MA232A: Euclidean and non-Euclidean Geometry
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Notes on the Sections of Gauss's *General Investigations of Curved Surfaces*

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A Gauss's *General Investigations*: The Differential Geometry of Curved Surfaces

A.1 The Significance of Gauss's *General Investigations on Curved Surfaces*

Mike Spivak begins Chapter 3 of Volume II of his five volume work entitled *A Comprehensive Introduction to Differential Geometry* with the following sentence:—

> "The single most important work in the history of differential geometry is Gauss' paper of 1827, *Disquisitiones generales circa superficies curvas* (in Latin)."

This paper was presented to the Royal Society of Göttingen in 1727 and published in 1728.

A.2 Remarks on Sections 1 and 2 of the *General Investigations*

Section 2 of the paper is not really concerned with the differential geometry of surfaces, but contains a derivation of some formulae in spherical trigonometry. For a discussion of these see the separate *Notes on Vector Algebra and Spherical Trigonometry.*

A.3 Remarks on Section 3 of the *General Investigations*

The current theory of real analysis developed through the nineteenth century, but the standard "epsilon–delta" definitions of continuity etc. did not become firmly established until the lecture courses delivered by Karl Weierstrass in Berlin from 1859 onwards, over thirty years after the publication of Gauss's *Disquisitiones Generales circa Superficies Curvas.* Later parts of the *Disquisitiones Generales* in particular assume that relevant functions can be expanded as power series.

It seems advisable to assume (unless explicitly noted otherwise) that all relevant surfaces, functions etc. are smooth, at least around the point at which the local geometry of a surface is considered. The functions that define the surface and describe its geometry will then have continuous derivatives of all orders, and standard rules of multivariable calculus, including in particular the Chain Rule for partial derivatives will be applicable.

The modern approach to the theory of smooth surfaces, regarding these as smooth submanifolds of three-dimensional Euclidean space is discussed in the separate *Notes on Smooth Surfaces*. These notes contain in particular a discussion of local coordinate systems on a surface, tangent spaces to smooth surfaces and differentials of smooth functions on surfaces.

A.4 Remarks on Section 4 of the *General Investigations*

Much of this section is concerned with the definition and properties of the Gauss map of a smooth surface in three-dimensional space. The *Gauss Map* of a smooth surface sends a point $\mathbf{p}$ on a smooth surface Σ to the point of the unit sphere S^2 in $\mathbb{R}^2$ whose position vector is the unit normal vector $\mathbf{n}(\mathbf{p})$ to the surface Σ at the point $\mathbf{p}$. This normal vector is orthogonal to the tangent plane to the surface at the point $\mathbf{p}$. Gauss denotes the Cartesian components of this normal vector $\mathbf{n}(\mathbf{p})$ by X, Y and Z, where $\sqrt{X^2 + Y^2 + Z^2} = 1$. Further details of the definition and properties of the Gauss map are to be found in the separate *Notes on the Gauss Map*.