Euclid's *Elements of Geometry* Book III (T.L. Heath's Edition)

Transcribed by D. R. Wilkins

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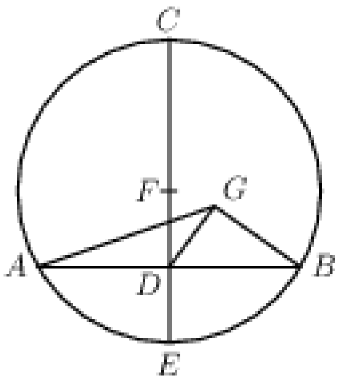
Proposition 1

To find the centre of a given circle.

Let ABC be the given circle;

thus it is required to find the centre of the circle ABC.



Let a straight line AB be drawn through it at random, and let it be bisected at the point D;

from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F;

I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre,

and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common,

the two sides AD, DG are equal to the two sides BD, DG respectively;

and the base GA is equal to the base GB, for they are radii;

therefore the angle ADG is equal to the angle DGB. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the the equal angles is right; [I Def. 10]

therefore the angle GDB is right.

But the angle FDB is also right;

Therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

Porism. From this, it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line.

Q.E.F.

Proposition 2

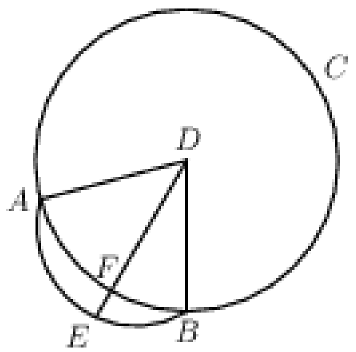
If on the circumference of a given circle two points be taken at random, the straight line joining the points wil fall within the circle.

Let ABC be a circle, and let two points A and B be taken at random on its circumference;

I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB;

let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.



Then since DA is equal to DB,

the angle DAE is also equal to the angle DBE. [I. 5] And, since one side AEB of the triangle DAE is produced,

the angle DEB is greater than the angle DAE. [I. 16]

But the angle DAE is equal to the angle DBE;

therefore the angle DEB is greater than the angle DBE. And the greater angle is subtended by the greater side; [I. 19]

therefore DB is greater than DE.

But DB is equal to DF;

therefore DF is greater than DE,

the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within. Therefore etc.

Q.E.D.

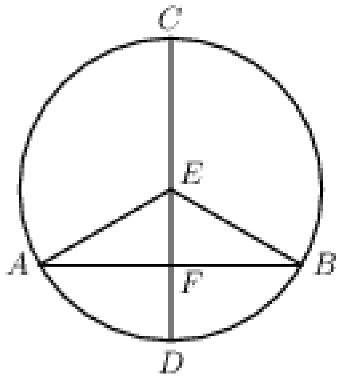
Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F;

I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common,

two sides are equal to two sides;

and the base EA is equal to the base EB;

therefore the angle AFE is equal to the angle BFE. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;

I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction,

since EA is equal to EB,

the angle EAF is also equal to the angle EBF. [I. 5]

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

therefore AF is equal to FB.

Therefore etc.

Q.E.D.

Proposition 4

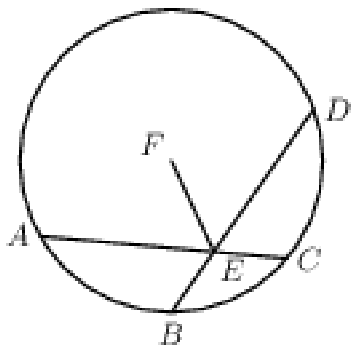
If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another in E;

I say that they do not bisect one another.

For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED;

let the centre of the circle $ABCD$ be taken [III. 1], and let it be F; let FE be joined.



Then, since a straight line FE through the centre bisects a straight line AC not through the centre;
it also cuts it at right angles; [III. 3]

therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD, it also cuts it at right angles; [III. 3]

therefore the angle FEB is right.

But the angle FBA was also proved right;

therefore the angle FEA is euqal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc.

Q.E.D.

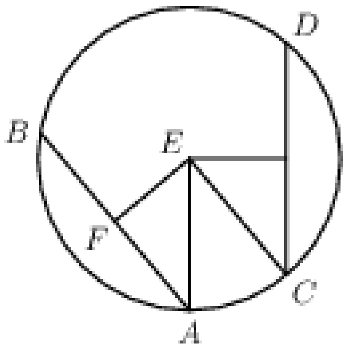
Proposition 14

In a circle equal straight lines are equally distant from the centre, and those which are equally distant from the centre are equal to one another.

Let $ABDC$ be a circle, and let AB, CD be equal straight lines in it;

I say that AB, CD are equally distant from the centre.

For let the centre of the circle $ABDC$ be taken [III. 1], and let it be E; from E let EF, EG be drawn perpendicular to AB, CD, and let AE, EC be joined.



Then, since a straight line EF through the centre cuts a straight line AB not through the centre at right angles, it also bisects it. [III. 3]

Therefore AF is equal to FB;

therefore AB is the double of AF.

For the same reason

CD is also the double of CG;

and AB is equal to CD;

therefore AF is also equal to CG.

And, since AE is equal to EC,

the square on AE is also equal to the square on EC.

But the squares on AF, EF are equal to the square on AE, for the angle at F is right;

and the squares on EG, GC are equal to the square on EC, for the angle at G is right; [I. 47]

therefore the squares on AF, FE are equal to the squares on CG, GE,

of which the square on AF is equal to the square on CG, for AF is equal to CG;

therefore the square on FE which remains is equal to the square on EG,

therefore EF is equal to EG.

But in a circle straight lines are said to be equally distant from the centre; that is, let EF be equal to EG.

I say that AB is also equal to CD.

For, with the same construction, we can prove, similarly, that AB is double of AF, and CD of CG.

And, since AE is equal to CE,

the square on AE is equal to the square on CE.

But the squares on EF, FA are equal to the square on AE, and the squares on EG, GC equal to the square on CE. [I. 47]

Therefore the squares on EF, FA are equal to the squares on EG, GC,

of which the square on EF is equal to the square on EG, for EF is equal to EG;

therefore the square on AF which remains is equal to the square on CG;

therefore AF is equal to CG.

And AB is double of AF, and CD double of CG;

therefore AB is equal to CD.

Therefore etc.

Q.E.D.

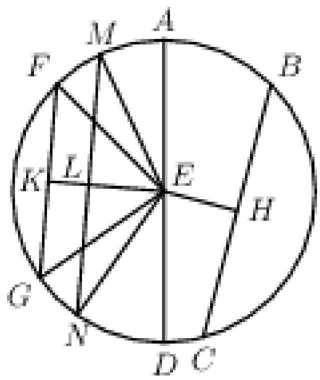
Proposition 15

Of straight lines in a circle the diameter is greatest, and of the rest the nearer to the centre is always greater than the more remote.

Let $ABCD$ be a circle, let AD be its diameter and E the centre; and let BC be nearer to the diameter AD, and FG more remote;

I say that AD is the greatest and BC greater than FG.

For from the centre E let EH, EK be drawn perpendicular to BC, FG.



Then, since BC is nearer to the centre and FG more remote, EK is greater than EH. [III. Def. 5]

Let EL be made equal to EH, through L let LM be drawn at right angles to EK and crried through to N, and let ME, EN, FE, EG be joined.

Then, since EH is equal to EL,

BC is also equal to MN. [III. 14]

Again, since AE is equal to EM, and ED to EN,

AD is equal to ME, EN.

But ME, EN are greater than MN, and MN is equal to BC;

therefore AD is greater than BC.

And since the two sides ME, EN are equal to the two sides FE, EG,

and the angle MEN greater than the angle FEG,

therefore the base MN is greater than the base FG. [I. 24]

But MN was proved equal to BC.

Therefore the diameter AD is the greatest and BC greater than FG.

Therefore etc.

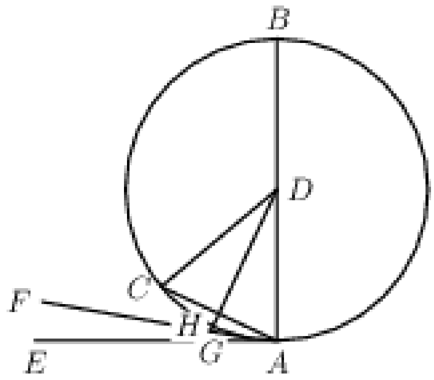
Q.E.D.

Proposition 16

The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle.

Let ABC be a circle about D as centre and AB as diameter; I say that the straight line drawn from A at right angles to AB from its extremity will fall outside the circle.

For suppose it does not, but, if possible, let it fall within as CA, and let DC be joined.



Since DA is equal to DC,

the angle DAC is also equal to the angle ACD. [I. 5]

But the angle DAC is right;

therefore the angle ACD is also right:

thus, in the triangle ACD, the two angles DAC, ACD are equal to two right angles: which is impossible. [I. 17]

Therefore the straight line drawn from the point A at right angles to BA will not fall within the circle.

Similarly we can prove that neither will it fall on the circumference;

therefore it will fall outside.

Let it fall as AE;

I say next that into the space between the straight line AE and the circumference CHA another straight line cannot be interposed.

For, if possible, let another straight line be so interposed, as FA, and let DG be drawn from the point D perpendicular to FA.

Then, since the angle AGD is right,

and the angle DAG is less than a right angle,

AD is greater than DG. [I. 19]

But DA is equal to DH;

therefore DH is greater than DG, the less than the greater, which is impossible.

Therefore another straight line cannot be interposed into the space between the straight line and the circumference.

I say further that the angle of the semicircle contained by the straight line BA and the circumference CHA is greater than any acute rectilineal angle,

and the remaining angle contained by the circumference CHA and the straight line AE is less than any acute rectilinear angle.

For, if there is any rectilineal angle greater than the angle contained by the straight line BA and the circumference CHA, and any rectilineal angle less than the angle contained by the circumference CHA and the straight line AE, then into the space between the circumference and the straight line AE a straight line will be interposed such as will make an angle contined by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, and another angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.

But such a straight line cannot be interposed;

therefore there will not be any acute angle contained by straight lines which is greater than the angle contained by the straight line BA and the circumference CHA, nor yet any acute angle contained by straight lines which is less than the angle contained by the circumference CHA and the straight line AE.—

PORISM. From this it is manifest that the straight line drawn at right angles to the diameter of a circle from its extremity touches the circle.

Q.E.D.