Some Comments on Leaving Certificate Geometry
Alternate Angles (Theorem 3)
Work in Progress

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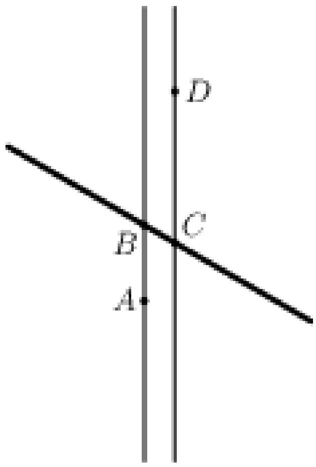
Alternate Angles

Suppose that a railway crosses the equator of the earth running from due south to due north across a flat featureless plane that extends in all directions as far as the eye can see. The rails give the appearance of maintaining a constant distance from one another. At the point where the railway crosses the equator, a bridge crosses over the railway: the centre of the bridge is exactly halfway between the two rails of the railway and lies exactly on the equator, and crosses the equator at an angle of 30° from the northwest quadrant to the southeast quadrant, as shown in the diagram.



We suppose that the inhabitants in this region do not know whether the surface of the earth is flat or curved. They have debated both possibilities, but they have not journeyed far enough away from their home to determine the geometry of the surface of the world on which they live. But it is taken for granted that all geometric properties of the surface of the earth are preserved under a central inversion about a point that has the effect of a rotation of the earth's surface about that point through an angle of 180°.

Sheila, one of the local inhabitants, stands at the centre of the bridge, looking due north. She observes that the rail on her left makes an obtuse interior angle of 120° with the bridge, and the rail on her right makes an acute interior angle of 60° with the bridge. She swivels round to look due south, and again observes that the rail on her left makes an obtuse interior angle of 120° with the bridge, and the rail on her right makes an acute interior angle of 60° with the bridge. She also notes that of course the distance between the points on the bridge directly over the two rails is the same, whether she is looking north or south. And indeed the view looks exactly the same whether she looks north or south. So although she does not know whether or not the rails going northward would meet or not, assuming each rail ran constantly northwards without turning to the left or right, she nevertheless concludes from the symmetry of the situation that *if* the rails running due north were to converge and eventually intersect on being continued due north without turning to the left or right, *then* the rails running due south would also have to converge and eventually intersect on each being continued due south without turning to the left or right. Because it is taken for granted that the conclusions regarding the geometry of their world are preserved under central inversion, it is not possible for them to conceive that the rails, continued without turning to the left or right, would meet on one side of the bridge, but not on the other.

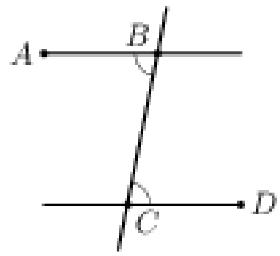
And indeed on the surface of our planet, ideally conceived to be a perfect sphere, the rails, if continued due northward and due southward, would meet at both the south pole and the north pole. This exemplifies a basic fact about spherical geometry that distinguishes spherical geometry from both flat and hyperbolic geometry: there exist pairs of conjugate points between which there exist multiple geodesics (following great circles of the sphere, such as the meridians along which longitude is constant) that join one conjugate point to the other. Thus, in spherical geometry, an axiom that states that between any two points there exists exactly one (length-minimizing) geodesic would not hold. Something different happens in "elliptic geometry", which is the geometry of real projective spaces (with the metric properties inherited from the sphere). In a world with "elliptic geometry" the point at which the rails running northward converge would coincide with the point at which the rails running southward would converge. The geodesic following the line of the bridge, produced indefinitely in both directions, would not separate the surface into two disconnected parts. Thus in "elliptic geometry" there exist geodesics (the "straightest possible" routes) that start out from a point and return to that point.

The scenario described above exhibits the basic symmetry property of complete two-dimensional surfaces of constant curvature that underlies the result of Theorem 3 of the *Project Maths* curriculum.

Theorem 3 (LCG). (Alternate Angles)
Suppose that A and D are on opposite sides of the line BC.

(1) If $|\angle ABC| = |\angle BCD|$, then $AB \parallel CD$. In other words, if a transversal makes equal alternate angles on two lines, then the lines are parallel.

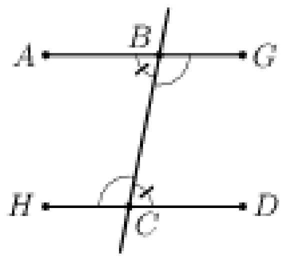
(2) Conversely, if $AB \parallel CD$, then $|\angle ABC| = |\angle BCD|$. In other words, if two lines are parallel, then any transversal will make equal alternate angles with them.



Proof (*DRW, based on LCG proof in NCCA 2013 syllabus*). First we prove (1). Suppose that

$$|\angle ABC| = |\angle BCD|.$$

Let G be a point taken on the line AB so that B lies between A and G,

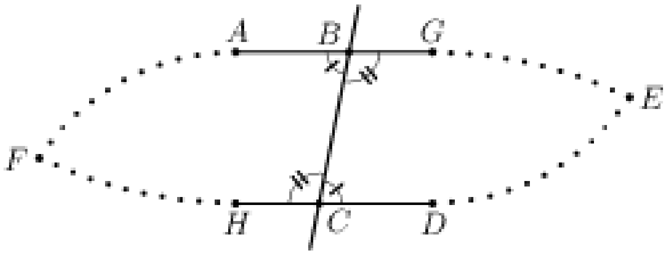


and let H be a point taken on the line DC so that C lies between H and D. Then $\angle ABC$ and $\angle GBC$ are supplementary angles, and also $\angle BCD$ and $\angle BCH$ are supplementary angles, and therefore

$$\begin{aligned} |\angle BCH| &= 180^\circ - |\angle BCD| \\ &= 180^\circ - |\angle ABC| \\ &= |\angle GBC|. \end{aligned}$$

Thus both $|\angle ABC| = |\angle BCD|$ and $|\angle GBC| = |\angle BCH|$.

We now prove that if the lines AB and CD were to intersect at some point on the same side of the line BC as the point D, then they would also have to intersect on the other side of BC. Thus suppose that the lines BG and CD were to meet at E, where E is on the same side of BC as the points D and G. Then a point F would lie on the line AB, on the same side of the line BC as the point A, with $|FB| = |EC|$. Join F to C and compare the triangles FBC and ECB.



The equality of the alternate angles would ensure that

$$|\angle FBC| = |\angle ABC| = |\angle BCD| = |\angle ECB|.$$

The Congruence Axiom (Axiom 4, SAS) would then ensure the congruence of the triangles FBC and ECB, because $|FB| = |EC|$, $|BC| = |CB|$ and $|\angle FBC| = |\angle ECB|$. It would then follow that $|\angle FCB| = |\angle EBC|$. But then the points F and H would both lie on the same side of the line BC, and

$$|\angle EBC| = |\angle GBC| = |\angle HCB|.$$

The Protractor Axiom (Axiom 3) would then ensure the collinearity of the points C, H and F. It would then follow that there would exist two distinct straight line segments joining F to E: one would pass through the points A, B and G and the other would pass through the points H, C and D. Moreover the points F and E would be distinct, because they would lie on opposite sides of the line BC. But the Two Points Axiom (Axiom 1) requires that there be only one line segment joining any two distinct points. Thus the assumption that the lines AB and CD intersect at some point on the same side of BC as the point D would lead to a contradiction. Moreover applying this result with A, B, C and D replaced by D, C, B and A respectively shows that a contradiction would also arise were the lines AB and CD to intersect on the same side of CD as the point A. Therefore the lines AB and CD must be parallel.

Conversely suppose that the lines AB and CD are parallel. We must show that $|\angle ABC| = |\angle BCD|$. Now the Protractor Axiom (Axiom 3) ensures the existence of a point K, on the same side of BC as the point A, such that

$|\angle KBC| = |\angle BCD|$. It follows from what we have already proved that $KB \parallel CD$. But the Axiom of Parallels (Axiom 5) ensures that there is only one line through the point B parallel to CD. It follows that the points A, B and K must be collinear, and therefore $|\angle ABC| = |\angle KBC| = |\angle BCD|$, as required.

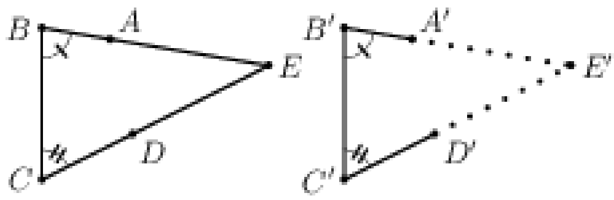
Q.E.D.

Remark (*DRW*). The result that if the alternate angles made by two lines on a transversal are equal then the lines are parallel is the result of Proposition 27 in Book I of Euclid's *Elements of Geometry*.

Congruence and Intersection

Proposition 3C (DRW). *Let A, B, C, D, A', B', C' and D' be distinct points of the plane, where the points A and D both lie away from the line BC and on one side of that line, and the points A' and D' both lie away from the line $B'C'$ and on one side of that line. Suppose also that $|BC| = |B'C'|$, $|\angle ABC| = |\angle A'B'C'|$ and $|\angle BCD| = |\angle B'C'D'|$. Then the rays $[BA$ and $[CD$ intersect if and only if the rays $[B'A'$ and $[C'D'$ intersect.*

Proof (*DRW*). Suppose that the rays $[BA$ and $[CD$ intersect at E. The Ruler Axiom (Axiom 2) that we can take a point E' on the ray $[B'A'$ for which $|B'E'| = |BE|$. Join $C'E'$.



The triangles $\triangle EBC$ and $\triangle E'B'C'$ must then be congruent. Indeed $|EB|$ and $|BC|$ are equal to $|E'B'|$ and $|B'C'|$ respectively and

$$|\angle EBC| = |\angle ABC| = |\angle A'B'C'| = |\angle E'B'C'|.$$

It follows from the Congruence Axiom (Axiom 4, SAS) that the two triangles are congruent and therefore $|\angle B'C'E'| = |\angle BCE|$. But then

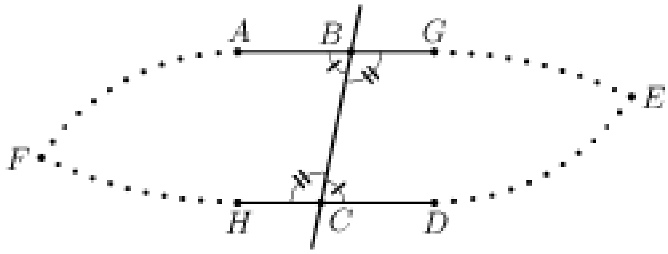
$$|\angle B'C'D'| = |\angle BCD| = |\angle BCE| = |\angle B'C'E'|.$$

Moreover the points D' and E' both lie on the same side of the line $B'C'$. It follows from the Protractor Axiom (Axiom 3) that the points C', D' and E' are colinear, and therefore the rays $[B'A'$ and $[C'D'$ intersect at E'.

We have proved that if the rays $[BA$ and $[CD$ intersect, then the rays $[B'A'$ and $[C'D'$ intersect. The converse follows directly on interchanging the roles of the two triangles. This completes the proof.

Q.E.D.

Remark (*DRW*). The above proposition (Proposition 3C [not included in the LCG syllabus]) is intended to capture the essence of the lengthy *reductio ad absurdum* section of the proof of Theorem 3 given previously. The proof of the first part of Theorem 3 assuming Proposition 3C [not included in the LCG syllabus] proceeds as follows.



We label points etc. as in the figure. The alternate angles $\angle ABC$ and $\angle BCD$ are equal. Therefore the corresponding supplementary angles are $\angle GBC$ and $\angle BCH$ are also equal, as explained in the first part of the given proof of Theorem 3. It then follows from Proposition 3C [not included in the LCG syllabus] that if the rays $[CD$ and $[BG$ were to intersect at some point E (on the right of the figure) then the rays $[BA$ and $[CH$ would have to intersect at some point F of the figure. We would thus obtain a result contradicting the Two Points Axiom (Axiom 1).