(\gamma(t))}{dt}\right|_{t=t_0} \\
&= \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} \left.\frac{dx(\gamma(t))}{dt}\right|_{t=t_0} + \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} \left.\frac{dy(\gamma(t))}{dt}\right|_{t=t_0}
\end{aligned}$$

$$
\begin{aligned}
&+ \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}} \left.\frac{dz(\gamma(t))}{dt}\right|_{t=t_0} \\
&= b_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}.
\end{aligned}
$$

Similarly

$$
\left.\frac{dv(\gamma(t))}{dt}\right|_{t=t_0} = b_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}.
$$

The result follows. ∎

6.8 Tangent Spaces to Smooth Surfaces

Definition Let $\gamma\colon I \to \mathbb{R}^3$ be a smooth curve parameterized by an open interval I in $\mathbb{R}$, (so that $\gamma(t)$ is defined for all $t \in I$ and is a smooth function of t). The *velocity vector* $\gamma'(t_0)$ at $t = t_0$ is defined for any $t_0 \in I$ such that

$$
\gamma'(t_0) = \left.\frac{d(\gamma(t))}{dt}\right|_{t=t_0}.
$$

Proposition 6.4 *Let Σ be a smooth surface in three-dimensional Euclidean space $\mathbb{R}^3$, let $\mathbf{p}$ be a point of Σ, and let (U, V, W) be a smooth curvilinear coordinate system with domain Ω, where $\mathbf{p} \in \Omega$ and*

$$
\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.
$$

Let $\mathbf{b} \in \mathbb{R}^3$ be a vector in $\mathbb{R}^3$ with components (b_x, b_y, b_z). Then the vector $\mathbf{b}$ is the velocity vector at $t = 0$ of some smooth curve $\gamma\colon I \to \Sigma$ in the surface Σ, where $0 \in I$ and $\gamma(0) = \mathbf{p}$, if and only if

$$
b_x \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}} = 0.
$$

Proof Suppose that $\mathbf{b} = \gamma'(0)$ for some smooth $\gamma\colon I \to \Sigma$ in the surface Σ for which $0 \in I$ and $\gamma(0) = \mathbf{p}$. Then

$$
\begin{aligned}
\left.\frac{\partial(x(\gamma(t)))}{dt}\right|_{t=0} &= b_x, \\
\left.\frac{\partial(y(\gamma(t)))}{dt}\right|_{t=0} &= b_y, \\
\left.\frac{\partial(z(\gamma(t)))}{dt}\right|_{t=0} &= b_z.
\end{aligned}
$$

Now $W(\gamma(t)) = 0$ for all $t \in I$. It follows from the Chain Rule for differentiating compositions of differentiable functions of several real variables that

$$b_x \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}} = 0.$$

Conversely suppose that the equation

$$b_x \frac{\partial W}{\partial x} + b_y \frac{\partial W}{\partial y} + b_z \frac{\partial W}{\partial z} = 0$$

is satisfied at the point $\mathbf{p}$. Let $J_{\mathbf{p}}$ be the invertible 3×3 matrix with real coefficients that is the value of the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{pmatrix}$$

at the point $\mathbf{p}$. Then

$$J_p \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} h \\ k \\ 0 \end{pmatrix},$$

where

$$h = b_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}$$

and

$$k = b_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}.$$

The Cartesian coordinate functions x, y, z are expressible as smooth functions of U, V, W in a neighbourhood of the point $\mathbf{p}$. It follows that there exists a strictly positive real number δ_0 and a smooth curve $\gamma\colon (-\delta_0, \delta_0) \to \mathbb{R}^3$ characterized by the requirements that $\gamma(t) \in \Omega$,

$$U(\gamma(t)) = ht, \quad V(\gamma(t)) = kt \quad \text{and} \quad W(\gamma(t)) = 0$$

for all real numbers t satisfying $|t| < \delta_0$. Let $\gamma'(t) = (c_x, c_y, c_z)$. Then

$$h = \left.\frac{\partial(U(\gamma(t)))}{dt}\right|_{t=0}$$

$$
\begin{aligned}
&= c_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + c_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + c_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}, \\
k &= \left.\frac{\partial (V(\gamma(t)))}{dt}\right|_{t=0} \\
&= c_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + c_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + c_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}, \\
0 &= \left.\frac{\partial (W(\gamma(t)))}{dt}\right|_{t=0} \\
&= c_x \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} + c_y \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} + c_z \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}}.
\end{aligned}
$$

It follows that

$$
J_p \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} h \\ k \\ 0 \end{pmatrix} = J_p \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}.
$$

Now the Jacobian matrix J_p is invertible, because (U, V, W) is a smooth curvilinear coordinate system around the point $\mathbf{p}$. It follows that

$$
\begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix},
$$

and therefore $\gamma'(0) = \mathbf{b}$. The result follows. ∎

Definition Let Σ be a smooth surface in $\mathbb{R}^3$, let $\mathbf{p}$ be a point of Σ, and let $\mathbf{b}$ be a vector in $\mathbb{R}^3$. The vector $\mathbf{b}$ is said to be a *tangent vector* to the surface at the point $\mathbf{b}$ if and only if

$$
\mathbf{b} = \left.\frac{d(\gamma(t))}{dt}\right|_{t=0},
$$

where $\gamma\colon I \to \Sigma$ is a smooth curve wholly contained in the surface Σ and parameterized by an open interval I that contains zero.

Let Σ be a smooth surface in $\mathbb{R}^3$, and let $\mathbf{p}$ be a point of Σ. It follows from Proposition 6.4 that the tangent vectors to the surface at the point $\mathbf{p}$ constitute a vector subspace $T_{\mathbf{p}}\Sigma$ of $\mathbf{R}^3$.

Definition Let Σ be a smooth surface in $\mathbb{R}^3$. The *tangent space* $T_{\mathbf{p}}\Sigma$ to the surface Σ at the point $\mathbf{p}$ is the two-dimensional vector subspace of $\mathbb{R}^3$ consisting of those vectors $\mathbf{b}$ in $\mathbb{R}^3$ that are expressible in the form

$$
\mathbf{b} = \left.\frac{d(\gamma(t))}{dt}\right|_{t=0},
$$

where $\gamma\colon I \to \Sigma$ is some smooth curve in the surface Σ for which $0 \in I$ and $\gamma(0) = \mathbf{p}$.

Lemma 6.5 *Let Σ be a smooth surface in $\mathbb{R}^3$, let $\mathbf{p}$ be a point of Σ, and let $T_{\mathbf{p}}\Sigma$ be the tangent space to the surface Σ at the point $\mathbf{p}$. Let (U, V, W) be a smooth curvilinear coordinate system with domain Ω which is adapted to the surface Σ, so that Ω is an open set in $\mathbb{R}^3$ and*

$$\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.$$

Then

$$T_{\mathbf{p}}\Sigma = \left\{ (b_x, b_y, b_z) \in \mathbb{R}^3 : b_x \left.\frac{\partial W}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial W}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial W}{\partial z}\right|_{\mathbf{p}} = 0 \right\}.$$

Proof The result follows immediately from Proposition 6.4 and the definition of the tangent spaces to a smooth surface. ∎

The tangent plane to a smooth surface Σ at a point $\mathbf{p}$ is the flat plane that best approximates to the surface at the point Σ. It is said to *touch* the surface at the point $\mathbf{p}$.

6.9 Differentials of Smooth Functions on Surfaces

The language of differential forms was initially developed by the differential geometer Élie Cartan from 1899 onwards. One significant benefit gained from the development of the theory of differential forms and the exterior derivative is that this theory provided a language and a conceptual framework in which to reinterpret, within the framework of (standard) analysis and modern differential geometry, much of the work of earlier mathematicians who had published their work in a form that made frequent use of "infinitely small" or "infinitesimally small' quantities.

For example, an ellipsoid with principle axes of lengths $2a$, $2b$ and $2c$ may be represented, with suitable choice of Cartesian coordinates, by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Differentiating this equation in the manner that was customary throughout the eighteenth and nineteenth centuries leads to equation

$$\frac{2x}{a^2}\,dx + \frac{2y}{b^2}\,dy + \frac{2z}{c^2}dz = 0.$$

This would then be an relation between the "infinitesimal" quantities dx, dy and dz.

We now discuss how to interpret such identities without appealing to notions of "infinitesimal" quantities, but instead associating to each differentiable real-valued function on a surface a corresponding linear functional each tangent space to the surface that represents the derivative of the function.

Let Σ be a smooth surface in $\mathbb{R}^3$, let $\mathbf{p}$ be a point on the surface, and let $T_{\mathbf{p}}\Sigma$ be the tangent space to the surface Σ at $\mathbf{p}$. Then there exists a smooth curvilinear coordinate system (U, V, W) with domain Ω for which $\mathbf{p} \in \Omega$ and

$$\Sigma \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\},$$

The curvilinear coordinate functions then restrict to functions u and v that together constitute a smooth local coordinate system (u, v) defined over the open region $\Sigma_{(u,v)}$ of the surface, where $\Sigma_{(u,v)} = \Sigma \cap \Omega$.

Let $\mathbf{b}$ be a tangent vector to the smooth surface Σ at the point $\mathbf{p}$, and let $\mathbf{b} = (b_x, b_y, b_z)$. Then there exists a smooth curve $\gamma\colon I \to \Sigma$ in the surface Σ parameterized by an open interval I for which $0 \in I$, $\mathbf{p} = \gamma(0)$ and $\mathbf{b} = \gamma'(0)$. It then follows from Proposition 6.3 that

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} \left.\frac{du(\gamma(t))}{dt}\right|_{t=0} + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} \left.\frac{dv(\gamma(t))}{dt}\right|_{t=0},$$

for all differentiable real-valued functions f defined around the point $\mathbf{p}$, where

$$\begin{aligned}
\left.\frac{du(\gamma(t))}{dt}\right|_{t=0} &= b_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}, \\
\left.\frac{dv(\gamma(t))}{dt}\right|_{t=0} &= b_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}.
\end{aligned}$$

It follows from this that the value of $\left.\dfrac{df(\gamma(t))}{dt}\right|_{t=0}$ is completely determined by the velocity vector $\gamma'(0)$ of the curve γ when that curve passes through the point $\mathbf{p}$ and by the partial derivatives of the function f with respect to the smooth local coordinates u and v. Thus if $\tilde{\gamma}\colon I \to \Sigma$ is a smooth curve in the surface Σ parameterized by an open interval I, and if $\mathbf{p} = \tilde{\gamma}(0)$ and $\mathbf{b} = \tilde{\gamma}'(0)$, then

$$\left.\frac{d(f(\tilde{\gamma}(t)))}{dt}\right|_{t=0} = \left.\frac{df(\gamma(t))}{dt}\right|_{t=0}.$$

Thus the value of $\left.\dfrac{df(\gamma(t))}{dt}\right|_{t=0}$ is the same for all smooth curves $t \mapsto \gamma(t)$ in the surface that pass through the point $\mathbf{p}$ when $t = 0$ with velocity vector $\mathbf{b}$.

Therefore, given any differentiable real-valued function f defined around the point $\mathbf{p}$, and given any tangent vector $\mathbf{b}$ at the point $\mathbf{p}$, there exists a well-defined real number $(df)_{\mathbf{p}}(\mathbf{b})$ with the property that

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=0} = (df)_{\mathbf{p}}(\mathbf{b})$$

for all smooth curves $t \mapsto \gamma(t)$ in the surface that pass through the point $\mathbf{p}$ when $t = 0$ with velocity vector $\mathbf{b}$. Moreover the formulae quoted above from the statement of Proposition 6.3 ensure that

$$(df)_{\mathbf{p}}(\mathbf{b}) = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} (du)_{\mathbf{p}}(\mathbf{b}) + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} (dv)_{\mathbf{p}}(\mathbf{b}),$$

where

$$\begin{aligned} (du)_{\mathbf{p}}(\mathbf{b}) &= b_x \left.\frac{\partial U}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial U}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial U}{\partial z}\right|_{\mathbf{p}}, \\ (dv)_{\mathbf{p}}(\mathbf{b}) &= b_x \left.\frac{\partial V}{\partial x}\right|_{\mathbf{p}} + b_y \left.\frac{\partial V}{\partial y}\right|_{\mathbf{p}} + b_z \left.\frac{\partial V}{\partial z}\right|_{\mathbf{p}}. \end{aligned}$$

It follows that $(df)_{\mathbf{p}}: T_{\mathbf{p}}\Sigma \to \mathbb{R}$ is a linear transformation from the tangent space $T_{\mathbf{p}}\Sigma$ to the field $\mathbb{R}$ of real numbers.

Definition Let Σ be a smooth surface in $\mathbb{R}^3$, and let $f: \Sigma \to \mathbb{R}$ be a smooth real-valued function defined throughout Σ. The *differential* df of the function f is the correspondence that associates to each point $\mathbf{p}$ of Σ the linear functional $(df)_{\mathbf{p}}: T_{\mathbf{p}}\Sigma \to \mathbb{R}$ on the tangent space $T_{\mathbf{p}}\Sigma$ to the surface at the point $\mathbf{p}$ characterized by the property that

$$(df)_{\mathbf{p}}(\gamma'(t_0)) = \left.\frac{df(\gamma(t))}{dt}\right|_{t=t_0}$$

for all smooth curves $\gamma: I \to \Sigma$ in the surface Σ for which $t_0 \in I$ and $\gamma(t_0) = \mathbf{p}$.

Lemma 6.6 *Let Σ be a smooth surface in $\mathbb{R}^3$, and let (u, v) be smooth local coordinates defined over an open region $\Sigma_{(u,v)}$ of the surface. Then*

$$df = \frac{\partial f}{\partial u}\, du + \frac{\partial f}{\partial v}\, dv$$

throughout $\Sigma_{(u,v)}$.

Proof Let $\mathbf{p}$ be a point belonging to the domain $\Sigma_{(u,v)}$ of the smooth local coordinate system (u, v), and let $\mathbf{b}$ be an element of the tangent space $T_{\mathbf{p}}\Sigma$ to the surface Σ at the point $\mathbf{p}$. As previously noted, it follows from Proposition 6.3 that

$$(df)_{\mathbf{p}}(\mathbf{b}) = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} (du)_{\mathbf{p}}(\mathbf{b}) + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} (dv)_{\mathbf{p}}(\mathbf{b}).$$

The result follows. ∎

The Cartesian coordinate functions x, y and z on $\mathbb{R}^3$ restrict to smooth functions on the surface Σ whose differentials at the point $\mathbf{p}$ are determined in the following lemma.

Lemma 6.7 *Let Σ be a smooth surface in $\mathbb{R}^3$, let $\mathbf{p}$ be a point of Σ, let $T_{\mathbf{p}}\Sigma$ be the tangent space to the surface Σ at the point $\mathbf{p}$, and let $\mathbf{b} \in T_{\mathbf{p}}\Sigma$, where $\mathbf{b} = (b_x, b_y, b_z)$. Then*

$$(dx)_{\mathbf{p}}(\mathbf{b}) = b_x, \quad (dy)_{\mathbf{p}}(\mathbf{b}) = b_y \quad \textit{and} \quad (dz)_{\mathbf{p}}(\mathbf{b}) = b_z.$$

Proof Let $\mathbf{b} = \gamma'(0)$, where $\gamma: I \to \Sigma$ is a smooth curve in Σ paramaterized by an open interval I that contains zero. It then follows from the differentials of the coordinate functions x, y and z on the surface Σ that

$$\begin{aligned} (dx)_{\mathbf{p}}(\mathbf{b}) &= \left.\frac{dx(\gamma(t))}{dt}\right|_{t=0} = b_x, \\ (dy)_{\mathbf{p}}(\mathbf{b}) &= \left.\frac{dy(\gamma(t))}{dt}\right|_{t=0} = b_y, \\ (dz)_{\mathbf{p}}(\mathbf{b}) &= \left.\frac{dz(\gamma(t))}{dt}\right|_{t=0} = b_z, \end{aligned}$$

as required. ∎

Lemma 6.8 *Let Σ be a smooth surface in $\mathbb{R}^3$, let $\mathbf{p}$ be a point of Σ, and let $T_{\mathbf{p}}\Sigma$ be the tangent space to the surface Σ at the point $\mathbf{p}$. Let $\tilde{f}: \Omega \to \mathbb{R}$ be a smooth function defined over an open set Ω in $\mathbb{R}^3$, where $\mathbf{p} \in \Omega$, and let $f: \Sigma \cap \Omega \to \mathbb{R}$ be the restriction of the smooth function $\tilde{f}$ to the surface Σ. Then the differentials of f and the restrictions of the coordinate functions x, y, z, to the surface satisfy the identity*

$$df = \frac{\partial \tilde{f}}{\partial x}\, dx + \frac{\partial \tilde{f}}{\partial y}\, dy + \frac{\partial \tilde{f}}{\partial z}\, dz$$

at all points of $\Sigma \cap \Omega$.

Proof Let $\mathbf{b}$ be an element of the tangent space $T_{\mathbf{p}}\Sigma$ to the surface Σ at the point $\mathbf{p}$. Then there exists a smooth curve $\gamma: I \to \Sigma$ such that $\mathbf{p} = \gamma(t_0)$ and $\mathbf{b} = \gamma'(t_0)$. Let $\mathbf{b} = (b_x, b_y, b_z)$. It then follows from Lemma 6.7 that

$$\begin{aligned}
(df)_{\gamma(t_0)}(\gamma'(t_0)) &= \frac{d}{dt}\Big(f(\gamma(t))\Big)\Big|_{t=t_0} = \frac{d}{dt}\Big(\tilde{f}(\gamma(t))\Big)\Big|_{t=t_0} \\
&= b_x \left.\frac{\partial \tilde{f}}{\partial x}\right|_{\gamma(t_0)} + b_y \left.\frac{\partial \tilde{f}}{\partial y}\right|_{\gamma(t_0)} + b_z \left.\frac{\partial \tilde{f}}{\partial z}\right|_{\gamma(t_0)} \\
&= \left.\frac{\partial \tilde{f}}{\partial x}\right|_{\gamma(t_0)} (dx)_{\mathbf{p}} + \left.\frac{\partial \tilde{f}}{\partial y}\right|_{\gamma(t_0)} (dy)_{\mathbf{p}} + \left.\frac{\partial \tilde{f}}{\partial z}\right|_{\gamma(t_0)} (dz)_{\mathbf{p}}.
\end{aligned}$$

The result follows. ∎

6.10 Directional Derivatives of Smooth Functions along Tangent Vectors

Let $f: \Sigma \to \mathbb{R}$ be a smooth function defined on a smooth surface Σ. We shall give the definition of the *directional derivative* of the function f along a tangent vector to the surface. The definition is justified by the following sequence of results.

Definition Let $f: \Sigma \to \mathbb{R}$ be a smooth function defined on a smooth surface Σ, let $\mathbf{p}$ be a point of the surface, and let $\mathbf{b}$ be a vector belonging to the tangent space $T_{mathbfp}\Sigma$ to the surface Σ at the point $\mathbf{p}$. The *directional derivative* $(\partial_{\mathbf{b}} f)_{\mathbf{p}}$ of f along the vector $\mathbf{b}$ at the point $\mathbf{p}$ is defined so that

$$(\partial_{\mathbf{b}} f)_{\mathbf{p}} = (df)_{\mathbf{p}}(\mathbf{b}).$$

It follows from the definition of the differential df of f on the surface Σ (and from the discussion preceding that definition) that, given any smooth curve $\gamma: I \to \Sigma$ paramaterized by an open interval I that satisfies $\gamma(t) = \mathbf{p}$ and $\gamma'(t_0) = \mathbf{b}$ for some $t_0 \in I$, the derivative of f along the curve γ satisfies

$$\left.\frac{df(\gamma(t))}{dt}\right|_{t=t_0} = (df)_{\mathbf{p}}(\mathbf{b}) = (\partial_{\mathbf{b}} f)_{\mathbf{p}}.$$

Proposition 6.9 *Let Σ be a smooth surface, let $\mathbf{p}$ be a point of Σ, let $\mathbf{b}$ be a tangent vector to the surface at the point $\mathbf{b}$, and let f be a smooth function on the surface defined around the point $\mathbf{p}$. Let (u, v) be a smooth local coordinate* *system for the surface Σ whose domain $\Sigma_{(u,v)}$ is an open neighbourhood of* $\mathbf{p}$ *in Σ. Then*

$$(\partial_{\mathbf{b}} f)_{\mathbf{p}} = b_u \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} + b_v \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}}.$$

where

$$b_u = (\partial_{\mathbf{b}} u)_{\mathbf{p}} \quad \textit{and} \quad b_v = (\partial_{\mathbf{b}} v)_{\mathbf{p}}.$$

Proof It follows from Lemma 6.6 that

$$\begin{aligned}(\partial_{\mathbf{b}} f)_{\mathbf{p}} &= (df)_{\mathbf{p}}(\mathbf{b}) = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} (du)_{\mathbf{p}}(\mathbf{b}) + \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} (dv)_{\mathbf{p}}(\mathbf{b}) \\ &= b_u \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} + b_v \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}},\end{aligned}$$

as required. ∎

Let $\mathbf{b}$ be a tangent vector to a smooth surface Σ at a point $\mathbf{p}$ of that surface. Then

$$\partial_{\mathbf{b}}(rf + sg)_{\mathbf{p}} = r(\partial_{\mathbf{b}} f)_{\mathbf{p}} + s(\partial_{\mathbf{b}} g)_{\mathbf{p}}.$$

for all smooth real-valued functions f and g on the surface defined around the point $\mathbf{p}$. The Product Rule for differentiation ensures that

$$\partial_{\mathbf{b}}(f.g)_{\mathbf{p}} = g(\mathbf{p})(\partial_{\mathbf{b}} f)_{\mathbf{p}} + f(\mathbf{p})(\partial_{\mathbf{b}} g)_{\mathbf{p}},$$

where $f.g$ denotes the product of the smooth real-valued functions f and g. Moreover if the functions f and g are equal throughout some open set in Σ that contains the point $\mathbf{p}$, then $(\partial_{\mathbf{b}} f)_{\mathbf{p}} = (\partial_{\mathbf{b}} g)_{\mathbf{p}}$.

Theorem 6.10 *Let Σ be a smooth surface in $\mathbb{R}^3$ and let $\mathbf{p}$ be a point of Σ. Let L be an operator that associates to each smooth real-valued function f on the surface defined around the point $\mathbf{p}$ a real number $L[f]$ so as to satisfy the following conditions:—*

(i) if f and g are smooth real-valued functions on the surface defined around the point $\mathbf{p}$, and if the functions f and g are equal throughout some open set in Σ that contains the point $\mathbf{p}$, then $L[f] = L[g]$;

(ii) $L[rf + sg] = rL[f] + sL[g]$ for all real numbers r and s and for all smooth real-valued functions f and g on the surface defined around the point $\mathbf{p}$;

(iii) $L[f.g] = g(\mathbf{p})L[f] + f(\mathbf{p})L[g]$ *for all smooth real-valued functions* f *and* g *on the surface defined around the point* $\mathbf{p}$*, where* $f.g$ *denotes the product of the functions* f *and* g*.*

Then there exists a tangent vector $\mathbf{b}$ *to the surface* Σ *at the point* $\mathbf{p}$ *such that* $L[f] = (\partial_{\mathbf{b}} f)_{\mathbf{p}}$.

Proof Let (u, v) be smooth local coordinates on the surface around the point $\mathbf{p}$, where $(u, v) = (0, 0)$ at the point $\mathbf{p}$ itself, and let f be a smooth real-valued function on the surface defined around the. point $\mathbf{p}$. Then there exists a positive number δ_0 and a smooth real-valued function F of two-real variables, defined throughout the open disk of radius δ_0 $\mathbb{R}^2$ centered on the point $(0, 0)$, such that $f(\mathbf{p}') = F(u(\mathbf{p}'), v(\mathbf{p}'))$ for all points $\mathbf{p}'$ on the surface for which $u^2 + v^2 < \delta_0^2$. Let

$$G_j(s_1, s_2) = \int_0^1 \left.\frac{\partial F}{\partial s_j}\right|_{(ts_1, ts_2)} dt$$

for $j = 1, 2$. Then

$$G_j(0, 0) = \left.\frac{\partial F}{\partial s_j}\right|_{(0,0)}$$

for $j = 1, 2$, and

$$\begin{aligned} F(s_1, s_2) &= F(0, 0) + \int_{t=0}^1 \frac{d}{dt} \left(F(ts_1, ts_2)\right) dt \\ &= F(0, 0) + s_1 G_1(s_1, s_2) + s_2 G_2(s_1, s_2) \end{aligned}$$

for all $(s_1, s_2) \in \mathbb{R}^2$ satisfying $s_1^2 + s_2^2 < \delta_0^2$. It follows that $f = f(\mathbf{p}) + u.g_1 + v.g_2$ around the point $\mathbf{p}$, where $g_j = G_j(u, v)$ for $j = 1, 2$. Moreover

$$g_1(\mathbf{p}) = \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}}, \quad \text{and} \quad g_2(\mathbf{p}) = \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}}.$$

If we apply the operator L to the constant function with value 1, we find that

$$L[1] = L[1.1] = L[1] + L[1],$$

and therefore $L[1] = 0$. It follows that $L[c] = 0$ for any constant function c. Also the smooth local coordinate system (u, v) has been chosen such that $u(\mathbf{p}) = 0$ and $v(\mathbf{p}) = 0$. It follows that

$$\begin{aligned} L[f] &= L[u.g_1] + L[v.g_2] \\ &= g_1(\mathbf{p})L[u]u(\mathbf{p})L[g_1] + g_2(\mathbf{p})L[v] + v(\mathbf{p})L[g_2] \\ &= g_1(\mathbf{p})L[u] + g_2(\mathbf{p})L[v] \\ &= L[u] \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} + L[v] \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}}. \end{aligned}$$

Let $\gamma\colon (-\delta_1, \delta_1) \to \Sigma$ be defined on the open interval $(-\delta_1, \delta_1)$, for some sufficiently small positive real number δ_1, such that $u(\gamma(t)) = b_u t$ and $v(\gamma(t)) = b_v t$ for $|t| < \delta_1$, where $b_u = L[u]$ and $b_v = L[v]$. It follows from Proposition 6.9 that

$$(\partial_{\mathbf{b}} f)_{\mathbf{p}} = b_u \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} + b_v \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} = L[u] \left.\frac{\partial f}{\partial u}\right|_{\mathbf{p}} + L[v] \left.\frac{\partial f}{\partial v}\right|_{\mathbf{p}} = L[f]$$

for all smooth real-valued functions f on the surface defined around the point $\mathbf{p}$. The result follows. ∎

6.11 Smooth Surfaces and the Inverse Function Theorem

We now state a result that is essentially the three-dimensional case of the Inverse Function Theorem of real analysis.

Theorem 6.11 (Inverse Function Theorem in Three Dimensions) *Let $\mathbf{p}$ be a point in three-dimensional Euclidean space $\mathbb{R}^3$, let U, V and W be smooth functions defined throughout some open neighbourhood Ω_0 of the point $\mathbf{p}$ in $\mathbb{R}^3$, and let $\varphi\colon \Omega_0 \to \mathbb{R}^3$ be the smooth vector-valued function on Ω_0 with components U, V and W, so that*

$$\varphi(x, y, z) = (U(x, y, z),\ \ V(x, y, z),\ \ W(x, y, z))$$

for all $(x, y, z) \in \Omega_0$. Suppose that the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{pmatrix}$$

is invertible at the point $\mathbf{p}$. Then there exists an open neighbourhood Ω of the point $\mathbf{p}$ contained in Ω_0 and smooth real-valued functions ξ, η and ζ defined around $\varphi(\mathbf{p})$, such that $\varphi(\Omega)$ is an open set in $\mathbb{R}^3$, the functions ξ, η and ζ are defined throughout $\varphi(\Omega)$, and

$$x = \xi(U, V, W), \quad y = \eta(U, V, W) \quad \text{and} \quad z = \zeta(U, V, W)$$

at all points (x, y, z) of Ω.

Corollary 6.12 *Let U, V, W be smooth real-valued functions defined throughout some neighbourhood of a point $\mathbf{p}$ in $\mathbb{R}^3$. Suppose that the Jacobian matrix*

$$\begin{pmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{pmatrix}$$

is invertible at the point $\mathbf{p}$. Then there exists an open neighbourhood Ω of the point $\mathbf{p}$ such that the restriction of the smooth real-valued functions U, V, W to the open set Ω determines a smooth curvilinear coordinate system with domain Ω.

Proof The existence of the open neighbourhood Ω of the point $\mathbf{p}$ over which the requirements for a smooth curvilinear coordinate system are satisfied follows directly from the three-dimensional Inverse Function Theorem (Theorem 6.11). ∎

Proposition 6.13 *Let $\mathbf{p}$ be a point of $\mathbb{R}^3$ with Cartesian coordinates (x_0, y_0, z_0), and let W be a smooth real-valued function defined over an open neighbourhood of the point $\mathbf{p}$ in $\mathbb{R}^3$. Suppose that $W = 0$ and $\dfrac{\partial W}{\partial z} \neq 0$ at the point $\mathbf{p}$. Then there exists an open neighbourhood Ω of $\mathbf{p}$ and a smooth function f of two real variables, defined around (x_0, y_0) in $\mathbb{R}^2$, for which*

$$\{(x, y, z) \in \Omega : W(x, y, z) = 0\} = \{(x, y, z) \in \Omega : z = f(x, y)\}.$$

Proof Let $U(x, y, z) = x$ and $V(x, y, z) = y$ for all real numbers (x, y, z). Then

$$\begin{vmatrix} \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \\ \dfrac{\partial V}{\partial x} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{\partial W}{\partial x} & \dfrac{\partial W}{\partial y} & \dfrac{\partial W}{\partial z} \end{vmatrix} = \frac{\partial W}{\partial z} \neq 0$$

throughout some open neighbourhood of the point $\mathbf{p}$. It follows from Corollary 6.12 that there exists some open neighbourhood Ω_1 of the point $\mathbf{p}$ such that the restrictions of the smooth functions U, V and W to Ω_1 are the components of a smooth curvilinear coordinate system with domain Ω_1. Let $\varphi\colon \Omega_1 \to \mathbb{R}^3$ be defined such that

$$\varphi(x, y, z) = (U(x, y, z),\ V(x, y, z),\ W(x, y, z))$$

for all $(x, y, z) \in \Omega_1$. Then the definition of smooth curvilinear coordinate systems ensures that $\varphi(\Omega_1)$ is an open set in $\mathbb{R}^3$ containing the point $(x_0, y_0, 0)$. There therefore exists some positive number δ such that

$$\{(u, v, w) \in \mathbb{R}^3 : |u - u_0| < \delta,\ \ |v - v_0| < \delta \text{ and } |w| < \delta\} \subset \varphi(\Omega_1).$$

Let

$$\Omega = \{(x, y, z) \in \Omega : |x - x_0| < \delta,\ \ |y - y_0| < \delta \text{ and } |W(x, y, z)| < \delta\}.$$

We may then suppose, without loss of generality, that the domain of definition of the smooth real-valued functions U, V, W is this open set Ω. Then (U, V, W) is a smooth curvilinear coordinate system with domain Ω, and therefore there exist smooth real-valued functions ξ, η and ζ defined on $\varphi(\Omega)$ such that $x = \xi(\varphi(x, y, z))$, $y = \eta(\varphi(x, y, z))$ and $z = \zeta(\varphi(x, y, z))$ for all $x, y, z \in \Omega$. Let

$$D = \{(x, y) \in \mathbb{R}^2 : |x - x_0| < \delta \text{ and } |y - y_0| < \delta\},$$

and let $f(x, y) = \zeta(x, y, 0)$. If $(x, y, z) \in \Omega$ satisfies $W(x, y, z) = 0$ then

$$z = \zeta(U(x, y, z), V(x, y, z), W(x, y, z)) = \zeta(x, y, 0) = f(x, y).$$

To complete the proof, we must also show that if $(x, y, z) \in \Omega$ satisfies $z = f(x, y)$ then $W(x, y, z) = 0$. Now $w = W(\sigma(u, v, w))$ for all $(u, v, w) \in \varphi(\Omega)$, where

$$\sigma(u, v, w) = (\xi(u, v, w),\ \eta(u, v, w),\ \zeta(u, v, w)\,)$$

(see Lemma 6.1). Also $\xi(x, y, 0) = x$, $\eta(x, y, z) = y$ and $\zeta(x, y, 0) = f(x, y)$. It follows that

$$0 = W(\xi(x, y, 0), \eta(x, y, 0), \zeta(x, y, 0)) = W(x, y, f(x, y)).$$

for all $(x, y) \in D$. Thus if $(x, y, z) \in \Omega$ and $z = f(x, y)$ then $W(x, y, z) = 0$, as required. ∎

Remark Proposition 6.13 is a particular case of the *Implicit Function Theorem.*

Corollary 6.14 *Let W be a smooth real-valued function defined over some open set Ω in $\mathbb{R}^3$, and let*

$$\Sigma = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.$$

Suppose that the gradient

$$\left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z} \right)$$

of W is non-zero at each point of Σ. Then Σ is a smooth surface in $\mathbb{R}^3$.

Proof Given any point $\mathbf{p}$, at least one of the partial derivatives $\dfrac{\partial W}{\partial x}$, $\dfrac{\partial W}{\partial y}$, $\dfrac{\partial W}{\partial z}$ is non-zero at the point $\mathbf{p}$. We show that Σ is a smooth surface throughout some open neighbourhood of the point $\mathbf{p}$. We may assume, without loss of generality, that $\dfrac{\partial W}{\partial z} \neq 0$ at the point $\mathbf{p}$. It follows from Proposition 6.13 that there exists an open subset $\tilde{\Omega}$ of Ω and a smooth function $f\colon D \to \mathbb{R}$, where $(x, y) \in D$ for all $(x, y, z) \in \tilde{\Omega}$, such that

$$\Sigma \cap \tilde{\Omega} = \{(x, y, z) \in \tilde{\Omega} : z = f(x, y)\}.$$

Let $\tilde{U}(x, y, z) = x$, $\tilde{V}(x, y, z) = y$ and $\tilde{W}(x, y, z) = z - f(x, y)$ for all $(x, y, z) \in \Omega$. Then $(\tilde{U}, \tilde{V}, \tilde{W})$ is a smooth coordinate system with domain $\tilde{\Omega}$. Moreover

$$\Sigma \cap \tilde{\Omega} = \{(x, y, z) \in \tilde{\Omega} : \tilde{W}(x, y, z) = 0\}.$$

It follows that Σ is a smooth surface throughout some open neighbourhood of the point $\mathbf{p}$, as required. ∎

Proposition 6.15 *Let $\chi\colon D \to \mathbb{R}^3$ be a smooth function defined over an open set D in $\mathbb{R}^2$ that expresses the Cartesian coordinates (x, y, z) of an image point in R^3 as smooth functions of Cartesian coordinates u and v on D. Let (u_0, v_0) be a point of D. Suppose that the vectors*

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right) \quad \text{and} \quad \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right)$$

are linearly independent when $u = u_0$ and $v = v_0$. Then there exists a smooth curvilinear coordinate system (U, V, W) with domain Ω, where Ω is an open set in $\mathbb{R}^3$ containing the point $\chi(u_0, v_0)$, such that

$$U(\chi(u, v)) = u \quad \text{and} \quad V(\chi(u, v)) = v$$

for all $(u, v) \in D \cap \chi^{-1}(\Omega)$ and

$$\chi(D) \cap \Omega = \{(x, y, z) \in \Omega : W(x, y, z) = 0\}.$$

Proof Let (p,q,r) be a vector in $\mathbb{R}^3$ chosen so that the three vectors

$$\left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) \quad \text{and} \quad (p,q,r)$$

are linearly independent when $u = u_0$ and $v = v_0$. Let

$$\psi(u,v,w) = \chi(u,v) + (wp, wq, qr).$$

for all $u, v \in D$ and $w \in \mathbb{R}$. Letting $(x,y,z) = \psi(u,v,w)$, we see that

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & p \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & q \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & r \end{pmatrix} \neq 0$$

when $u = u_0$, $v = v_0$ and $w = 0$. It then follows from the three-dimensional Inverse Function Theorem (Theorem 6.11) that there exists an open set D_0 in $\mathbb{R}^2$, where $(u_0, v_0) \in D_0$ and $D_0 \subset D$, and a positive real number δ_0 such that the smooth map ψ maps $D_0 \times (-\delta_0, \delta_0)$ onto an open set Ω in $\mathbb{R}^3$ on which are defined smooth real-valued functions U, V, W such that

$$u = U(\psi(u,v,w)), \quad v = V(\psi(u,v,w)), \quad w = W(\psi(u,v,w))$$

for all $(u,v) \in D_0$ and for all real numbers w satisfying $|w| < \delta_0$. Then $D_0 = D \cap \chi^{-1}(\Omega)$ and $\chi(u,v) = \psi(u,v,0)$ for all $(u,v) \in D_0$. It follows that

$$U(\chi(u,v)) = u, \quad V(\chi(u,v)) = v \quad \text{and} \quad W(\chi(u,v)) = 0$$

for all $(u,v) \in D \cap \chi^{-1}(\Omega)$. It follows that

$$\chi(D) \cap \Omega \subset \{(x,y,z) \in \Omega : W(x,y,z) = 0\}.$$

Now let (x,y,z) be a point of Ω for which $W(x,y,z) = 0$. Then there exist $(u,v) \in D_0$ and a real number w satisfying $|w| < \delta_0$ for which $(x,y,z) = \psi(u,v,w)$. Then

$$w = W(\psi(u,v,w)) = W(x,y,z) = 0,$$

and thus $(x,y,z) = \chi(u,v)$. Thus

$$\chi(D) \cap \Omega = \{(x,y,z) \in \Omega : W(x,y,z) = 0\}.$$

This completes the proof. ∎