Material included in the NCCA Post-Primary Geometry Syllabus: Theorem 12
Work in Progress

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April 27, 2016

Theorem 12 (NCCA). *Let $\triangle ABC$ be a triangle. If a line ℓ is parallel to BC and cuts $[AB]$ in the ratio $s : t$, then it also cuts $[BC]$ in the same ratio.*

Proof (*NCCA*). We prove only the commensurable case.

Let ℓ cut $[AB]$ in D in the ratio $m : n$ with natural numbers m, n. Thus there are points (Figure NCCA:17) equally spaced along $[AB]$, i.e., the segments

$$[D_0D_1],\ \ [D_1D_2], \ldots [D_iD_{i+1}], \ldots [D_{m+n-1}D_{m+n}]$$

have equal length.

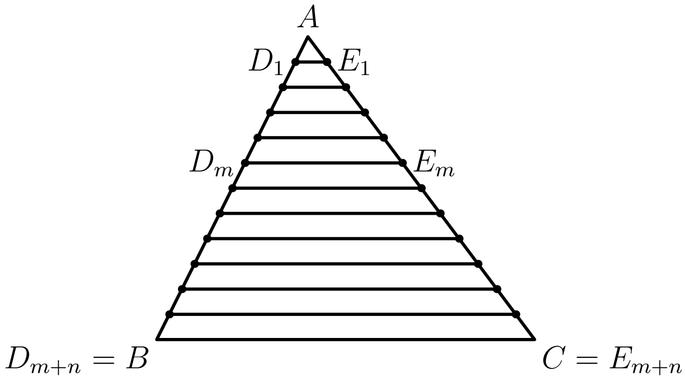


Figure NCCA:17

Draw lines $D_1E_1, D_2E_2, \ldots$ parallel to BC with $E_1, E_2, \ldots$ on $[AC]$.

Then all the segments

$$[AE_1],\ [E_1E_2],\ [E_2E_3], \ldots,\ [E_{m+n-1}C]$$

have the same length [Theorem 11], and $E_m = E$ has the point where ℓ cuts $[AC]$ [Axiom of Parallels]. Hence E divides $[AC]$ in the ratio $m : n$.

Q.E.D.

Remark (*DRW*). The above proof of Theorem 12 is taken directly from the document *Geometry Course for Post-Primary Mathematics* prepared by Professor Anthony O'Farrell. It should be noted that only the commensurable case in which s and t are both natural numbers is covered.

We give below a proof covering both commensurable and incommensurable cases. The proof makes use of the results of the following lemmas.

Lemma 12A (DRW). *Let u and v be real numbers satisfying $0 < u < v$. Then there exist natural numbers p and q such that*

$$u < \frac{p}{q} < v.$$

Proof (*DRW*). Let q be a natural number chosen large enough to ensure that

$$q > \frac{1}{v-u},$$

and let p be the smallest natural number for which

$$\frac{p}{q} > u.$$

Then

$$\frac{p-1}{q} \leq u,$$

and therefore

$$\frac{p}{q} = \frac{p-1}{q} + \frac{1}{q} \leq u + \frac{1}{q} < u + (v-u) = v.$$

Thus

$$u < \frac{p}{q} < v,$$

as required.

Q.E.D.

We denote by $\mathbb{R}^+$ the set of all positive real numbers. A real number x is *positive* if and only if $x > 0$. Thus

$$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}.$$

Lemma 12B (DRW). *Let $f: \mathbb{R}^+ \to \mathbb{R}^+$ be a function mapping the set $\mathbb{R}^+$ of positive real numbers into itself. Suppose that $f(a) < f(b)$ for all real numbers a and b satisfying $0 < a < b$. Suppose also that $f(p/q) = p/q$ for all natural numbers p and q. Then $f(x) = x$ for all positive real numbers x.*

Proof (*DRW*). We prove the result by showing that contradictions would arise were there to exist a positive real number x for which either $f(x) > x$ or $f(x) < x$. Suppose that x were a positive real number for which $f(x) > x$. It would then follow from Lemma 12A that there would exist natural numbers p and q for which $x < p/q < f(x)$. But then $f(p/q) = p/q$ and therefore both $x < p/q$ and $f(x) > f(p/q)$, contradicting the requirement that $f(a) < f(b)$ for all real numbers a and b satisfying $0 < a < b$. Next suppose that x were a positive real number for which $f(x) < x$. It would then follow from Lemma 12A that there would exist natural numbers p and q for which $f(x) < p/q < x$. But then $f(p/q) = p/q$ and therefore both $p/q < x$ and $f(p/q) > f(x)$, contradicting the requirement that $f(a) < f(b)$

for all real numbers a and b satisfying $0 < a < b$. Thus there cannot exist any positive real number x for which either $f(x) > x$ or $f(x) < x$. It follows that $f(x) = x$ for all positive real numbers x, as required.

Q.E.D.

Lemma 12C (DRW). *Let s and t be positive real numbers, and let A and B be distinct points in the Euclidean plane. Then a point G of the interior of the line segment $[AB]$ divides that line segment in the ratio $s : t$ if and only if*

$$\frac{|AG|}{|AB|} = \frac{s}{s+t}.$$

Proof (*DRW*). Let

$$x = \frac{|AG|}{|AB|}.$$

Now $|AG| > 0$, $|GB| > 0$ and $|AB| = |AG| + |GB|$, because the point G lies in the interior of the line segment $[AB]$. It follows that

$$|GB| = |AB| - |AG| = (1-x)|AB|,$$

and therefore

$$\frac{|AG|}{|GB|} = \frac{x}{1-x}.$$

It follows that the point G divides the line segment $[AB]$ in the ratio $s : t$ if and only if

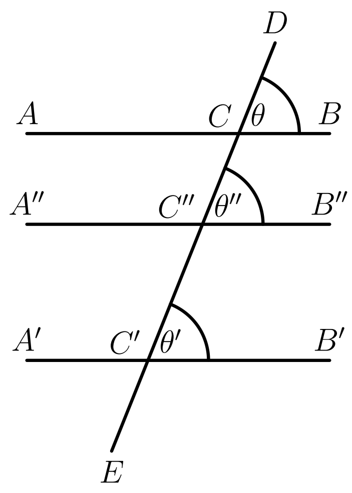
$$\frac{s}{t} = \frac{x}{1-x}.$$

But

$$\begin{aligned}
\frac{s}{t} = \frac{x}{1-x} \quad &\iff \quad \frac{(1-x)s}{t} = x \\
&\iff \quad \frac{s}{t} = \left(1 + \frac{s}{t}\right) x \\
&\iff \quad \frac{s}{t} = \left(\frac{s+t}{t}\right) x \\
&\iff \quad s = (s+t)x \\
&\iff \quad x = \frac{s}{s+t}.
\end{aligned}$$

The result follows.

Q.E.D.



Lemma 12D (DRW). (*Euclid*, Book I, Proposition 30). *Straight lines parallel to the same straight line are also parallel to one another.*

Proof (*DRW, following Euclid*). Let AB, $A'B'$ and $A''B''$ be straight lines, let DE be a transversal, intersecting AB at C, $A'B'$ at C' and $A''B''$ at C'', and let $\theta = \angle BCD$, $\theta' = \angle B'C'D$ and $\theta'' = \angle B''C''D$. Suppose that the lines AB and $A'B'$ are both parallel to the line $A''B''$. Then θ, θ' and θ'' are corresponding angles with respect to the traversal LM. The Corresponding Angles Theorem (Theorem 5) ensures that $\theta = \theta''$ and $\theta' = \theta''$. It follows that $\theta = \theta'$. The Corresponding Angles Theorem then ensures that the lines AB and $A'B'$ are parallel, as required.

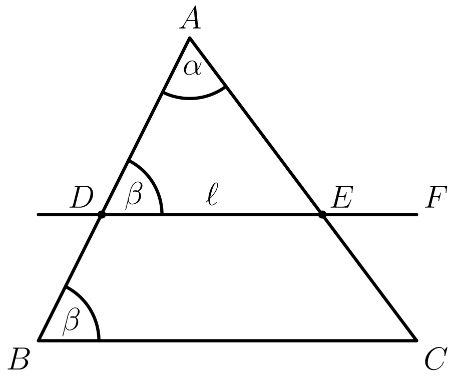
Q.E.D.

Remark (*DRW*). The statement for Lemma 12D is taken from Heath (1908a, p. 314).

Lemma 12E (DRW). *Let $\triangle ABC$ be a triangle, let D be a point of the ray $[AB$, where $D \neq A$, and let ℓ be the line through the point D parallel to BC. Then ℓ intersects the ray $[AC$ at some point E of that ray, where $E \neq A$.*

Proof (*DRW*). Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. It follows from Theorem 4 (*Angle Sum*) that $\alpha + \beta < 180°$.

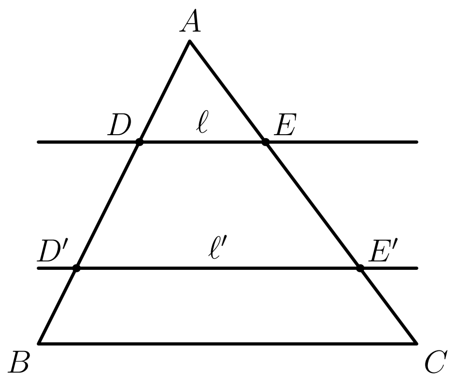
Let F be a point on the line ℓ through the point D parallel to BC that lies on the same side of A as the point C. Now the line AB is a transversal cutting the lines BC and DF. Moreover the lines BC and DF are parallel. It follows from Theorem 5 (*Corresponding Angles*) that the corresponding angles determined by these lines at B and D respectively are equal, and therefore $\angle ADF = \angle ABC = \beta$. The line AB also cuts the lines AC and



DF. But, because $\alpha + \beta < 180^0$, the corresponding angles at A and D are unequal. It follows from Theorem 5 (*Corresponding Angles*) that the lines AC and DF are not parallel. These lines must therefore intersect at some point E. Moreover, because $\angle DAC + \angle ADF = \alpha + \beta < 180°$, the point E must lie on the same side of the line AB as the point C, and therefore the point E must lie on the ray $[AC$. The result follows.

Q.E.D.

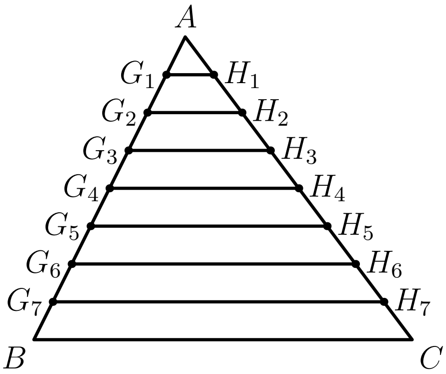
Lemma 12F (DRW). *Let $\triangle ABC$ be a triangle, and let D and D' be points of the ray $[AB$, where the points A, D and D' are distinct and D lies between A and D'. Let E and E' be the points on the ray $[AC$ at which that ray is cut by the lines ℓ and ℓ' parallel to BC that pass through the points D and D' respectively. Then the points A, E and E' are distinct, and E lies between A and E'.*



Proof (*DRW*). The lines ℓ and ℓ' are both parallel to BC. It follows from Lemma 12D above that the lines ℓ and ℓ' are parallel to one another, and therefore the line segment $[D'E']$ does not intersect the line ℓ. The points D' and E' must therefore lie on the same side of the line ℓ. Moreover the line segment $[AD']$ intersects the line ℓ at the point D and therefore the points A and D' lie on opposite sides of the line ℓ. It follows that the points A and E' must lie on opposite sides of the line ℓ, and therefore the line segment $[AE']$ must intersect the line ℓ at some point in the interior of that line segment. The result follows.

Q.E.D.

Lemma 12G (DRW). *Let $\triangle ABC$ be a triangle, let G be a point distinct from A that lies on the ray $[AC$, and, for each natural number j, let G_j be the unique point lying on the ray $[AC$ for which $[AG_j] = j[AG]$, let ℓ_j be the line parallel to BC that passes through the point D_j, and let H_j be the point at which the line ℓ_j cuts the ray $[AC$. Then $[AH_j] = j[AH_1]$ for all natural numbers j.*



Proof (*DRW*). The definition of G_j for all natual numbers j ensures that

$$|AG_1| = |G_1G_2| = |G_2G_3| = |G_3G_4| = \cdots.$$

It then follows from Theorem 11 that

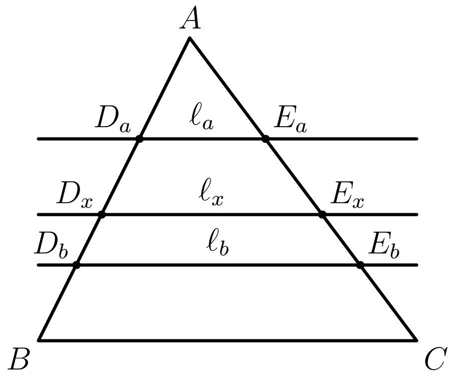
$$|AH_1| = |H_1H_2| = |H_2H_3| = |H_3H_4| = \cdots$$

and therefore $|AH_j| = j|AH_1|$ for all natural numbers j, as required.

Q.E.D.

Proof of Theorem 12 (*DRW*). We construct a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ from the set $\mathbb{R}^+$ of all positive real numbers to itself as follows.

Given any positive real number x, there exists a unique point D_x of the ray $[AB$ for which $|AD_x| = x|AB|$ (Axiom 2). A line ℓ_x can then be constructed through D_x parallel to BC (Axiom 5, see also Construction 5). It follows



from Lemma 12E above that the line ℓ_x must intersect the ray $[AC$ at some point E_x. We define

$$f(x) = \frac{|AE_x|}{|AC|}.$$

Then $f(x) > 0$ for all positive real numbers x. We shall prove that $0 < f(a) < f(b)$ for all real numbers a and b satisfying $0 < a < b$. We shall also prove that $f(p/q) = p/q$ for all natural numbers p and q. It will then follow from Lemma 12B that $f(x) = x$ for all positive real numbers x, and the required result will then follow.

Let a and b be real numbers satisfying $0 < a < b$, and let D_a and D_b be the unique points on the ray $[AB$ for which $|AD_a| = a|AB|$ and $|AD_b| = b|AB|$. Then $0 < |AD_a| < |AD_b|$, and therefore the points A, D_a and D_b are distinct, and the point D_a lies between A and D_b. Let E_a and E_b be the points at which the ray $[AC$ is cut by the lines ℓ_a and ℓ_b respectively, where ℓ_a and ℓ_b are parallel to BC, the line ℓ_a cuts $[AB$ at D_a and the line ℓ_b cuts $[AB$ at D_b. It follows from Lemma 12F above that the points A, E_a and E_b are distinct, and E_a lies between A and E_b, and thus $0 < |AE_a| < |AE_b|$. It follows that $0 < f(a) < f(b)$ for all real numbers a and b satisfying $0 < a < b$.

Let p and q be natural numbers and let $u = 1/q$. Then $B = D_{qu}$ and $C = E_{qu}$. Moreover $|AD_{ju}| = j|AD_u|$ for all natural numbers j. It follows from Lemma 12G that $|AE_{ju}| = j|AE_u|$ for all natural numbers j, and therefore

$$f(p/q) = f(pu) = \frac{|AE_{pu}|}{|AC|} = \frac{|AE_{pu}|}{|AE_{qu}|} = \frac{p|AE_u|}{q|AE_u|} = \frac{p}{q}.$$

We have now shown that the function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $0 < f(a) < f(b)$ for all real numbers a and b satisfying $0 < a < b$. We have also shown that $f(p/q) = p/q$ for all natural numbers p and q. It follows from Lemma 12B that $f(x) = x$ for all positive real numbers x, and thus

$$\frac{|AD_x|}{|AB|} = \frac{|AE_x|}{|AC|}$$

for all positive real numbers x. It now follows from Lemma 12C that, for all real numbers x satisfying $0 < x < 1$, the points D_x and E_x at which the line ℓ_x intersects the line segments $[AB]$ and $[AC]$ divide those line segments in the same ratio, as required.

Q.E.D.

Bibliography

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