Some Comments on Leaving Certificate Geometry
Work in Progress

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Axioms for Leaving Certificate Geometry

In 1932, George D. Birkhoff published a paper entitled *A set of postulates for plane geometry, based on scale and protractor* [*Annals of Mathematics*, Second Series, Vol. 33, No. 2 (1932), pp. 329–345. JSTOR stable URL: `http://www.jstor.org/stable/1968336`.] The first three axioms chosen for the development of synthetic geometry on the Leaving Certificate syllabus from 2013 onwards clearly derive from some of Birkhoff's axioms.

Axiom 1 (Two Points Axiom, LCG). *There is exactly one line through any two given points. (The line through A and B is denoted by AB.)*

Remark (*DRW*). The "Two Points Axiom" (Axiom 1) corresponds to Birkhoff's "Point-Line Postulate".

Remark (*DRW*). Postulate 1 in Book I of Euclid's *Elements of Geometry* postulates that one can draw a straight line from any point to any other point. Many 17th, 18th and 19th century editions of Euclid's *Elements* replace the 4th and 5th postulates and the five "common notions" that are presumed to constitute Euclid's own axiom system with twelve "axioms". The 10th of these axioms asserts that "two straight lines cannot enclose a space". This axiom (which is presumed to have been interpolated into manuscripts of Euclid by later editors and copyists) ensures that at most one line segment can join any two given points.

Axiom 2 (Ruler Axiom, LCG). *The distance between points has the following properties:*

1. *the distance $|AB|$ is never negative;*
2. $|AB| = |BA|$*;*
3. *if C lies on AB, between A and B, then $|AB| = |AC| + |CB|$;*
4. *(marking off a distance) given any ray from A, and given any real number $k \geq 0$, there is a unique point B on the ray whose distance from A is k.*

Remark (*DRW*). The "Ruler Axiom" (Axiom 2) derives from Birkhoff's "Postulate of Line Measure".

Remark (*DRW*). Propositions 1, 2 and 3 in Book I of Euclid's *Elements of Geometry* together provide a construction for cutting off from a ray an initial segment equal (in length) to a given line segment.

Axiom 3 (Protractor Axiom, LCG). *The number of degrees in an angle (also known as its degree-measure) is always a number between* $0°$ *and* $360°$. *The number of degrees in an ordinary angle is less than* $180°$. *It has these properties:*

1. *a straight line has* $180°$*;*
2. *given a ray* $[AB$*, and a number* d *between* 0 *and* 180*, there is exactly one ray from* A *on each side of the line* AB *that make an (ordinary) angle having* d *degrees with the ray* AB*;*
3. *if* D *is a point within an angle* $\angle BAC$*, then*

$$|\angle BAC| = |\angle BAD| + |\angle DAC|.$$

Remark (*DRW*). The "Protractor Axiom" (Axiom 3) derives from Birkhoff's "Postulate of Angle Measure".

Remark (*DRW*). Given a point A on a line AB, the existence of rays making a given angle with the ray $[AB$ is the result of Proposition 23 in Book I of Euclid's *Elements*. The proof of this proposition and the preceding propositions show how such rays can be constructed.

Axiom 4 (Congruence Axiom, SAS+ASA+SSS, LCG). *If*

(SAS) $\quad |AB| = |A'B'|$, $|AC| = |A'C'|$ *and* $|\angle A| = |\angle A'|$,

or

(ASA) $\quad |BC| = |B'C'|$, $|\angle B| = |\angle B'|$ *and* $|\angle C| = |\angle C'|$,

or

(SSS) $\quad |AB| = |A'B'|$, $|BC| = |B'C'|$ *and* $|CA| = |C'A'|$

then the triangles $\triangle ABC$ *and* $\triangle A'B'C'$ *are congruent.*

Remark (*DRW*). The the "SAS" case of the Congruence Axiom above is related to Birkhoff's "Similarity Postulate" which, when translated into the notation of the Leaving Certificate syllabus, states that, given triangles ABC and $A'B'C'$, if there exists some positive real constant k such that

$$|A'B'| = k|AB|,\ |A'C'| = k|AC| \text{ and } |\angle A'| = |\angle A|,$$

then

$$|B'C'| = k|BC|,\ |\angle B'| = |\angle B| \text{ and } |\angle C'| = |\angle C|,$$

The triangles ABC and $A'B'C'$ are then similar.

Remark (*DRW*). The "SAS" case of the Congruence Axiom (Axiom 4) is the result of Proposition 4 in Book I of Euclid's *Elements of Geometry.* Euclid's proof of this proposition notoriously uses the technique of "applying" one triangle to another, superposing parts of one triangle on corresponding parts of the other, so that one can assume, without loss of generality, that the vertex A coincides with the vertex A', the ray $[AB$ coincides with the ray $[A'B'$ and the points C and C' lie on the same side of the line AB. Euclid then applies his Postulates and Common Notions to deduce that the vertices and edges of the first triangle coincide with the corresponding vertices and edges of the second triangle, which then ensures that lengths of corresponding sides are equal and that corresponding angles are equal. The possibility of "applying" a triangle to another in this way of course assumes that the image of a line or angle under a "Euclidean motion" resulting from a translation, rotation or reflection of the plane is equal to the line or angle that is mapped onto it.

The "SSS" case of the Congruence Axiom is the result of Proposition 8 in Book I of Euclid's *Elements.* The "ASA" case, together with the other ("AAS") case in which the measure of two angles and one side of the first triangle agree with the corresponding angles and side of the second triangle, constitute the result of Proposition 26 in Book I of Euclid's *Elements.*

Axiom 5 (Axiom of Parallels, LCG). *Given any line l and a point P, there is exactly one line through P that is parallel to l.*

Remark (*DRW*). The "Axiom of Parallels" (Axiom 5) is *Playfair's Axiom*, introduced by John Playfair into his edition of Euclid's *Elements of Geometry*, published in 1795.

Euclid himself, in his Fifth Postulate, asserted that "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles". This postulate appears as the Twelfth Axiom in many editions of Euclid published in the 17th, 18th and 19th centuries.

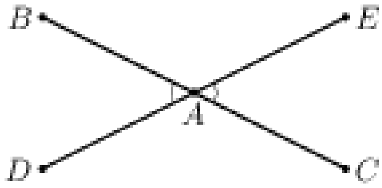
Sides of a Line

Axiom PSA (Plane Separation Axiom, [not included in the LCG syllabus]). *Any line in the Euclidean plane separates the remainder of the plane into exactly two sides that are opposite one another, so that if a point of the plane does not lie on the given line then it lies within exactly one of the two sides of the line. Moreover if two points A and B both lie on the same side of a line l then the line segment $[AB]$ joining A to B does not intersect l. But if those points A and B lie on opposite sides of the line l then the line segment $[AB]$ joining A to B intersects the line l.*

Remark (*DRW*). The statement of Theorem 3 on the Leaving Certificate Geometry syllabus refers to points being on opposite sides of line.

Vertically Opposite Angles

Theorem 1 (LCG). (Vertically-opposite Angles)
Vertically opposite angles are equal in measure.



Proof (*DRW*). Let A be a point that lies on distinct lines BC and DE. We must prove that $|\angle BAD| = |\angle CAE|$.

The angles $\angle BAD$ and $\angle BAE$ are supplementary angles and therefore $|\angle BAD| + |\angle BAE| = 180°$. Similarly the angles $\angle CAE$ and $\angle BAE$ are supplementary angles and therefore $|\angle CAE| + |\angle BAE| = 180°$. On subtracting $|\angle BAE|$ from both sides of these equalities, we find that $|\angle BAD| = |\angle CAE|$, as required.

Q.E.D.

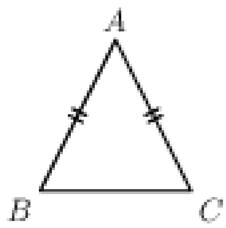
Remark (*DRW*). The statement and proof in the Leaving Certificate geometry syllabus correspond to the above proof, and are essentially identical to the statement and proof of Proposition 15 in Book I of Euclid's *Elements*. Euclid's proof makes use of Proposition 13, which ensures that supplementary angles always add up to two right angles.

Isosceles Triangles

Theorem 1 (LCG). (Isosceles Triangles)

(1) *In an isosceles triangle the angles opposite the equal sides are equal.*

(2) *Conversely, if two angles are equal, then the triangle is isosceles.*



Proof (*LCG*). (1) Suppose the triangle $\triangle ABC$ has $|AB| = |AC|$ (as in the figure). Then
$\triangle ABC$ is congruent to $\triangle ACB$ [SAS],
therefore $\angle B = \angle C$.

(2) Suppose now that $|\angle B| = |\angle C|$. Then
$\triangle ABC$ is congruent to $|triangle ACB$ [ASA],
therefore $|AB| = |AC|$, and thus $\triangle ABC$ is isosceles.

Q.E.D.

Remark (*DRW*). The result of part (1) is that of Proposition 4 in Book I of Euclid's *Elements of Geometry.* This is the famous *Pons Asinorum* or "Bridge of Asses". The proof method presented in the Leaving Certificate geometry syllabus was described by Proclus (412–485 A.D.) who attributed it to Pappus (4th century A.D.).

Pappus's proof, as presented by Proclus, was translated by T.L. Heath (*The Thirteen Books of Euclid*, Vol 1, 2nd edition, 1908, p. 254, as follows.

> "Let ABC be an iscosceles triangle, and AB equal to AC.
>
> "Let us conceive this one triangle as two triangles, and let us argue in this way.
>
> "Since AB is equal to AC, and AC to AB, the two sides AB, AC are equal to the two sides AC, AB.
>
> "And the angle BAC is equal to the angle CAB, for it is the same.
>
> "Therefore the corresponding parts (in the triangles) are equal, namely

BC to BC,
the triangle ABC to the triangle ABC (i.e., ACB),
the angle ABC to the angle ACB,
and the angle ACB to the angle ABC,
(for these are the angles subtended by the equal sides AB, AC).

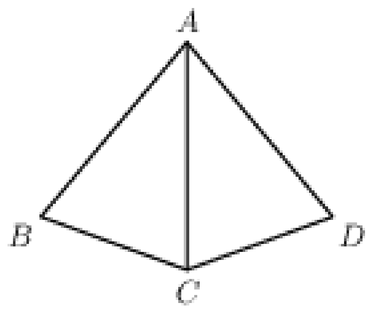
"Therefore in isosceles triangles the angles at the base are equal."

However the validity of this argument was rejected by many down towards the end of the 19th century. The proof in Euclid's *Elements* were far more complicated, as were variant proofs by Proclus and others.

The second part of the statement of Theorem 2 above is the result of Proposition 6 in Book I of Euclid's *Elements*.

Remark (*DRW*). In his edition of the first six books of Euclid's *Elements*, An edition of Euclid's *Elements of Geometry* was published in 1885 by John Casey, a graduate of Trinity College Dublin who became Professor of the Higher Mathematics and of Mathematical Physics in the Catholic University of Ireland, and subsequently Lecturer in Mathematics at University College, Dublin. Casey offers (on p.12), the following proof that the angles of an isosceles triangle opposite the equal sides are equal:—

> "The following is a very easy proof of this Proposition. Conceive the $\triangle ACB$ to be turned, without alteration, round the line AC, until it falls on the other side. Let ACD be its new position; then the angle ADC of the displaced triangle is evidently equal to the angle ABC, with which it originally coincided. Again, the



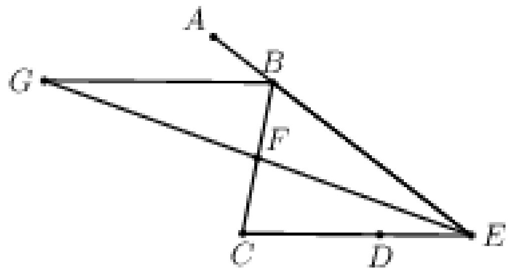
> two $\triangle$s BAC, CAD have the sides BA, AC of one respectively equal to the sides AC, AD of the other, and the included angles equal; therefore [Proposition 4] the angle ACB opposite the side AB is equal to the angle ADC opposite the side AC; but the angle ADC is equal to ABC; therefore ACB is equal to ABC."

External Angles exceed Internal Angles

Proposition 3A (DRW). *Suppose that A and D are on opposite sides of the line BC, and that the lines AB and CD intersect at E, where E lies on the same side of BC as D. Then $|\angle BCD| < |\angle ABC|$.*

Proof (*DRW, axiomatic, using LCG axioms*).

It follows from the Ruler Axiom (LCG, Axiom 2, (4)) that there exists a point F on the ray $[BC$ for which $|BF| = \frac{1}{2}|BC|$. There then exists a point G on the ray $[EF$ for which $|EG| = 2|EF|$. (This also follows from the Ruler Axiom (LCG, Axiom 2, (4).) The point F then the midpoint of the line segments $[BC]$ and $[EG]$, and lies in the interior of those line segments.



Now the points C and F both lie on the same side of the line AB, because they both lie on a ray $[BC$ whose endpoint B lies on AB. Similarly the points F and G both lie on the same side of the line AB, because they both lie on a ray $[EF$ whose endpoint E lies on the line AB. Therefore the points C and G are both on the same side of the line AB, because they are both on that side of the line AB that contains the point F.

Also the line segments $[EG]$ and $[BC]$ intersect at the point F, and therefore the point C lies in the interior of the ordinary angle between the rays $[BE$ and $[BG$. It follows from the Protractor Axiom (LCG, Axiom 3, (1) and (3)) that

$$|\angle EBC| + |\angle GBC| = |\angle EBG| < 180^\circ.$$

Now angles $\angle EBC$ and $\angle ABC$ are supplementary angles, because the point E, B and A are collinear, and therefore

$$|\angle EBC| + |\angle ABC| = 180^\circ.$$

(see LCG, Definition 2 and LCG, Definition 16). It follows that

$$|\angle EBC| + |\angle CBG| < |\angle EBC| + |\angle ABC|.$$

On subtracting $|\angle EBC|$ from both sides of this inequality, we find that $|\angle CBG| < |\angle ABC|$.

Now $|FB| = |FC|$ and $|FE| = |FG|$, because F is the midpoint of the line segments $[BC]$ and $[EG]$. Moreover $\angle BFG$ and $\angle CFE$ are vertically opposite angles, and therefore $|\angle BFG| = |\angle CFE|$ (LCG, Theorem 1). It therefore follows from the Congruence Axiom (LCG, Axiom 4, SAS) that the triangle BFG is congruent to the triangle CFE, and therefore $|\angle ECF| = |\angle GBF|$, But $\angle BCD = \angle FCE = \angle ECF$ and $\langle GBC = \langle GBF$, because F lies on $[BC]$ and D lies on $[CE]$. It follows that $|\angle BCD| = |\angle GBC|$. But we showed earlier that $|\angle GBC| < |\angle ABC|$. It follows that $|\angle BCD| < |\angle ABC|$, as required.

Q.E.D.

Remark (*DRW*). The statement and proof of Proposition 3A are adapted from the statement and proof of Euclid, Book I, Proposition 16.

The proof has been written out at length, to seek to ensure that the assumptions made are clearly set out. But the basic ideas can be summarized as follows.

From the basic construction, it is clear that the line segments $[CF]$ and $[FG]$ do not intersect the line AB. Therefore the points C, F and G must all lie on the same side of the line AB. It follows that $\angle EBG$ must be an ordinary angle that contains the point F and therefore contains the point C that lies on the ray $[BF$. It follows that

$$|\angle EBC| + |\angle CBG| < 180^\circ.$$

But $\angle EBC$ and $\angle ABC$ are supplementary angles, and therefore

$$|\angle EBC| + |\angle ABC| = 180^\circ.$$

It follows that $|\angle CBG| < |\angle ABC|$.

But the construction has been designed to ensure that the triangles BFG and CFE are congruent. Moreover it is clear from the construction that $\angle FBG = \angle CBG$ and $\angle FCE = \angle BCD$. Therefore

$$|\angle BCD| = |\angle CBG| < |\angle ABC|.$$

Corollary 3B (DRW). *An exterior angle of a triangle always exceeds the interior and opposite angles.*

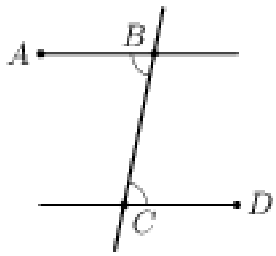
Remark (*DRW*). The above corollary is Proposition 16 in Book I of Euclid's *Elements of Geometry*.

Alternate Angles

Theorem 3 (LCG). (Alternate Angles)
Suppose that A and D are on opposite sides of the line BC.

(1) If $|\angle ABC| = |\angle BCD|$, then $AB \parallel CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.

(2) Conversely, if $AB \parallel CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.



Proof (*DRW*). First we prove (1). Suppose that $|\angle ABC| = |\angle BCD|$. If it were the case that the lines AB and CD were not parallel then they would intersect on one or other side of the line BC. If they intersected on the side of the line containing the point D then the point D would lie between the point C and the intersection point E of the lines AB and CD. It would then follow from Proposition 3A [not included in the LCG syllabus] that $|\angle BCD| < |\angle ABC|$. Similarly if they intersected on the other side of the line BC, then one could apply Proposition 3A [not included in the LCG syllabus] (with A, B, C and D in the statement of the proposition replaced by D, C, B and A respectively) to deduce that $|\angle ABC| < |\angle BCD|$. But the alternate angles $|\angle ABC|$ and $|\angle BCD|$ are equal. It follows that the lines AB and CD cannot intersect on either side of the line BC, and therefore these lines are parallel.

Conversely suppose that the lines AB and CD are parallel. We must show that $|\angle ABC| = |\angle BCD|$. Now the Protractor Axiom (Axiom 3) ensures the existence of a point E, on the same side of BC as the point A, such that $|\angle EBC| = |\angle BCD|$. It follows from what we have already proved that $EB \parallel CD$. But the Axiom of Parallels (Axiom 5) ensures that there is only one line through the point B parallel to CD. It follows that the points A, B and E must be collinear, and therefore $|\angle ABC| = |\angle EBC| = |\angle BCD|$, as required.

Q.E.D.

Remark (*DRW*). The result that if the alternate angles made by two lines on a transversal are equal then the lines are parallel is the result of Proposition 27 in Book I of Euclid's *Elements of Geometry*. The deduction of this result from Proposition 3A [not included in the LCG syllabus] follows the standard deduction of Proposition 27 from Proposition 16 of Book I of Euclid's *Elements of Geometry*.

The proof that alternate angles determined by parallel lines are equal closely follows John Playfair's proof of Proposition 29 in Book I of his edition of Euclid's *Elements of Geometry*, published in 1795. Proposition 23 in Book I of Euclid ensures the existence of the point E for which $|\angle EBC| = |\angle BCD|$. The proof is completed using *Playfair's Axiom*, which is Axiom 5 of the Leaving Certificate Geometry syllabus.

Remark (*DRW*). The proof given in the Leaving Certificate geometry syllabus (2013) uses the method of *reductio ad absurdum* in a fairly long sequence of steps to draw conclusions after successive steps that are very much at variance with the visual appearance of the accompanying diagram.