MA232A: Euclidean and non-Euclidean Geometry
Michaelmas Term 2015
The Hyperbolic Plane

David R. Wilkins

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8 The Hyperbolic Plane

8.1 Metric Tensors on Subsets of the Euclidean Plane

Let Σ be a smooth surface in $\mathbb{R}^3$ and let (p, q) be a smooth local coordinate system on a portion of Σ. Let $\mathbf{r}(p, q)$ represent the position vector of a point of Σ as a smooth function of the local coordinates p and q, and let

$$\begin{aligned} E &= \left|\frac{\partial \mathbf{r}}{\partial p}\right|^2, \\ F &= \frac{\partial \mathbf{r}}{\partial p} \cdot \frac{\partial \mathbf{r}}{\partial q}, \\ G &= \left|\frac{\partial \mathbf{r}}{\partial q}\right|^2. \end{aligned}$$

Proposition 7.7 shows that the Gaussian curvature k of the surface is expressible in terms of the functions E, F, G and their partial derivatives of first and second order with respect to the local coordinates p and q by means of the following formula:

$$\begin{aligned} 4(EG - F^2)^2 k &= E\left(\frac{\partial E}{\partial q}\frac{\partial G}{\partial q} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial q} + \left(\frac{\partial G}{\partial p}\right)^2\right) \\ &\quad + F\Big(\frac{\partial E}{\partial p}\frac{\partial G}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial q}\frac{\partial F}{\partial q} \\ &\quad + 4\frac{\partial F}{\partial p}\frac{\partial F}{\partial q} - 2\frac{\partial F}{\partial p}\frac{\partial G}{\partial p}\Big) \\ &\quad + G\left(\frac{\partial E}{\partial p}\frac{\partial G}{\partial p} - 2\frac{\partial E}{\partial p}\frac{\partial F}{\partial q} + \left(\frac{\partial E}{\partial q}\right)^2\right) \\ &\quad - 2(EG - F^2)\left(\frac{\partial E}{\partial q^2} - 2\frac{\partial^2 F}{\partial p\,\partial q} + \frac{\partial G}{\partial p^2}\right). \end{aligned}$$

The local coordinates (p, q) are defined over some open subset of smooth surface Σ, and their values represent points in an open subset D of the plane $\mathbb{R}^2$. There is then a smooth map $\chi\colon D \to \Sigma$ that maps a corresponding point of the smooth surface. Thus if (u, v) is the standard Cartesian coordinate system on $\mathbb{R}^2$ then the corresponding coordinate functions p and q on the surface Σ are related to u and v through the equations $p(\chi(u, v)) = u$ and $q(\chi(u, v)) = v$. Moreover the definition of partial derivatives of functions on the surface Σ with respect to p and q are defined so that

$$\left.\frac{\partial f}{\partial p}\right|_{\chi(u_0,v_0)} = \left.\frac{\partial (f \circ \chi)}{\partial u}\right|_{(u_0,v_0)} \left.\frac{\partial f}{\partial p}\right|_{\chi(u_0,v_0)} = \left.\frac{\partial (f \circ \chi)}{\partial u}\right|_{(u_0,v_0)}.$$

One can then define a "metric" on the subset D of the plane, induced by the embedding of D as a smooth surface Σ in $\mathbb{R}^{\nvDash}$ through the map $\chi\colon D \to \mathbb{R}^3$ so that the length of a smooth curve $\gamma\colon [a,b] \to D$ in D parameterized by a closed interval $[a,b]$ with respect to the induced metric on D is equal to

$$\int_a^b \left| \frac{d(\chi(\gamma))}{dt} \right| \, dt,$$

where

$$\int_a^b \left| \frac{d(\chi(\gamma))}{dt} \right| \, dt = \int_{a,b} \sqrt{E \left(\frac{dp}{dt}\right)^2 + 2F \frac{dp}{dt}\frac{dq}{dt} + G \left(\frac{dp}{dt}\right)^2} \, dt.$$

The dependence of the various quantities occurring in the integrand on the various mappings and local coordinate systems involved can be expressed more fully as follows:—

$$\begin{aligned} E \left(\frac{dp}{dt}\right)^2 &= E(\chi(\gamma(t))) \left(\frac{dp(\chi(\gamma(t)))}{dt}\right)^2, \\ F \frac{dp}{dt}\frac{dq}{dt} &= F(\chi(\gamma(t))) \frac{dp(\chi(\gamma(t)))}{dt} \frac{dq(\chi(\gamma(t)))}{dt}, \\ G \left(\frac{dq}{dt}\right)^2 &= G(\chi(\gamma(t))) \left(\frac{dq(\chi(\gamma(t)))}{dt}\right)^2 \end{aligned}$$

In order to reduce the complexity of notation involved, we use the letters E, G and G to denote the functions on the open set D induced by the corresponding functions on the smooth surface, so that

$$E(u,v) = E(\chi(u,v)), \quad F(u,v) = F(\chi(u,v)), \quad G(u,v) = G(\chi(u,v)).$$

These functions determine a *metric tensor* g which associates to each point (u,v) of D a symmetric bilinear form $g_{(u,v)}$, where

$$g_{u,v}((z_1,w_1),(z_2,w_2)) = E(u,v)z_1z_2 + F(u,v)(z_1w_2 + w_1z_2) + G(u,v)w_1w_2$$

for all $(z_1,w_1),(z_2,w_2) \in \mathbb{R}^2$. Moreover

$$E(u,v)z^2 + F(u,v)(zw + wz) + G(u,v)w^2 > 0$$

for all non-zero $(z, w) \in \mathbb{R}^2$, and thus

$$g_{u,v}((z, w), (z, w)) > 0.$$

for all non-zero $(z, w) \in \mathbb{R}^2$. The quadratic form $q_{(u,v)}$ is then positive definite for all $(u, v) \in D$.

Definition A *Riemannian metric* g on an open set D in $\mathbb{R}^2$ assigns to each point (u, v) of V a positive definite bilinear symmetric form $g_{(u,v)}$ whose coefficients are smooth real-valued functions on D.

It follows from this definition that, given any Riemannian metric on an open set D in the plane, there exist smooth real-valued functions E, F and G on D such that

$$g_{u,v}((z_1, w_1), (z_2, w_2)) = E(u, v)z_1 z_2 + F(u, v)(z_1 w_2 + w_1 z_2) + G(u, v)w_1 w_2$$

for all $(z_1, w_1), (z_2, w_2) \in \mathbb{R}^2$ and

$$E(u, v)z^2 + F(u, v)(zw + wz) + G(u, v)w^2 > 0$$

for all non-zero $(z, w) \in \mathbb{R}^2$.

Let g be a Riemannian metric on an connected open set D in the plane. Let $\gamma\colon [a, b] \to D$ be a smooth curve in D parameterized by a closed interval $[a, b]$, and let $\gamma(t) = (u(t), v(t))$ for all $t \in [a, b]$. The length $\mathrm{length}_g(\gamma)$ of the curve γ with respect to the induced metric g on D is then defined such that

$$\begin{aligned} \mathrm{length}_g(\gamma) &= \int_a^b \sqrt{E(\gamma)u'(t)^2 + 2F(\gamma)u'(t)v'(t) + G(\gamma)v'(t)^2}\, dt \\ &= \int_a^b \sqrt{g\left(\frac{d\gamma(t)}{dt}, \frac{d\gamma(t)}{dt}\right)}\, dt \end{aligned}$$

The distance between two points of D with respect to the metric g is then defined to be the greatest lower bound (or infinimum) of the lengths of all smooth curves that join the first point to the second. This distance function satisfies all the axioms that are required to be satisfied by the distance function on a metric space. A connected open set in the plane provided with a Riemannian metric therefore a metric space.

It is customary to specify a Riemannian metric with coefficients E, F and G on an open set with coordinate system (u, v) by writing

$$ds^2 = E du^2 + 2F du dv + G dv^2.$$

Historically, ds was conceptualized as an infinitesimal representing an element of arclength, and the above formula expressed the manner in which infinitesimal increments of arclength are determined by infinitesimal increments du and dv of the coordinates u and v.

Definition Let g be a Riemannian metric on an open subset D in the plane. The *curvature* k of g is then determined according to the formula

$$\begin{aligned} 4(EG-F^2)^2 k &= E\left(\frac{\partial E}{\partial v}\frac{\partial G}{\partial v} - 2\frac{\partial F}{\partial u}\frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u}\right)^2\right) \\ &\quad + F\Bigl(\frac{\partial E}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial v}\frac{\partial F}{\partial v} \\ &\qquad + 4\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - 2\frac{\partial F}{\partial u}\frac{\partial G}{\partial u}\Bigr) \\ &\qquad + G\left(\frac{\partial E}{\partial u}\frac{\partial G}{\partial u} - 2\frac{\partial E}{\partial u}\frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v}\right)^2\right) \\ &\qquad - 2(EG-F^2)\left(\frac{\partial E}{\partial v^2} - 2\frac{\partial^2 F}{\partial u\,\partial v} + \frac{\partial G}{\partial u^2}\right). \end{aligned}$$

where u and v are local coordinates, and E, F and G are the coefficients of the metric tensor g with respect to these local coordinates, so that

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

8.2 Curvature of Conformally Flat Metrics

Definition Let D be a connected subset of the plane. A *conformally flat* metric on D is a Riemannian metric g on D that takes the form

$$g_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = \lambda(\mathbf{p})^2 \mathbf{v} \,.\, \mathbf{w},$$

where x and y are the standard Cartesian coordinates on D, $\lambda\colon D \to \mathbb{R}$ is a smooth real-valued function that is strictly positive throughout D, and $\mathbf{v}.\mathbf{w}$ is the usual two-dimensional scalar product of the vectors $\mathbf{v}$ and $\mathbf{w}$, defined so that

$$(v_1, v_2) \,.\, (w_1, w_2) = v_1 w_1 + v_2 w_2.$$

The metric tensor of a conformally-flat metric determined by an real-valued function λ that is positive throughout the connected open set D can be specified in traditional notation as follows:

$$ds^2 = \lambda^2(dx^2 + dy^2).$$

Let $\lambda\colon D \to \mathbb{R}$ be a smooth real-valued function that is positive throughout some connected open set D in the plane, and let g be the conformally-flat metric determined by λ, so that

$$g_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = \lambda(\mathbf{p})^2 \mathbf{v}.\mathbf{w}$$

at each point $\mathbf{p}$ of D. Let $\gamma\colon [a,b] \to D$ be a smooth curve in D parameterized by a closed interval $[a,b]$, and let $\gamma(t) = (u(t), v(t))$ for all $t \in [a,b]$. The length length(γ) of the smooth curve γ is then given by the formula

$$\text{length}_g(\gamma) = \int_a^b \lambda(\gamma(t))|\gamma'(t)|\,dt.$$

Proposition 8.1 *Let $\lambda\colon D \to \mathbb{R}$ be a smooth real-valued function that is positive throughout some connected open set D in the plane, and let g be the conformally-flat metric determined by λ, so that*

$$g_{\mathbf{p}}(\mathbf{v}, \mathbf{w}) = \lambda(\mathbf{p})^2 \mathbf{v}.\mathbf{w}$$

at each point $\mathbf{p}$ of D. Then the curvature k of this conformally flat metric satisfies

$$k = -\frac{1}{\lambda^2}\left(\frac{\partial^2(\log\lambda)}{\partial x^2} + \frac{\partial^2(\log\lambda)}{\partial y^2}\right).$$

Proof Let E, F and G denote the components of the conformally-flat metric tensor g, so that

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G dy^2.$$

Then $E = G = \lambda^2$ and $F = 0$. Substituting into the formula that specifies the curvature of a two-dimensional Riemannian metric, we find that

$$\begin{aligned}
4\lambda^8 k &= 2\lambda^2\left(\left(\frac{\partial(\lambda^2)}{\partial x}\right)^2 + \left(\frac{\partial(\lambda^2)}{\partial y}\right)^2\right) - 2\lambda^4\left(\frac{\partial^2(\lambda^2)}{\partial x^2} + \frac{\partial^2(\lambda^2)}{\partial y^2}\right) \\
&= 8\lambda^4\left(\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right) \\
&\qquad - 2\lambda^4\left(2\lambda\frac{\partial^2\lambda}{\partial x^2} + 2\left(\frac{\partial\lambda}{\partial x}\right)^2 + 2\lambda\frac{\partial^2\lambda}{\partial y^2} + 2\left(\frac{\partial\lambda}{\partial y}\right)^2\right),
\end{aligned}$$

and therefore

$$k = \frac{-1}{\lambda^3}\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) + \frac{1}{\lambda^4}\left(\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right).$$

Now

$$\begin{aligned}
\left(\frac{\partial^2(\log\lambda)}{\partial x^2} + \frac{\partial^2(\log\lambda)}{\partial y^2}\right) &= \frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial y}\right) \\
&= \frac{1}{\lambda}\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) - \frac{1}{\lambda^2}\left(\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right).
\end{aligned}$$

It follows that
$$k = -\frac{1}{\lambda^2}\left(\frac{\partial^2(\log\lambda)}{\partial x^2} + \frac{\partial^2(\log\lambda)}{\partial y^2}\right),$$
as required. ∎

Corollary 8.2 *The upper half plane* $\{(x, y) : y > 0\}$ *with metric*
$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$$
has constant Gaussian curvature equal to -1*.*

Proof The metric is conformally-flat, and the function λ that determines the metric is given by $\lambda(x, y) = y^{-1}$. It follows that $\log\lambda = -\log y$, and therefore
$$\frac{\partial^2(\log\lambda)}{\partial x^2} = 0, \qquad \frac{\partial^2(\log\lambda)}{\partial y^2} = \frac{1}{y^2} = \lambda^2.$$

The general formula for the curvature of a conformally flat metric (Proposition 8.1) ensures that $k = -1$, as required. ∎

Corollary 8.3 *The open unit disk* $\{(x, y) : x^2 + y^2 < 1\}$ *with metric*
$$ds^2 = \frac{4}{(1 - x^2 - y^2)^2}(dx^2 + dy^2)$$
has constant Gaussian curvature equal to -1*.*

Proof In this case $\lambda(x, y) = 2(1 - x^2 - y^2)^{-1}$ and therefore
$$\begin{aligned}
\log\lambda &= \log 2 - \log(1 - x^2 - y^2), \\
\frac{\partial(\log\lambda)}{\partial x} &= \frac{2x}{1 - x^2 - y^2}, \\
\frac{\partial(\log\lambda)}{\partial y} &= \frac{2y}{1 - x^2 - y^2}, \\
\frac{\partial^2(\log\lambda)}{\partial x^2} &= \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}, \\
\frac{\partial^2(\log\lambda)}{\partial y^2} &= \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2},
\end{aligned}$$

and therefore
$$\frac{\partial^2(\log\lambda)}{\partial x^2} + \frac{\partial^2(\log\lambda)}{\partial y^2} = \frac{4}{(1 - x^2 - y^2)^2} = \lambda^2.$$
The result therefore follows from the general formula for the curvature of a conformally flat metric (Proposition 8.1). ∎

8.3 Möbius Transformations of the Upper Half Plane

A *Möbius transformation* determined by complex numbers a, b, c, d is a transformation from $\mathbb{C} \setminus \{-d/c\}$ that sends z to

$$\frac{az+b}{cz+d}$$

for all complex numbers z satisfying $cz + d \neq 0$.

We consider in this section the action on the open upper half plane of those Möbius transformations for which the parameters a, b, c and d are real numbers and satisfy $ad - bc = 1$. We shall show that such Möbius transformations map the open upper half plane H onto itself, where

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}.$$

Let a, b, c, d be real numbers satisfying $ad - bc = 1$, let z be a complex number not equal to $-d/c$, and let $w = \frac{az+b}{cz+d}$. Let $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$. Multiplying the numerator and denominator of the fraction defining w by $c\overline{z}+d$, where $\overline{z}$ denotes the complex conjugate of z, and using the fact that a, b, c and d are real numbers satisfying $ad-bc = 1$, we find that

$$w = \frac{(az+b)(c\overline{z}+d)}{(cz+d)(c\overline{z}+d)} = \frac{ac|z|^2 + bd + (ad+bc)x + iy}{|cz+d|^2}.$$

It follows that $w = u + iv$, where

$$u = \frac{ac|z|^2 + bd + (ad+bc)x}{|cz+d|^2}.$$

and

$$v = \frac{y}{|cz+d|^2}.$$

Moreover $v = 0$ when $y = 0$ and $v > 0$ when $y > 0$. It follows that the map that sends $z \in H$ to $(az+b)/(cz+d)$ maps the open upper half plane H into itself. Also $cwz - az = dw - b$, and therefore

$$z = \frac{dw-b}{a-cw}.$$

It follows that the map that sends $z \in H$ to $(az+b)/(cz+d)$ is surjective, and thus maps the open upper half plane H onto itself.

Let $\mathrm{SL}(2,\mathbb{R})$ denote the group of all 2×2 matrices with determinant equal to one, where the group operation on $\mathrm{SL}(2,\mathbb{R})$ is matrix multiplication. Then

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}.$$

Given any 2×2 matrix A with real coefficients satisfying $\det A = 1$, we denote by $\mu_A\colon H \to H$ the Möbius transformation of the open upper half plane H defined such that

$$\mu_A(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are real numbers satisfying $ad - bc = 1$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Lemma 8.4 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, and, given any $A \in \mathrm{SL}(2,\mathbb{R})$, where*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

let Möbius(H) denote the group of all Möbius transformations that map the open upper half plane H onto itself, and, for each $A \in \mathrm{SL}(2,\mathbb{R})$, let $\mu_A\colon H \to H$ denote the mapping of the upper half plane H to itself defined such that

$$\mu_A(z) = \frac{az+b}{cz+d}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then $\mu_B(\mu_A(z)) = \mu_{BA}(z)$ for all $A, B \in \mathrm{SL}(2,\mathbb{R})$ and therefore the mapping

$$\mu\colon \mathrm{SL}(2,\mathbb{R}) \to \text{Möbius}(H)$$

sending $A \in \mathrm{SL}(2,\mathbb{R})$ to $\mu_A\colon H \to H$ is a homomorphism of groups. The kernel of this homomorphism is the normal subgroup of $\mathrm{SL}(2,\mathbb{R})$ of order two consisting of the matrices I and $-I$, where I denotes the identity 2×2 matrix.

Proof Let a, b, c, d, e, f, g and h be real numbers satisfying $ad - bc = 1$ and $eh - fg = 1$, let $z \in H$, and let $w = (az+b)/(cz+d)$. Then

$$\frac{ew+f}{gw+h} = \frac{e(az+b)+f(cz+d)}{g(az+b)+h(cz+d)} = \frac{(ea+fc)z+(eb+fd)}{(ga+hc)z+(gb+hd)}.$$

Moreover
$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}.$$
It follows that $\mu_{BA}(z) = \mu_B(\mu_A(z))$, where
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Suppose that the matrix A belongs to the kernel of μ. Then
$$\frac{az+b}{cz+d} = z$$
for all $z \in H$. But then $cz^2 + (d-a)z - b = 0$ for all $z \in H$, and therefore $c = b = 0$ and $d = a$. But $ad - bc = ad = 1$. It follows that either $a = d = 1$ or $a = d = -1$. The result follows. ∎

The *projective special linear group* $\mathrm{PSL}(2,\mathbb{R})$ is defined to be the quotient group
$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{I, -I\}$$
that is the quotient of the group $\mathrm{SL}(2,\mathbb{R})$ by the normal subgroup consisting of the matrices I and $-I$. It follows from Lemma 8.4 that the group of Möbius transformations of the upper half plane H is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

Next we compute the derivative of $\mu_A\colon H \to H$, where $A \in \mathrm{SL}(2,\mathbb{R})$, where we take the real and imaginary parts of a complex number belonging to H as its local coordinates.

Proposition 8.5 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, let a, b, c and d be real numbers satisfying $ad - bc = 1$, and let $\mu\colon H \to H$ be the Möbius transformation of the upper half plane H defined such that*
$$\mu(z) = \frac{az+b}{cz+d}$$
for all $z \in H$. Let x and y be real variables, where $y > 0$, and let $u(x,y)$ and $v(x,y)$ be the smooth functions of x and y on H determined so that
$$u(x,y) + iv(x,y) = \mu(x+iy),$$
where $i = \sqrt{-1}$. Then
$$u = \frac{ac(x^2+y^2) + (ad+bc)x + bd}{(cx+d)^2 + c^2y^2},$$

$$\begin{aligned} v &= \frac{y}{(cx+d)^2 + c^2y^2}, \\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = \frac{(cx+d)^2 - c^2y^2}{((cx+d)^2 + c^2y^2)^2} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = \frac{2c(cx+d)}{((cx+d)^2 + c^2y^2)^2} \end{aligned}$$

and

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \frac{1}{((cx+d)^2 + c^2y^2)^2} = \frac{v^2}{y^2}.$$

Also the Jacobian matrix of the smooth map sending (x, y) *to* (u, v) *for all real numbers* x *and positive real numbers* y *satisfies*

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \frac{v}{y} \begin{pmatrix} \cos\theta_{x,y} & -\sin\theta_{x,y} \\ \sin\theta_{x,y} & \cos\theta_{x,y} \end{pmatrix}.$$

where $\theta_{x,y}$ *is the angle that the vector* $\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)$ *makes with the vector* $(1, 0)$.

Proof Let $z = x + iy$. Then

$$\begin{aligned} \mu(z) &= \frac{az+b}{cz+d} = \frac{(az+b)(c\overline{z}+d)}{|cz+d|^2} \\ &= \frac{(ax+b+iay)(cx+d-icy)}{(cx+d)^2 + c^2y^2} \\ &= \frac{ac(x^2+y^2) + (ad+bc)x + bd + iy}{(cx+d)^2 + c^2y^2}. \end{aligned}$$

and thus $\mu(x + iy) = u + iv$, where

$$u(x, y) = \frac{ac(x^2+y^2) + (ad+bc)x + bd}{(cx+d)^2 + c^2y^2} \quad \text{and} \quad v(x, y) = \frac{y}{(cx+d)^2 + c^2y^2}.$$

It then follows from direct calculation that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{(2acx + ad + bc)(c^2x^2 + 2cdx + d^2 + c^2y^2)}{((cx+d)^2 + c^2y^2)^2} \\ &\quad - \frac{(acx^2 + acy^2 + adx + bcx + bd)(2c^2x + 2cd)}{((cx+d)^2 + c^2y^2)^2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(ad-bc)(c^2x^2+2cdx-c^2y^2+d^2)}{((cx+d)^2+c^2y^2)^2} \\
&= \frac{(cx+d)^2-c^2y^2}{((cx+d)^2+c^2y^2)^2}, \\
\frac{\partial u}{\partial y} &= \frac{(2acy)(c^2x^2+2cdx+d^2+c^2y^2)}{((cx+d)^2+c^2y^2)^2} \\
&\quad - \frac{(acx^2+acy^2+adx+bcx+bd)(2c^2y)}{((cx+d)^2+c^2y^2)^2} \\
&= \frac{2(ad-bc)c(cx+d)}{((cx+d)^2+c^2y^2)^2} \\
&= \frac{2c(cx+d)y}{((cx+d)^2+c^2y^2)^2} \\
\frac{\partial v}{\partial x} &= \frac{-2c(cx+d)y}{((cx+d)^2+c^2y^2)^2} = -\frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial y} &= \frac{(cx+d)^2-c^2y^2)}{((cx+d)^2+c^2y^2)^2} = \frac{\partial u}{\partial x}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 &= \frac{((cx+d)^2-c^2y^2)^2+4c^2(cx+d)^2y^2}{((cx+d)^2+c^2y^2)^4} \\
&= \frac{1}{((cx+d)^2+c^2y^2)^2} = \frac{v^2}{y^2}.
\end{aligned}
$$

The result concerning the Jacobian matrix of the transformation then follows from the previous identities. ∎

Definition The *hyperbolic metric* on H is the Riemannian metric defined by the formula

$$ds^2 = \frac{1}{y^2}(dx^2+dy^2).$$

It follows from Corollary 8.2 that this metric has constant curvature equal to -1.

The length $\mathrm{hylen}(\gamma)$ of a smooth curve $\gamma\colon [p,q] \to H$ with respect to the hyperbolic metric is then given by the formula

$$\mathrm{hylen}(\gamma) = \int_p^q \frac{1}{y(\gamma(t))} |\gamma'(t)|\, dt.$$

Corollary 8.6 *Let $H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}$, let a, b, c and d be real numbers satisfying $ad - bc = 1$, and let $\mu\colon H \to H$ be the Möbius transformation of the upper half plane H defined such that*

$$\mu(z) = \frac{az + b}{cz + d}$$

for all $z \in H$. Then

$$\mathrm{hylen}(\mu \circ \gamma) = \mathrm{hylen}(\gamma)$$

for all smooth curves γ in H, where $\mathrm{hylen}(\gamma)$ *denotes the hyperbolic length of the curve γ.*

Proof Let u and v be the real-valued functions on H defined so that $\mu(x + iy) = u(x, y) + iv(x, y)$ for all real numbers x and y for which $y > 0$. Let $\gamma\colon [t_1, t_2] \to H$ be a smooth curve in the upper half plane H parameterized by a closed bounded interval $[t_1, t_2]$. Then $(\mu \circ \gamma)'(t) = J(\gamma(t))\gamma'(t)$ for all real numbers t in the interior of $[t_1, t_2]$, where $J(\gamma(t))$ is the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

of the smooth map expressing u and v as functions of x and y on H so that $u + iv = \mu(x + iy)$ for all $(x, y) \in H$, where $i = \sqrt{-1}$. It follows from Proposition 8.5 that

$$|(\mu \circ \gamma)'(t)| = \frac{v(\gamma(t))}{y(\gamma(t))}|\gamma'(t)| = \frac{y(\mu(\gamma(t)))}{y(\gamma(t))}|\gamma'(t)|,$$

and therefore

$$\frac{1}{y(\mu(\gamma(t)))}|(\mu \circ \gamma)'(t)| = \frac{1}{y(\gamma(t))}|\gamma'(t)|,$$

for all $t \in I$. It follows that

$$\begin{aligned} \mathrm{hylen}(\mu \circ \gamma) &= \int_{t_1}^{t_2} \frac{1}{y(\mu(\gamma(t)))}|(\mu \circ \gamma)'(t)|\, dt \\ &= \int_{t_1}^{t_2} \frac{1}{y(\gamma(t))}|\gamma'(t)|\, dt \\ &= \mathrm{hylen}(\gamma) \end{aligned}$$

as required. ∎

Definition A smooth curve $\gamma\colon [t_1, t_2] \to H$ in the upper half plane H parameterized by an closed interval $[t_1, t_2]$ is said to be *parameterized by hyperbolic arclength* if the velocity vector of the curve has hyperbolic length equal to one, so that

$$\frac{1}{y(\gamma(t))}\sqrt{\left(\frac{dx(\gamma(t))}{dt}\right)^2 + \left(\frac{dy(\gamma(t))}{dt}\right)^2} = 1$$

for all $t \in I$.

Definition A smooth curve $\gamma\colon I \to H$ parameterized by hyperbolic arclength in the upper half plane H is said to be a *geodesic* if, for all $t_1 \in I$, and all t_2 belonging to some sufficiently small open neighbourhood of t_1 in the parameterizing interval I, the portion $\gamma|[t_1, t_2]$ of the curve γ parameterized by the subinterval $[t_1, t_2]$ minimizes hyperbolic length amongst all piecewise smooth curves from $\gamma(t_1)$ to $\gamma(t_2)$.

Lemma 8.7 *For all real numbers x_0 and t_0 the curve $t \mapsto x_0 + \sqrt{-1}e^t$ is a geodesic in the upper half plane H with respect to the hyperbolic metric. Moreover, given any geodesic passing through the points (x_0, y_1) and (x_0, y_2), there exists a real constant t_0 such that the geodesic is either of the form $t \mapsto x_0 + \sqrt{-1}e^{t-t_0}$ or else is of the form $t \mapsto x_0 + \sqrt{-1}e^{t_0-t}$.*

Proof Let x_0 be chosen, and let $\gamma\colon \mathbb{R} \to H$ be defined such that $\gamma(t) = (x_0, e^t)$ for all real numbers t. Then $\gamma'(t) = (0, e^t)$ and therefore

$$\frac{1}{y(\gamma(t))}|\gamma'(t)| = 1$$

(where $|\gamma'(t)|$ here denotes the length of the vector $\gamma'(t)$ with respect to the Euclidean metric). It follows that the curve is parametrized by hyperbolic arclength. Let $\eta\colon [t_1, t_2] \to H$ be a piecewise smooth curve in H from (x_0, y_1) to (x_0, y_2), where $y_1 < y_2$. Then

$$\begin{aligned}
\text{hylen}(\eta) &= \int_{t_1}^{t_2} \frac{1}{y(\gamma(t))}\sqrt{\left(\frac{dx(\gamma(t))}{dt}\right)^2 + \left(\frac{dy(\gamma(t))}{dt}\right)^2}\, dt \\
&\geq \int_{t_1}^{t_2} \frac{1}{y(\gamma(t))}\left|\frac{dy(\gamma(t))}{dt}\right| dt \\
&\geq \int_{t_1}^{t_2} \frac{1}{y(\gamma(t))}\frac{dy(\gamma(t))}{dt}\, dt = \int_{t_1}^{t_2} \frac{d(\log(y(\gamma(t)))}{dt}\, dt \\
&= \log y_2 - \log y_1.
\end{aligned}$$

Now $\gamma(\log y_1) = (x_0, y_1)$ and $\gamma(\log y_2) = (x_0, y_2)$ and the hyperbolic length of $\gamma|[\log y_1, \log y_2]$ is $\log y_2 - \log y_1$. It follows that γ minimizes length amongst all piecewise smooth curves from (x_0, y_1) and (x_0, y_2). This proves that $\gamma\colon \mathbb{R} \to H$ is a geodesic.

The calculation carried out above shows that the real part will have to be constant on any geodesic that passes through the points (x_0, y_1) and (x_0, y_2). Moreover geodesics are parameterized by hyperbolic arclength. It follows that every geodesic passing through these points is of one of the forms stated in the proposition. ∎

Let p and q be real numbers, and let $\mu\colon H \to H$ be the Möbius transformation defined so that

$$\mu(z) = \frac{qz + p}{1 + z} = \frac{az + b}{cz + d}.$$

where

$$a = \frac{q}{\sqrt{q - p}}, \quad b = \frac{p}{\sqrt{q - p}}, \quad c = \frac{1}{\sqrt{q - p}}, \quad d = \frac{1}{\sqrt{q - p}}.$$

(Note that $ad - bc = 1$.)

Every Möbius transformation of the upper half plane preserves hyperbolic lengths of piecewise-smooth curves. Let $\gamma(t) = \mu(ie^t)$ for all $t \in \mathbb{R}$, where $i = \sqrt{-1}$. Now the curve $t \mapsto ie^t$ is a geodesic in H. It follows that the curve γ is also a geodesic.

Now

$$\begin{aligned} \mu(iy) - \tfrac{1}{2}(p + q) &= \frac{2qiy + 2p - (p + q + piy + qiy)}{2 + 2iy} \\ &= \frac{(q - p)(iy - 1)}{2 + 2iy} \\ &= \tfrac{1}{2}(q - p)\frac{iy - 1}{iy + 1} \end{aligned}$$

and therefore

$$|\mu(iy) - \tfrac{1}{2}(p + q)| = \tfrac{1}{2}(q - p)$$

for all real numbers y. It follows that

$$|\gamma(t) - \tfrac{1}{2}(p + q)| = \tfrac{1}{2}|q - p|$$

for all real numbers t. Thus the geodesic γ is contained in the semicircle that is the intersection of the upper half plane with the circle of radius $\frac{1}{2}(q - p)$ centred on the point $\frac{1}{2}(p + q)$. This circle intersects the real axis at the points p and q and is orthogonal to the real axis there.

Theorem 8.8 *Let*

$$H = \{z \in \mathbb{C} : \mathrm{Im}[z] > 0\}.$$

Let $\eta: I \to H$ be a smooth curve in H that is a geodesic with respect to the hyperbolic metric on H. If there exist two points on η at which the real parts coincide then there exist real constants x_0 and t_0 such that

$$\eta(t) = x_0 + ie^{t-t_0} \quad \text{or} \quad \eta(t) = x_0 + ie^{t_0-t}$$

for all real numbers t. Otherwise there exist real constants p, q and t_0 such that

$$\eta(t) = \frac{qie^{t-t_0} + p}{1 + ie^{t-t_0}}$$

for all $t \in \mathbb{R}$.

Proof Let z_1 and z_2 be distinct points of H that lie on the geodesic η. If the real parts of z_1 and z_2 coincide then let $x_0 = \mathrm{Re}[z_1]$. It then follows directly from Lemma 8.7 that the geodesic takes one of the two stated form.

If the real parts of z_1 and z_2 do not coincide then z_1 and z_2 lie on a circle centred on a point that lies on the real axis. Let that circle cut the real axis at p and q. Then there exists a Möbius transformation μ that maps the semicircle of radius $\frac{1}{2}|q-p|$ centred on $\frac{1}{2}(p+q)$ onto the upper imaginary axis $\{iy : y > 0\}$. This Möbius transformation will map the geodesic η onto a geodesic that passes through two distinct points of the positive imaginary axis. By the previous case, that geodesic must be contained in the upper imaginary axis, and therefore η must be contained in the circle of radius $\frac{1}{2}|q-p|$ about the point $\frac{1}{2}(p+q)$. The result then follows by a straightforward argument. ∎