Euclid's *Elements of Geometry* Book I (Lardner's Edition)

Transcribed by D. R. Wilkins

November 2, 2017

PREFACE.

Two thousand years have now rolled away since Euclid's Elements were first used in the school of Alexandria, and to this day they continue to be esteemed the best introduction to mathematical science. They have been adopted as the basis of geometrical instruction in every part of the globe to which the light of science has penetrated; and, while in every other department of human knowledge there have been almost as many manuals as scheools, in this, and in this only, one work has, by common consent, been adopted as an universal standard. Euclid has been translated into the languages of England, France, Germany, Spain, Italy, Holland, Sweden, Denmark, Russia, Egypt, Turkey, Arabia, Persia nad China. This unprecedented unanimity in the adoption of one work as the basis of instruction has not arisen from the absence of other treatises on the same subject. Some of the most eminent mathematicians have written, either original Treatises, or modifications and supposed improvements of the Elements; but still the "Elements" themselves have been invariably preferred. To what can a preference so universal be attributed, if not to that singular perspicuity of arrangement, and that rigorous exactitude of demonstration, in which this celebrated Treatise has never been surpassed? 'To this', says Playfair, 'is added every association which can render a work venerable. It is the production of a man distinguished among the first instructors of the human race. It was almost the first ray of light which pervaded the darkness of the middle ages; and men still view with gratitude and affection the torch which rekindled the sacred fire, when it was nearly extinguished upon earth.'

It must not, however, be concealed, that, excellent as this Work is, many, whose opinions are entitled to respect, conceive that it needs much improvement; and some even think that it might be superseded with advantage by other Treatises. The Elements, as Dr. Robert Simson left them, are certainly inadequate to the purposes of instruction, in the present improved state of science. The demonstration are characterised by prolixity, and are not always expressed in the most happy phraseology. The formalities and paraphernalia of rigour are so ostentatiously put forward, as almost to hide the reality. Endless and perplexing repetitions, which do not confer greater exactitude on the reasoning, render the demonstrations involved and obscure, and conceal from the view of the student the consecution of evidence. Independent of this defect, it is to be considered that the "Elements" contain only the naked leading truths of Geometry. Numerous inferences may be drawn, which, though not necessary as links of the great chain, and therefore subordinate in importance, are still useful, not only as exercises for the mind, but in many of the most striking physical applications. These however are wholly omitted by Simson, and not supplied by Playfair.

When I undertook to prepare an elementary geometrical text-book for students in, and preparing for, the University of London, I wished to render it useful in places of education generally. In this undertaking, an alternative was presented, either to produce an original Treatise on Geometry, or to modify Simson's Euclid, so as to supply all that was necessary, and remove all that was superfluous; to elucidate what was obscure, and to abridge what was prolix, to retain geometrical rigour and real exactitude, but to reject the obtrusive and verbose display of them. The consciousness of inability to originate any work, which would bear even a remote comparison with that of the ancient Greek Geometer, would have been reason sufficient to decide upon the part I should take, were there no other considerations to direct my choice. Other considerations, however, there were, and some when seemed of great weight. The question was not, whether an elementary Treatise might noe be framed superior to the "Elements" as given by Simson and Playfair; but whether an original Treatise could be produced superior to what these Elements would become, when all the improvements of which they were susceptible had been made, and when all that was found deficient had been supplied. Let us for the present admit, that a new work were written on a plan different from that of Euclid, constructed upon different principles, built on different data, and exhibiting the leading results of geometrical science of a different order. Let us was also the great improbability, that even an experienced instructor should execute a work superior to that which has been stamped with the approbation of ages, and consecrated, as it were, by the collected suffrage of the whole civilised globe. Still it may be questioned whether, on the whole, any real advantage would be gained. It is certain that all would not agree in their decision on the merits of such a work. Euclid once superceded, every teacher would esteem his own work the best, and every school would have its own class-book. All that rigour and exactitude, which have so long excited the admiration of men of science, would be at an end. These very words would lose all definite meaning. Every school would have a different standard: matter of assumption in one, being matter of demonstration in others; until, at length, GEOMETRY, in the ancient sense of the word, would be altogether frittered away, or be only considered as a particular application of Arithmetic and Algebra.

Independently of the disadvantages which would attend the introduction of a great number of different geometrical classbooks into the schools, nearly all of which must be expected to be of a very inferior order, inconveniences of another kind would, I conceive, be produced by allowing Euclid's Elements to fall into disuse. Hitherto Euclid has been a universal standard of geometrical science. His arrangement of principles is registered in the memory of every mathematician of the present times, and is referred to in the works of every mathematician of past ages. The Books of Euclid, and their propositions, are as familiar to the minds of all who have been engaged in scientific pursuits, as the letters of the alphabet. The same species of inconvenience, differing only in degree, would arise from disturbing this universal arrangement of geometrical principles, as would be produced by changing the names and power of the letters. It is very probably, nay, it is certain, that a better classification of simples sounds and articulations could be found than the commonly received vowels and consonants; yet who would advocate a change?

In expressing my sentiments respecting Euclid's Work, as compared with others which have been proposed to supercede it, I may perhaps be censured for an undue degree of confidence in a case where some respectable opinions are opoosed to mine. Were I not supported in the most unqualified degree by authorities ancient and modern, the force of which seems almost irresistible, I should feel justly obnoxious to this charge. The objections which have been from time to time brought against this work, and which are still sometimes repeated, may be reduced to two classes; those against the arrangement, and those against the reasoning. My business is not to show that Euclid is perfect either in the one respect or the other, but to show that no other elementary writer has approached so near to perfection in both. It is important to observe, that validity of reasoning and vigour of demonstration are objects which a geometer should never lose sight of, and to which arrangement and every other consideration must be subordinate. LEIBNITZ, an authority of great weight on such a subject, and not the less so as being one of the fathers of modern analysis, has declared that the geometers who have disapproved of Euclid's arrangement have vainly attempted to change it without weakening the force of the demonstrations. Their unavailing attempts he considers to be the strongest proof of the difficulty of substituting, for the chain formed by the ancient geometer, any other equally strong and valid.[1] WOLF also acknowledges how futile it is to attempt to arrange geometrical truths in a natural or absolutely methodical order, without either taking for granted what has not been previously established, or relaxing in a great degree the rigour of demonstration.[2] One of the favorite arrangements of those who object to that of Euclid, has consisted in establishing all the properties of straight lines considered without reference to their length, intersecting obliquely and at right angles, as well as the properties of parallel lines, before the more complex magnitudes called triangles are considered.

[1] Montucla, tom. i. p. 205.

[2] Element. Math. tom. v. c. 3. art. 8.

In attempting this, it is curious to observe the difficulties into which these authors fall, and the expedients to which they are compelled to resort. Some find it necessary to prove that every point on a perpendicular to a given right line is equally distant from two points taken on the given right line at equal distances from the point where the perpendicular meets it. 'They imagine,' says *Montucla*, 'that they prove this by saying that the perpendicular does not lean more to one side than the other.' Again to prove that equal chords of a circle subtend equal arcs, they say that the uniformity of the circle produces this effect: that two circles intersect in no more than two points, and that a perpendicular is the shortest distance of a point from a right line, are propositions which they dispose of very summarily, by appealing to the evidence of the senses. They prefer an imperfect demonstration, or no demonstration at all, to any infringement of the order which they have assumed.

'There is a kind of puerility in this affectation of not mentioning a marticular modification of magnitude,—triangles, for example,—until we have first treated of lines and angles; for if any degree of geometrical rigour be required, as many and as long demonstrations are necessary as if we had at once commenced with triangles, which, though more complex modifications of magnitude, are still so simple that the student does not require to be led by degrees to them. Some have even gone so far as to think that this affectation of a natural and absolutely methodical order contracts the mind, by habituating it to a process of investigation contrary to that of discovery.'[3]

The mathematicians who have attempted to improve the reasoning of Euclid, have not been more successful than those who have tried to reform his arrangemen. Of the various objections which have been brought against Euclid's reasoning, two only are worthy of notice; viz. those respecting the twelfth axiom of the first book, which is sometimes called *Euclid's Postulate*, and those which relate to his doctrine of proportion. On the former I have enlarged so fully in Appendix II. that little remains to be said here. I have there shown that what is really assumed by Euclid is, that 'two right lines which diverge from the same point cannot be both parallel to the same right line;' or that 'more than one parallel cannot be drawn through a given point to a given right line.' The geometers who have attempted to improve this theory, have all either committed illogicisms, or assumed theorems less evident tha nthat which has just been expressed, and which seems ot me as evident as several of the other axioms. In the Appendix, I have stated at length some of the theories of parallels which have been proposed to supercede that of Euclid, and have shown their defects. Numerous have been

[3]Montucla, p. 206.

the attempts to demonstrate the twelfth axiom by the aid of the first twenty-eight propositions. Ptolemy, Proclus, Nasireddin, Clavius, Wallis, Saccheri, and a cloud of editors and commentators of former and later times, have assailed the problem without success.

The second source of objection, on the score of reasoning, is the definition of four proportional magnitudes prefixed to the fifth book. By this definition, four magnitudes will be proportional, if there be any equimultiples of the first and third, which are respectively equal to equimultiplies of the second and fourth. This is the common popular notion of proportion. But it is necessary to render the term more general in its geometrical application. Four magnitudes are frequently so related, that no equimultiples of the first and third are equal respectively to other equimultiples of the second and fourth, and yet have all the other properties of proportional quantities, and therefore it is necessary that they should be brought under the same definition. Euclid adapted his definition to embrace these, by declaring four magnitudes to be proportional when every pair of equimultiplies of the first and third were both greater, equal to, or less than equimultiples of the second and fourth. I agree with Playfair, in thinking that no other definition has every been given from which the properties of proportionals can be deduced by reasonings, which, at the same that they are perfectly rigorous, are also simple and direct. Were we content with a definition which would only include commensurate magnitudes, no difficulty would remain. But such a definition would be useless: for in almost the first instance in which it should be applied, the reasoning would either be inconclusive, or the result would not be sufficiently general.

In the second and fifth books, in addition to Euclid's demonstrations, I have in most instances given others, which are rendered more clear and concise by the use of a few of the symbols of algebra, the signification of which is fully explained, and which the student will find no difficulty in comprehending. The nature of the reasoning, however, is essentially the same, the language alone in which it is expressed being different.

The commentary and deductions are distinguished from the text of the Elements by being printed in a smaller character, and those articles in each book which are marked thus $^*{*}^*$, the student is advised to omit until the second reading.

No part of Euclid's Elements has attained the same celebrity, or been so universally studied, as the first six books. The seventh, eighth, and ninth books treat of the Theory of Numbers, and the Tenth is devoted ot the Theory of Incommensurable Quantities. Instead of the eleventh an twelfth books, I have added a Treatise on Solid Geometry, more suited to the present state of mathematical knowledge. For much of the materials of this treatise

I am indebted to Legendre's Geometry.

Appendix I. contains a short Essay on the Ancient Geometrical Analysis, whihc may be read with advantage after the sixth book. The second Appendix contains an account of the Theories of Parallels.

I have directed that the cuts of this work shall be published separately, in a small size, for the convenience of students who are taught in classes where the use of the book itself is not permitted.

London, May 1828.

PREFACE
TO
THE FOURTH EDITION.

Since the publication of the first edition of this Work, various additions and corrections have been made in it; the demonstrations of the solid geometry have been improved; the symbols of arithmetic and algebra have been introduced, wherever they have been found by experience to facilitate the progress of the student. Teachers will find the short view of the Theory of Transversals, which has been added to the Appendix, and excellent exercise for the more advanced class of students; independently of which it is of extensive usefulness in various practical applications of geometry.

Through the kind attention fo professors and teachers who have used this work in schools and the universities, the Editor has been enabled to discover and correct a vast number of small errors, which arose in the process of printing, and which could scarcely have been detected by any other means. The present edition is free from these errors; and, as the work has been stereotyped, it is hoped that it will be found in future to be more than usually correct. If, however, any minute errors may have escaped attention, the Editor will feel obliged to any teacher or student who will communicate them to the publisher.

The following observations, supplied by Professor De Morgan, on the manner of studying Euclid, are recommended to the attention of the student.

"In order clearly to perceive the connection which exists between the parts of a proposition, it is necessary to separate those sentences which contain independent assertions. This must be done, in fact, whatever be the method which the student pursues, before he can be said to have a clear conception of the proposition; but as the shortest way to accustom his mind to the separation of a demonstration into its constituent parts, I would recommend him to commit to writing the propositions of the first three Books, at least, taking case to place in separate paragraphs the different assertions of which each demonstration consists, with some reference to the manner in which each assertion is established.

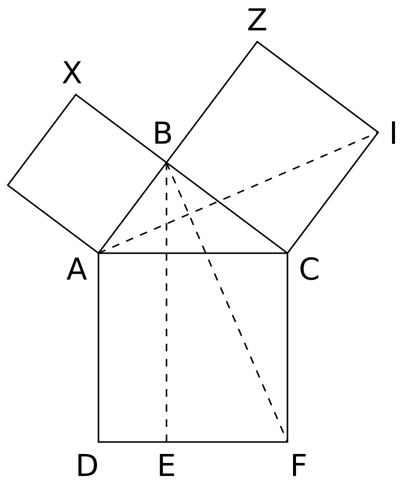
"To render this task more easy, I have subjoined an example, taken from the celebrated 47th proposition of the First Book, which he will here find treated in the manner in which it is desirable he should write each proposition. The number placed before each paragraph is intended for reference; and the student will see that to each assertion is attached the number of each previous one, by means of which it is established.

"Before the demonstration the student should write down briefly the enunciations of all the previous Theorems by means of which the one in hand is established; to these he may attach letters, by means of which he may refer to them in that part of the demonstration in which they become necessary.

The whole process is as follows:—

a If two triangles have two sides, and the included angle respectively equal, the two triangles are equal.

b If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallellogram is double of the triangle.



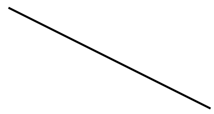
Proposition.			In a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the sides.
Hypothesis.	1.		A B C is a triangle,
Construction.	2.		Upon A B describe the square A X;
	3.		Upon B C describe the square B I;
	4.		Upon A C describe the square A F;
	5.		Draw B E parallel to C F or A D;
	6.		Join B and F;
	7.		Join A and I;
Demonstration.	8.	3. 4.	The angle I C B is equal to A C F;
	9.		Add the angle B C A to both;
	10.	8. 9.	I C A is equal to B C F;
	11.	3. 4.	Both I C and A C are respectively equal to B C and C F;
	12.	10. 11. *a*	the triangles A C I and B C F are equal;
	13.	3.	A Z is parallel to C I;
	14.	13. *b*	the parallelogram C Z is double of the triangle C A I;
	15.	5.	B E is parallel to C F;
	16.	15. *b*	The parallelogram C E is double of the triangle C B F;
	17.	12. 14. 16.	The figures C Z and C E are equal in area;
	18.		In like manner it can be shown that the figures A X and A E are equal in area;
	19.	17. 18.	Therefore the figure A F is equal to the sum of C Z and A X. Q. E. D.

"'This method may be considerably shortened by the use of some algebraical characters; but here the student must be cautious, as he may be very easily led into false, or at least unestablished, analogies, by the indiscriminate use of these symbols. For example: equal figures in geometry are those which can be made to coincide entirely; in algebraical language, two figures would be called equal which consist of the same number of square feet, though they could not be made to coincide. Therefore, if the student uses the symbolical notation, he must remember to express by different signs these different meanings of the word 'equality.' The word square has also different meanings in geometry and algebra; and, though custom has authorised the use of the *word* in two different senses, it is important that the beginner should attach one meaning only to the *sign*."

In the successive Editions through which this work has passed I have been much indebted to Mr. G. K. Gillespie, private teacher of the Classics and Mathematics, for various corrections which he has pointed out, and for several useful suggestions.

DEFINITIONS

(1) I. A *point* is that which has no parts.
(2) II. A *line* is length without breadth.
(3) III. The extremities of a line are points.
(4) IV. A *right line* is that which lies evenly between its extremities.



(5) V. A *surface* is that which has length and breadth only.
(6) VI. The extremities of a surface are lines.
(7) VII. A *plane surface* is that which lies evenly between its extremities.

(8) These definitions require some elucidation. The object of Geometry[4] is the properties of *figure*, and figure is defined to be the relation which subsists between the boundaries of space. Space or magnitude is of three kinds, *line*, *surface*, and *solid*. It may be here observed, once for all, that the terms used in geometrical science, are not designed to signify any real, material or physical existences. They signify certain abstracted notions or conceptions of the mind, derived, without doubt, originally from material objects by the senses, but subsequently corrected, modified, and, as it were, purified by the operations of the understanding. Thus, it is certain, that nothing exactly conformable to the geometrical notion of a right line ever existed; no edge, which the finest tool of an artist can construct, is so completely free from inequalities as to entitle it to be consisdered as a mathematical right line. Nevertheless, the first notion of such an edge being obtained by the senses, the process of mind by which we reject the inequalities incident upon the nicest mechanical production, and substitute for them, *mentally*, that perfect evenness which constitutes the essence of a right line, is by no means difficult. In like manner, if a pen be drawn over this paper an effect is produced, which, in common language, would be called a line, right or curved, as the case may be. This, however, cannot, in the strict geometrical sense of the term, be a *line* at all, since it has breadth as well as length; for if it had not it could not be made evident to the senses. But having first obtained this rude and incorrect notion of a line, we can imagine that, while its length remains unaltered, it may be infinitely attenuated until it ceases alteogether

[4]From γῆ, *terra*; and μέτρον, *mensura*.

to have breadth, and thus we obtain the exact conception of a mathematical line.

The different modes of magnitude are ideas so extremely uncompounded that their names do not admit of definition properly so called at all.[5] We may, however, assist the student to form correct notions of the true meaning of these terms, although we may not give rigorous logical definitions of them.

A notion being obtained by the senses of the smallest magnitude distinctly perceptible, this is called a *physical point*. If this point were indivisible even *in idea*, it would be strictly what is called a *mathematical point*. But this is not the case. No material substance can assume a magnitude so small that a smaller may not be imagined. The mind, however, having obtained the notion of an extremely minute magnitude, may proceed without limit in a mental diminution of it; and that state at which it would arrive if this diminution were infinitely continued is a *mathematical point*.[6]

The introduction of the idea of motion into geometry has been objected to as being foreign to that science. Nevertheless, it seems very doubtful whether we may not derive from motion the most distinct ideas of the modes of magnitude. If a mathematical point be conceived to move in space, and to mark its course by a trace or track, that trace or track will be a *mathematical line*. As the moving point has no magnitude, so it is evident that its track can have no breadth or thickness. The places of the point at the beginning and end of its motion, are the extremities of the line, which are therefore *points*. The third of the preceding definitions is not properly a definition, but a proposition, the truth of which may be inferred from the first two definitions.

As a *mathematical line* may be conceived to proceed from the motion of a *mathematical point*, so a *physical line* may be conceived to be generated by the motion of a *physical point*.

In the same manner as the motion of a point determines the idea of a line, the motion of a line may give the idea of a surface. If a mathematical line be conceived to move, and to leave in the space through which it passes a trace or track, this trace or track will be a surface; and since the line has no breadth, the surface can have no thickness. The initial and final positions of the moving line are two boundaries or extremities of the surface, and the other extremities are the lines traced by the extreme points of the line whose motion produced the surface.

The sixth definition is therefore liable to the same objection as the third.

[5]The name of a simple idea cannot be defined, because the general terms which compose the definition signifying several different ideas can by no means express an idea which has no manner of composition.—LOCKE.

[6]The Pythagorean definition of point is '*a monad having position.*'

It is not properly a definition, but a principle, the truth of which be derived from the fifth and preceding definitions.

It is scarcely necessary to observe, that the validity of the objection against introducing *motion* as a *principle* into the Elements of Geometry, is not here disputed, nor is it introduced as such. The preceding observations are designed merely as *illustrations* to assist the student in forming correct notions of the true mathematical significations of the different modes of magnitude. With the same view we shall continue to refer to the same mechanical ideas of motion, and desire our observations always to be understood in the same sense.

The fourth definition, that of a right or straight line, is objectionable, as being unintelligible; and the same may be said of the definition (seventh) of a plane surface. Those who do not know what the words 'straight line' and 'plane surface' mean, will never collect their meaning from these definitions; and to those who *do* know the meaning of those terms, definitions are useless. The meaning of the terms 'right line' and 'plane surface' are only to be made known by an appeal to experience, and the evidence of the senses, assisted, as was before observed, by the power of the mind called *abstraction*. If a perfectly flexible string be pulled by its extremities in opposite directions, it will assume, between the two points of tension, a certain position. Were we to speak without the rigorous exactitude of geometry, we should say that it formed a *straight line*. But upon consideration, it is plain that the string has weight, and that its weight produces a flexure in it, the convexity of which will be turned towards the surface of the earth. If we conceive the weight of the string to be extremely small, that flexure will be proportionably small, and if, by the process of abstraction, we conceive the string to have no weight, the flexure will altogether disappear, and the string will be accurately a straight line.

A straight line is sometimes defined 'to be the shortest way between two points.' This is the definition given by Archimedes, and after him by Legendre in his Geometry; but Euclid considers this as a property to be proved. In this sense, a straight line may be conceived to be that which is traced by one point moving towards another, which is quiescent.

Plato defines a straight line to be that whose extremity hides all the rest, the eye being placed in the continuation of the line.

Probably the best definition of a plane surface is, that it is such a surface that the right line, which joins every two points which can be assumed upon it, lies entirely in the surface. This definition, originally given by *Hero*, is substituted for Euclid's by R. Simson and Legendre.

Plato defined a plane surface to be one whose extremities hide all the intermediate parts, the eye being placed in its continuation.

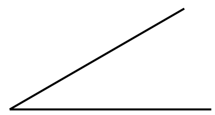
It has been also defined as ‘the smallest surface which can be contained between given extremities.’

Every line which is not a straight line, or composed of straight lines, is called a *curve*. Every surface which is not a plane, or composed of planes, is called a *curved surface*

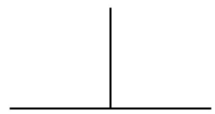
(9) VIII. A *plane angle* is the inclination of two lines to one another, in a plane, which meet together, but are not in the same direction.

This definition, which is designed to include the inclination of curves as well as right lines, is omitted in some editions of the Elements, as being useless.

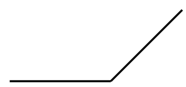
(10) IX. A *plane rectilinear angle* is the inclination of two right lines to one another, which meet together, but are not in the same right line.



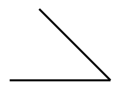
(11) X. When a right line standing on another right line makes the adjacent angles equal, each of these angles is called a *right angle*, and each of these lines is said to *perpendicular* to the other.



(12) XI. An *obtuse* angle is an angle greater than a right angle.

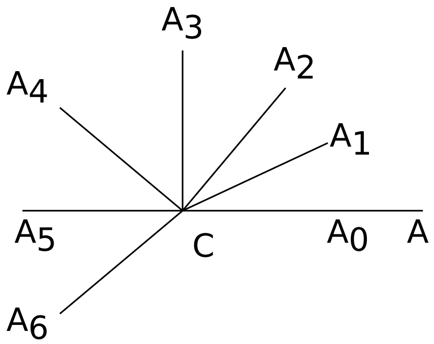


(13) XII. An *acute* angle is an angle less than a right angle.



(14) Angles might not improperly be considered as a fourth species of magnitude. Angular magnitude evidently consists of parts, and must therefore be admitted to be a species of quantity. The student must not suppose

that the magnitude of an angle is affeced by the length of the right lines which include it, and of whose mutual divergence it is the measure. These lines, which are called the *sides* or *legs* of the angle, are supposed to be of indefinite length. To illustrate the nature of angular magnitude, we shall again recur to motion. Let C be supposed to be the extremity of a right line C A,

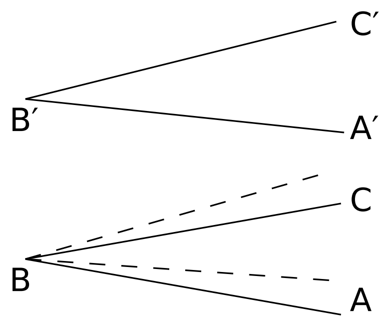


extending indefinitely in the direction C A. Through the same point C, let another indefinite straight line C A_0, be conceived to be drawn; and suppose this right line to revolve in the same plane round its extremity C, it being supposed at the beginning of its motion to coincide with C A. As it revolves from C A_0 to C A_1, C A_2, C A_3, &c., its divergence from C A or, what is the same, the *angle* it makes with C A, continually increases. The line continuing to revolve, and successively assuming the positions C A_1, C A_2, C A_3, C A_4, &c., will at length coincide with the continuation C A_5 of the line C A_0 on the opposite side of the point C. When it assumes this position, it is considered by Euclid to have no inclination to C A_0, and to form no *angle* with it. Nevertheless, when the student advances further in mathematical science, he will find, that not only the line C A_5 is considered to form an angle with C A_0, but even when the revolving line continues its motion past C A_6; and this angle is measured in the direction A_6, A_5, A_4, &c. to A_0.

The point where the sides of an angle meet is called the *vertex* of the angle.

Superposition is the process by which one magnitude may be conceived to be placed upon another, so as exactly to cover it, or so that every part of each shall exactly coincide with every part of the other.

It is evident that any magnitudes which admit of superposition must be equal, or rather this may be considered as the definition of equality. Two angles are therefore equal when they admit of superposition. This may be determined thus; if the angles A B C and A′ B′ C′ are those whose equality is to be ascertained, let the vertex B′ be conceived to be placed on the vertex

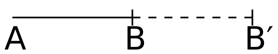


B, and the side B′ A′ on the side B A, and let the remaining side B′ C′ be placed on the same side of B A with B C. If under these circumstances B′ C′ lie upon, or coincide with B C, the angles admit of superposition, and are equal, but are otherwise not. If the side B′ C′ fall between B C and B A, the angle B′, is said to be *less* than the angle B, and if the side B C fall between B′ C′ and B A, the angle B′ is said to be greater than B.

As soon as the revolving line assumes such a position C A_3 that the angle A C A_3 is equal to the angle A_3 C A_5 each of those angles is called a *right angle*.

An angle is sometimes expressed simply by the letter placed at its vertex, as we have done in comparing the angles B and B'. But when the same point, as C, is the vertex of more angles than one, it is necessary to use the three letters expressing the sides as A C A_3, A_3 C A_5, the letter at the vertex being always placed in the middle.

When a line is extended, prolonged, or has its length increased, it is said to be *produced*, and the increase of length which it receives is called its *produced part*, or its *production*. Thus, if the right line A B be prolonged to



B′, it is said to be *produced through* the extremity B, and B B′ is called its *production* or *produced part*.

Two lines which meet and cross each other are said to *intersect*, and the point or points where they meet are called *points of intersection*. It is assumed as a self-evident truth, that two right lines can only intersect in one point. Curves, however, may intersect each other, or right lines, in several points.

Two right lines which intersect, or whose productions intersect, are said to be *inclined* to each other, and their inclination is measured by the angle which they include. The angle included by two right lines is sometimes called the angle *under* those lines; and right lines which include equal angles are said to be equally inclined to each other.

It may be observed, that in general when right lines and plane surfaces are spoken of in Geometry, there are considered as extended or *produced* indefinitely. When a determinate portion of a right line is spoken of, it is generally called a *finite* right line. When a right line is said to be given, it is generally meant that its position or direction on a plane is given. But when a *finite* right line is given, it is understood, that not only its position, but its length is given. These distinctions are not always rigorously observed, but it never happens that any difficulty arises, as the meaning of the words is always sufficiently plain from the context.

When the direction alone of a line is given, the line is sometimes said to be *given in position*, and when the length alone is given, it is said to be *given in magnitude*.

By the inclination of two finite right lines which do not meet, is meant the angle which would be contained under these lines if produced until they intersect.

(15) XIII. A term or boundary is the extremity of any thing.

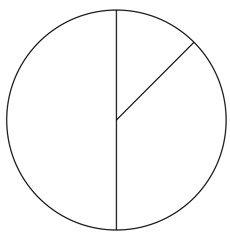
This definition might be omitted as useless.

(16) XIV. A figure is a surface, inclosed on all sides by a line or lines.

The entire length of the line or lines, which inclose a figure, is called its *perimeter*.

A figure whose surface is a plane is called a plane figure. The first six books of the Elements treat of plane figures only.

(17) XV. A *circle* is a plane figure, bounded by one continued line, called its *circumference* or *periphery*; and having a certain point within it, from which all right lines drawn to its circumference are equal.



If a right line of a given length revolve in the same plane round one of its extremities as a fixed point, the other extremity will describe the circumference of a circle, of which the centre is the fixed extremity.

(18) XVI. This point (from which the equal lines are drawn) is called the centre of the circle.

(19) A line drawn from the centre of a circle to its circumference is called a *radius*.

(20) XVII. A *diameter* of a circle is a right line drawn through the centre, terminated both ways in the circumference.

(21) XVIII. A semicircle is the figure contained by the diameter, and the part of the circle cut off by the diameter.

(22) From the definition of a circle, it follows immediately, that a line drawn from the centre to any point *within* the circle is less than the radius; and a line from the centre to any point *without* the circle is greater than the radius. Also, every point, whose distance from the centre is *less* than the radius, must be *within* the circle; every point whose distance from the centre is *equal* to the radius must be *on* the circle; and every point, whose distance from the centre is *greater* than the radius, is *without* the circle.

The word 'semicircle' in Def. XVIII., assumes, that a diameter divides the circle into two equal parts. This may be easily proved by supposing the two parts, into which the circle is thus divided, placed one upon the other, so that they shall lie at the same side of their common diameter: then if the arcs of the circle which bound them do not coincide, let a radius be supposed to be drawn, intersecting them. Thus, the radius of the one will be a part of the radius of the other; and therefore, two radii of the same circle are unequal, which is contrary to the definition of a circle (17.)

(23) XIX. A segment of a circle is a figure contained by a right line, and the part of the circumference which it cuts off.

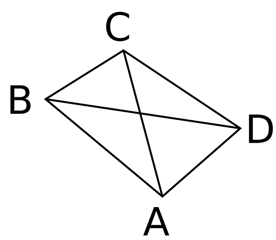
(24) XX. A figure contained by right lines only, is called a *rectilineal figure*.

The lines which include the figre are called its *sides*.

(25) XXI. A triangle is a rectilinear figure included by three sides.

A triangle is the most simple of all rectilinear figures, since less than three right lines cannot form any figure. All other rectilinear figures may be resolved into triangles by drawing right lines from any point within them to their several vertices. The triangle is therefore, in effect, the element of all rectilinear figures; and on its properties, the properties of all other rectilinear figures depend. Accordingly the greater part of the first book is devoted to the development of the properties of this figure.

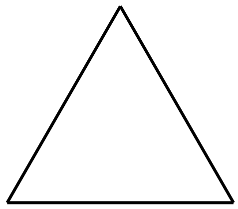
(26) XXII. A quadrilateral figure is one which is bounded by four sides. The right lines A C, B D, connecting the vertices of the opposite sides of a quadrilateral figure, are called its diagonals.



(27) XXIII. A polygon is a rectilinear figure, bounded by more than four sides.

Polygons are called pentagons, hexagons, heptagons, &c., according as they are bounded by five, six, seven or more sides. A line joining the vertices of any two angles which are not adjacent is called a diagonal of the polygon.

(28) XXIV. A triangle, whose three sides are equal, is said to be equilateral.

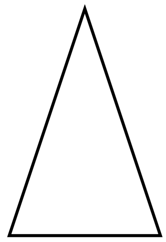


In general, all rectilinear figures whose sides are equal, may be said to be equilateral.

Two rectilinear figures, whose sides are respectively equal each to each, are said to be *mutually equilateral.* Thus, if two triangles have each sides of three, four, or five feet in length, they are *mutually equilateral*, although neither of them is *an equilateral triangle.*

In the same way a rectilinear figure having all its angles equal, is said to be *equiangular*, and two rectilinear figures whose several angles are equal each to each, are said to be *mutually equiangular*.

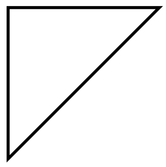
(29) XXV. A triangle which has only two sides equal is called an isosceles triangle.



The equal sides are generally called the *sides*, to distinguish them from the third side, which is called the *base*.

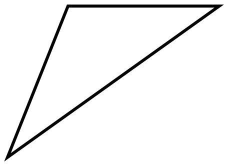
(30) XXVI. A *scalene triangle* is one which has no two sides equal.

(31) XXVII. A right-angled triangle is that which has a right angle.

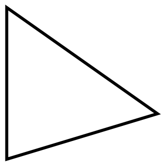


That side of a right-angled triangle which is opposite to the right angle is called the *hypotenuse*.

(32) XXVIII. An obtuse-angled triangle is that which has an obtuse angle.

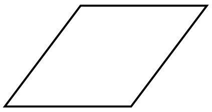


(33) XXIX. An acute-angled triangle is that which has an three acute angles.

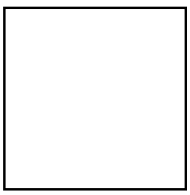


It will appear hereafter, that a triangle cannot have more than one angle right or obtuse, but may have all its angles acute.

(34) XXX. An equilateral quadrilateral figure is called a *lozenge*.

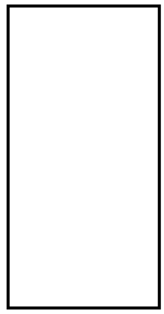


(35) XXXI. An equilateral lozenge is called a *square*.



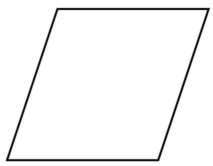
We have ventured to change the definition of a square as given in the text. A lozenge, called by Euclid a *rhombus*, when equiangular, must have all its angles right, as will appear hereafter. Euclid's definition, which is a 'a lozenge all whose angles are right,' therefore, contains more than sufficient for a definition, inasmuch as, had the angles been merely *defined* to be equal, they might be *proved* to be right. To effect this change in the definition of a square, we have transposed the order of the last two definitions. See (158).

(35) XXXII. An *oblong* is a quadrilateral, whose angles are all right, but whose sides are not equal.



This term is not used in the Elements, and therefore the definition might have been omitted. The same figure is defined in the second book, and called a *rectangle*. It would appear that this circumstance of defining the same figure twice must be an oversight.

(36) XXXIII. A *rhomboid* is a quadrilateral, whose opposite sides are equal.

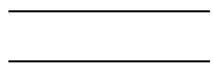


This definition and the term *rhomboid* are superceded by the term *parallelogram*, which is a quadrilateral, whose opposite sides are parallel. It will be proved hereafter, that if the opposite sides of a quadrilateral be equal, it must be a parallelogram. Hence, a distinct denomination for such a figure is useless.

(37) XXXIV. All other quadrilateral figures are called *trapeziums*.

As *quadrilateral figure* is a sufficiently concise and distinct denomination, we shall restrict the application of the term *trapezium* to those quadrilaterals which have two sides parallel.

(38) XXXV. Parallel right lines are such as are in the same plane, and which, being produced continually in both direction, would never meet.



It should be observed, that the circumstance of two right lines, which are produced indefinitely, never meeting, is not sufficient to establish their parallelism. For two right lines which are not in the same plane can never meet, and yet are not parallel. Two things are indispensably necessary to establish the parallelism of two right lines, 1°, that they be in the same plane, and 2°, that when indefinitely produced, they never meet. As in the first six books of the Elements all the lines which are considered are supposed to be in the same plane, it will be only necessary to attend to the latter criterion of parallelism.

POSTULATES

(39) I. Let it be granted that a right line may be drawn from any one point to any other point.

(40) II. Let it be granted that a finite right line may be produced to any length in a right line.

(41) III. Let it be granted that a circle may be described with any centre at any distance from that centre.

(42) The object of the postulates is to declare, that the only instruments, the use of which is permitted in Geometry, are the *rule* and *compass*. The *rule* is an instrument which is use to direct the pen or pencil in drawing a right line; but it should be observed, that the geometrical rule is not supposed to be *divided* or *graduated*, and, consequently, it does not enable us to draw a right line of any proposed length. Neither is it permitted to place any permanent mark or marks on any part of the rule, or we should be able by it to solve the second proposition of the first book, which is *to draw from a given point a right line equal to a another given right line*. This might be done by placing the rule on the given right line, and marking its extremities on the rule, then placing the mark corresponding to one extremity at the given point, and drawing the pen along the rule to the second mark. This, however, is not intended to be granted by the postulates.

The third postulate concedes the use of the compass, which is an instrument composed of two straight and equal legs united at one extremity by a joint, so constructed that the legs can be opened or closed so as to form any proposed angle. The other extremities are points, and when the legs have been opened to any degree of divergence, the extremity of one of them being fixed at a point, and the extremity of the other being moved around it in the same plane will describe a circle, since the distance between the points is supposed to remain unchanged. The fixed point is the centre; and the distance between the points, the radius of the circle.

It is not intended to be conceded by the third postulate that a circle can be described round a given centre with a radius of a given length; in other words, it is not granted that the legs of the compass can be opened until the distance between their points shall equal a given line.

AXIOMS

(43) I. Magnitudes which are equal to the same are equal to each other.

(44) II. If equals be added to equals the sums will be equal.

(45) III. If equals be taken away from equals the remainders will be equal.

(46) IV. If equals be added to unequals the sums will be unequal.

(47) V. If equals be taken away from unequals the remainders will be unequal.

(48) VI. The doubles of the same or equal magnitudes are equal.

(49) VII. The halves of the same or equal magnitudes are equal.

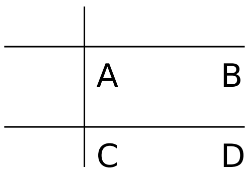
(50) VIII. Magnitudes which coincide with one another, or exactly fill the same space, are equal.

(51) IX. The whole is greater than its part.

(52) X. Two right lines cannot include a space.

(53) XI. All right angles are equal.

(54) XII. If two right lines (A B, C D) meet a third right line (A C) so as to make the two interior angles (B A C and D C A) on the same side less than two right angles, these two right lines will meet if they be produced on that side on which the angles are less than two right angles.



(55) The geometrical axioms are certain general propositions, the truth of which is taken to be self-evident, and incapable of being established by demonstration. According to the spirit of this science, the number of axioms should be as limited as possible. A proposition, however self-evident, has no title to be taken as an axiom, if its truth can be deduced from axioms already admitted. We have a remarkable instance of the rigid adherence to this principle in the twentieth proposition of the first book, where it is proved that 'two sides of a triangle taken together are greater than the third;' a proposition which is quite as self-evident as any of the received axioms, and much more self-evident than several of them.

On the other hand, if the truth of a proposition cannot be established by demonstration, we are compelled to take it as an axiom, *even though it be* *not self-evident*. Such is the case with the twelfth axiom. We shall postpone our observations on this axiom, however, for the present, and have to request that the student will omit it until he comes to read the commentary on the twenty-eighth proposition. See Appendix II.

Two magnitudes are said to be *equal* when they are capable of exactly covering one another, or filling the same space. In the most ordinary practical cases we use this test for determining equality; we apply the two things to be compared one to the other, and immediately infer their equality from their coincidence.

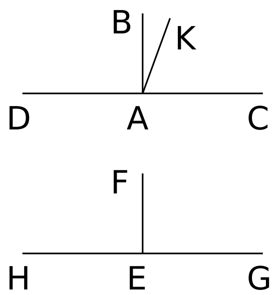
By the aid of this definition of equality we conceive that the second and third axioms might easily be deduced from the first. We shall not however pursue the discussion here.

*** The fourth and fifth axioms are not sufficiently definite. After the addition or subtraction of equal quantities, unequal quantities continue to be unequal. But it is also evident, that their difference, that is, the quantity by which the greater exceeds the less, will be the same after such addition or subtraction as before it.

The sixth and seventh axioms may very easily be inferred from the preceding ones.

The tenth axiom may be presented under various forms. It is equivalent to stating, that between any two points only one right line can be drawn. For if two different right lines could be drawn from one point to another, they would evidently enclose a space between them. It is also equivalent to stating, that two right lines being infinitely produced cannot intersect each other in more than one point; for if they intersected at two points, the parts of the lines between these points would enclose a space.

The eleventh axiom admits of demonstration. Let A B and E F be perpendicular to D C and H G. Take any equal parts E H, E G on H G measured from the point E, and on D C take parts from A equal to these (Prop. III. Book I.) Let the point H be conceived to be placed upon the point D.



The points G and C must then be in the circumference of a circle described round the centre D, with the distance D C or H G as radius. Hence, if the line H G be conceived to be turned round this centre D, the point G must in some position coincide with C. In such a position every point of the line H G must coincide with C D (ax. 10.), and the middle points A and E must evidently coincide. Let the perpendiculars E F and A B be conceived to be placed at the same side of D C. They must then coincide, and therefore the right angle F E G will be equal to the right angle B A C. For if E F do not coincide with A B, let it take the position A K. The right angle K A C is equal to K A D (11), and therefore greater than B A D; but B A D is equal to B A C (11), and therefore K A C is greater than B A C. But K A C is a part of B A C, and therefore less than it, which is absurd; and therefore E F must coincide with A B, and the right angles B A C and F E G are equal.

The postulates may be considered as axioms. The first postulated, which declares the possibility of one right line joining two given points, is as much an axiom as the tenth axiom, which declares the impossibility of more than one right line joining them.

In like manner, the second postulate, which grants the power of producing a line, may be considered as an axiom, declaring that every finite straight line may have another placed at its extremity so to form with it one continued straight line. In fact, the straight line thus placed will be its production. This postulate is assumed as an axiom in the fourteenth proposition of the first book.

(56) Those results which are obtained in geometry by a process of reasoning are called *propositions*. Geometrical propositions are of two species, *problems* and *theorems*.

A *problem* is a proposition in which something is proposed to be done; as a line to be drawn under some given conditions, some figure to be constructed, &c. The *solution* of the problem consists in showing how the thing required may be done by the aid of the rule and compass. The *demonstration* consists in proving that the process indicated in the solution really attains the required end.

A *theorem* is a proposition in which the truth of some principle is asserted. The object of the demonstration is to show how the truth of the proposed principle may be deduced from the axioms and definitions or other truths previously and independently established.

A problem is analogous to a postulate, and a theorem to an axiom.

A postulate is a problem, the solution of which is assumed.

An axiom is a theorem, the truth of which is granted without demonstration.

In order to effect the demonstration of a proposition, it frequently happens that other lines must be drawn besides those which are actually engaged in the enunciation of the proposition itself. The drawing of such lines is generally called the *construction*.

A *corollary* is an inference deduced immediately from a proposition.

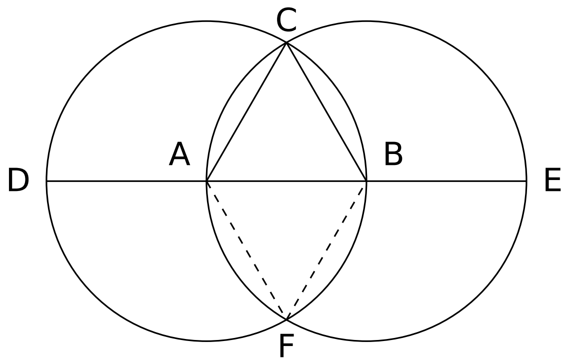
A *scholium* is a note or observation on a proposition not containing any inference, or, at least, none of sufficient importance to entitle it to the name of a *corollary*.

A *lemma* is a proposition merely introduced for the purpose of establishing some more important proposition.

PROPOSITION I. PROBLEM.

(57) *On a given finite right line* (A B) *to construct an equilateral triangle.*

SOLUTION



With the centre A and the radius A B let a circle B C D be described (41), and with the centre B and the radius B A let another circle A C E be described. From a point of intersection C of these circles let right lines be drawn to the extremities A and B of the given right line (39). The triangle A C B will be that which is required.

DEMONSTRATION.

It is evident that the triangle A C B is constructed on the given right line A B. But it is also equilateral; for the lines A C and A B, being radii of the same circle B C D, are equal (17), and also B C and B A, being radii of the same circle A C E, are equal. Hence the lines B C and A C, being equal to the same line A B, are equal to each other (43). The three sides of the triangle A B C are therefore equal, and it is an equilateral triangle (28).

(58) In the solution of this problem it is assumed that the two circles intersect, inasmuch as the vertex of the equilateral triangle is a point of intersection. This, however, is sufficiently evident if it be considered that a circle is a continued line which includes space, and that in the present instance each circle passing through the centre of the other must have a part of its circumference within that other, and a part without it, and must therefore intersect it.

It follows from the solution, that as many different equilateral triangles can be constructed on the same right line as there are points in which the two circles intersect. It will hereafter be proved that two circles cannot intersect in more than two points, but for the present this may be taken for granted.

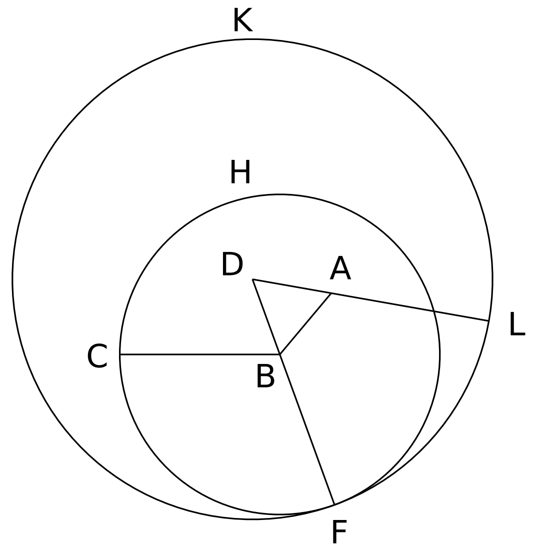
Since there are but two points of intersection of the circles, there can be but two equilateral triangles constructed on the same finite right line, and these are placed on opposite sides of it, their vertices being at the points C and F.

After having read the first book of the elements, the student will find no difficulty in proving that the triangles C F E and C D F are equilateral. These lines are not in the diagram, but may easily be supplied.

Proposition II. Problem.

(59) *From a given point (A) to draw a right line equal to a given finite right line (B C).*

Solution.



Let a right line be drawn from the given point A to either extremity B of the given finite right line B C (39). On the line A B let an equilateral triangle A D B be constructed (I). With the centre B and the radius B C let a circle be described (41). Let D B be produced to meet the circumference of this circle in F (40), and with the centre D and the radius D F let another circle F L K be described. Let the line D A be produced to meet the circumference of this circle in L. The line A L is then the required line.

Demonstration.

The lines D L and D F are equal, being radii of the same circle F L K (17). Also the lines D A and D B are equal, being sides of the equilateral triangle B D A. Taking the latter from the former, the remainders A L and B F are equal (45). But B F and B C are equal, being radii of the same circle F C H (17), and since A L and B C are both equal to B F, they are equal to each other (43), Hence A L is equal to B C, and is drawn from the given point A, and therefore solves the problem.

*** The different positions which the given right line and given point may have with respect to each other, are apt to occasion such changes in the diagram as to lead the student into error in the execution of the construction for the solution of this problem.

Hence it is necessary that in solving this problem, the student should be guided by certain *general* directions, which are independent of any particular arrangement which the several lines concerned in the solution may assume. If the student is governed by the following general directions, no change which the diagram can undergo will mislead him.

1° The given point is to be joined with *either* extremity of the given right line. (Let us call the extremity with which it is connected, the *connected extremity* of the given right line; and the line so connecting them, the *joining line*.)

2° The centre of the first circle is the *connected extremity* of the given right line; and its radius, the given right line.

3° The equilateral triangle may be constructed on *either side* of the joining line.

4° The side of the equilateral triangle which is produced to meet the circle, is that side which is opposite to the given point, and it is produced through the centre of the first circle till it meets its circumference.

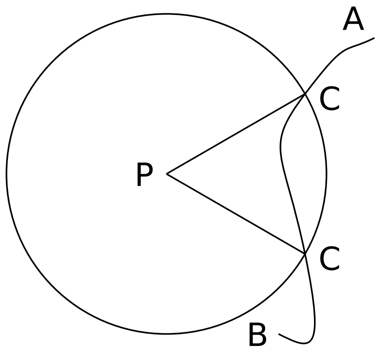
5° The centre of the second circle is that vertex of the triangle which is opposite to the joining line, and its radius is made up of that side of the triangle which is opposite to the given point, and its production which is the radius of the first circle. So that the radius of the second circle is the sum of the side of the triangle and the radius of the first circle.

6° The side of the equilateral triangle which is produced through the given point to meet the second circle, is that side which is opposite to the connected extremity of the given right line, and the production of this side is the line which solves the problem; for the sum of this line and the side of the triangle is the radius of the second circle, but also the sum of the given right line (which is the radius of the first circle) and the side of the triangle is equal to the radius of the second circle. The side of the triangle being taken away the remainders are equal.

As the given point may be joined with either extremity, there may be two different joining lines, and as the triangle may be constructed on either side of each of these, there may be four different triangles; so the right line and the point being given, there are four different constructions by which the problem may be solved.

If the student inquires further, he will perceive that the solution may be effected also by producing the side of the triangle opposite the given point, not through the extremity of the right line but through the vertex of the triangle. The various consequences of this variety in the construction we leave to the student to trace.

(60) By the second proposition a right line of a given length can be inflected from a given point P upon any given line A B. For from the point



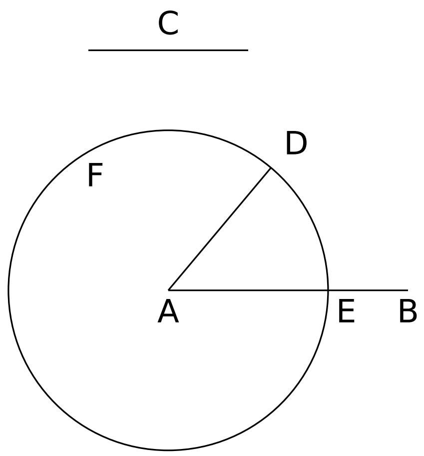
P draw a right line of the given length (II), and with P as centre, and that line as radius, describe a circle. A line drawn from P to any point C, where this circle meets the given line A B, will solve the problem.

By this proposition the first may be generalized; for an *isosceles* triangle may be constructed on a given line as base, and having its side of a given length. The construction will remain unaltered, except that the radius of each of the circles will be equal to the length of the side of the proposed triangle. If this length be not greater than half the base, the two circles will not intersect, and no triangle can be constructed, as will appear hereafter.

Proposition III. Problem.

(61) *From the greater* (A B)*, of two given right lines to cut off a part equal to the less* (C)*.*

Solution.



From either extremity A of the greater let a right line A D be drawn equal to the less C (II), and with the point A as centre, and the radius A D let a circle be described (41). The part A E of the greater cut off by this circle will be equal to the less C.

Demonstration.

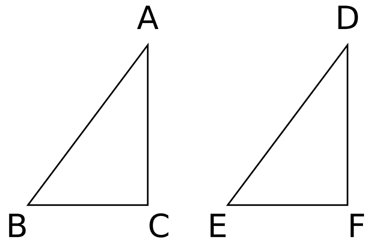
For A E and A D are equal, being radii of the same circle (17); and C and A D are equal by the construction. Hence A E and C are equal.

By a similar construction, the less might be produced until it equal the greater. From an extremity of the less let a line equal to the greater be drawn, and a circle be described with this line as radius. Let the less be produced to meet this circle.

Proposition IV. Theorem.

(62) *If two triangles (*B A C *and* E D F*) have two sides (*B A *and* A C*) in the one respectively equal to two sides (*E D *and* D F*) in the other, and the angles (*A *and* D*) included by those sides also equal; the bases or remaining sides (*B C *and* E F*) will be equal, also the angles (*B *and* C*) at the base of the one will be respectively equal to those (*E *and* F*) at the base of the other which are opposed to the equal sides (i. e.* B *to* E *and* C *to* F*).*

Let the two triangles be conceived to be so placed that the vertex of one of the equal angles D shall fall upon that of the other A, that one of the sides



D E containing the given equal angles shall fall upon the side A B in the other triangle to which it is equal, and that the remaining pair of equal sides A C and D F shall lie at the same side of those A B and D E which coincide.

Since then the vertices A and D coincide, and also the equal sides A B and D E, the points B and E must coincide. (If they did not the sides A B and D E would not be equal.) Also, since the side D E falls on A B, and the sides A C and D F are at the same side of A B, and the angles A and D are equal, the side D F must fall upon A C; (for otherwise the angles A and D would not be equal.)

Since the side D F falls on A C, and they are equal, the extremity F must fall on C. Since the extremities of the bases B C and E F coincide, these lines themselves must coincide, for if they did not they would include a space (52). Hence the sides B C and E F are equal (50).

Also, since the sides E D and E F coincide respectively with B A and B C, the angles E and B are equal (50), and for a similar reason the angles F and C are equal.

Since the three sides of the one triangle coincide respectively with the three sides of the other, the triangles themselves coincide, and are therefore equal (50).

In the demonstration of this proposition, the converse of the eighth axiom (50) is assumed. The axiom states, that 'if two magnitudes coincide they must be equal.' In the proposition it is assumed, that if they be equal they must under certain circumstances coincide. For when the point D is placed on A, and the side D E on A B, it is assumed that the point E must fall on B, because A B and D E are equal. This may, however, be proved by the combination of the eighth and ninth axioms; for if the point E did not fall upon B, but fell either above or below it, we should have either E D equal to a part of B A, or B A equal to a part of E D. In either case the ninth axiom would be contradicted, as we should have the whole equal to its part.

The same principle may be applied in proving that the side D F will fall upon A C, which is assumed in Euclid's proof.

In the superposition of the triangles in this proposition, three things are to be attended to:

1° The vertices of the equal angles are to be placed one on the other.

2° Two equal sides to be placed one on the other.

3° The other two equal sides are to be placed on the same side of those which are laid one upon the other.

From this arrangement the coincidence of the triangles is inferred.

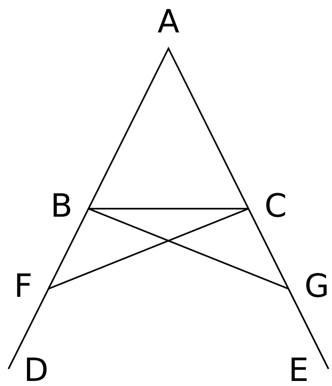
It should be observed, that this superposition is not assumed to be actually effected, for that would require other postulates besides the three already stated; but it is sufficient for the validity of the reasoning, if it be conceived to be possible that the triangles might be so placed. By the same principle of superposition, the following theorem must be easily demonstrated, 'If two triangles have two angles in one respectively equal to two angles in the other, and the sides lying between those angles also equal, the remaining sides and angles will be equal, and also the triangles themselves will be equal.' See prop. xxvi.

This being the first *theorem* in the Elements, it is necessarily deduced exclusively from the axioms, as the first problem must be from the postulates. Subsequent theorems and problems will be deduced from those previously established.

Proposition V. Theorem.

(63) *The angles (*B, C*) opposed to the equal sides (*A C *and* A B*) of an isosceles triangle are equal, and if the equal sides be produced through the extremities (*B *and* C*) of the third side, the angles (*D B C *and* E C B*) formed by their produced parts and the third side are equal.*

Let the equal sides A B and A C be produced through the extremities B, C, of the third side, and in the produced part B D of either let any point F



be assumed, and from the other let A G be cut off equal to A F (III). Let the points F and G so taken on the produced sides be connected by right lines F C and B G with the alternate extremities of the third side of the triangle.

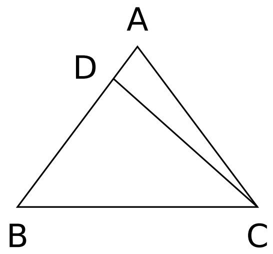
In the triangles F A C and G A B the sides F A and A C are respectively equal to G A and A B, and the included angle A is common to both triangles. Hence (IV), the line F C is equal to B G, the angle A F C to the angle A G B, and the angle A C F to the angle A B G. If from the equal lines A F and A G, the equal sides A B and A C be taken, the remainders B F and C G will be equal. Hence, in the triangles B F C and C G B, the sides B F and F C are respectively equal to C G and G B, and the angles F and G included by those sides are also equal. Hence (IV), the angles F B C and G C B, which are those included by the third side B C and the productions of the equal sides A B and A C, are equal. Also, the angles F C B and G B C are equal. If these equals be taken from the angles F C A and G B A, before proved equal, the remainders, which are the angles A B C and A C B opposed to the equal sides, will be equal.

(64) Cor.—*Hence, in an equilateral triangle the three angles are equal; for by this proposition the angles opposed to every two equal sides are equal.*

Proposition VI. Theorem.

(65) *If two angles (*B *and* C*) of a triangle (*B A C*) be equal, the sides (*A C *and* A B*) opposed to them are also equal.*

For if the sides be not equal, let one of them A B be greater than the other, and from it cut off D B equal to A C (III), and draw C D.



Then in the triangles D B C and A C B, the sides D B and B C are equal to the sides A C and C B respectively, and the angles D B C and A C B are also equal; therefore (IV) the triangles themselves D B C and A C B are equal, a part equal to the whole, which is absurd; therefore neither of the sides A B or A C is greater than the other; there are therefore equal to one another.

(66) Cor.—*Hence every equiangular triangle is also equilateral, for the sides opposed to every two equal angles are equal.*

In the construction for this proposition it is necessary that the part of the greater side which is cut off equal to the less, should be measured upon the greater side B A from vertex (B) of the equal angle, for otherwise the fourth proposition could not be applied to prove the equality of the part with the whole.

It may be observed generally, then when a part of one line is cut off equal to another, it should be distinctly specified from which extremity the part is to be cut.

This proposition is what is called by logicians the *converse* of the fifth. It cannot however be inferred from it by the logical operation called *conversion*; because, by the established principles of Aristotelian logic, *an universal affirmative admits no simple converse.* This observation applies generally to those propositions in the Elements which are converses of preceding ones.

The demonstration of the sixth is the first instance of indirect proof which occurs in the Elements. The force of this species of demonstration consists in showing that a principle is true, because some manifest absurdity would follow from supposing it to be false.

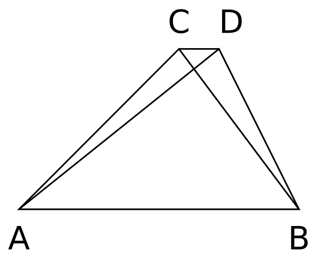
This kind of proof is considered inferior to *direct* demonstration, because it only proves that a thing *must* be so, but fails in showing *why* it must be so; whereas *direct* proof not only shows that the thing *is* so, but *why* it is so. Consequently, indirect demonstration is never used, except where no direct proof can be had. It is used generally in proving principles which are nearly self-evident, and in the Elements if oftenest used in establishing the *converse* propositions. Examples will be seen in the 14th, 19th, 25th and 40th propositions of this book.

Proposition VII. Theorem.

(67) *On the same right line (*A B*), and on the same side of it, there cannot be constructed two triangles, (*A C B, A D B*) whose conterminous sides (*A C *and* A D, B C *and* B D*) are equal.*

If it be possible, let the two triangles be constructed, and,

First,—Let the vertex of each of the triangles be without the other triangle, and draw C D.



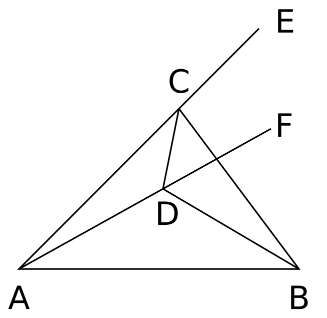
Because the sides A D and A C of the triangle C A D are equal (hyp.)[7] the angles A C D and A D C are equal (V); but A C D is greater than B C D (51), therefore A D C is greater than B C D; but the angle B D C is greater than A D C (51), and therefore B D C is greater than B C D; but in the triangle C B D, the sides B C and B D are equal (hyp.), therefore the angles B D C and B C D are equal (V); but the angle B D C has been proved to be greater than B C D, which is absurd: therefore the triangles constructed upon the same right line cannot have their conterminous sides equal, when the vertex of each of the triangles is without the other.

Secondly,—Let the vertex D of one triangle be within the other; produce the sides A C and A D, and join C D.

Because the sides A C and A D of the triangle C A D are equal (hyp.), the angles E C D and F D C are equal (V); but the angle B D C is greater than

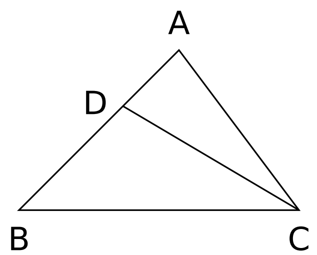
[7]The *hypothesis* means the *supposition*; that is, the part of the enunciation of the proposition in which something is supposed to be granted true, and from which the proposed conclusion is to be inferred. Thus in the seventh proposition the hypothesis is, that the triangles stand on the same side of their base, and that their conterminous sides are equal, and the conclusion is a manifest absurdity, which proves that the hypothesis must be false.

In the fourth proposition the hypothesis is, that two sides and the included angle of one triangle are respectively equal to two sides and the included angle of the other; and the conclusion deduced from this hypothesis is, that the remaining side and angles in the one triangle are respectively equal to the remaining side and angles in the other triangle.



F D C (51), therefore greater than E C D; but E C D is greater than B C D (51), and therefore B D C is greater than B C D; but in the triangle C B D, the sides B C and B D are equal (hyp.), therefore the angles B D C and B C D are equal (V); but the angle B D C has been proved to be greater than B C D, which is absurd: therefore triangles constructed on the same right line cannot have their conterminous sides equal, if the vertex of one of them is within the other.

Thirdly,—Let the vertex D of one triangle be on the side A B of the other, and it is evident that the sides A B and B D are not equal.



Therefore in no case can two triangles, whose conterminous sides are equal, be constructed at the same side of the given line.

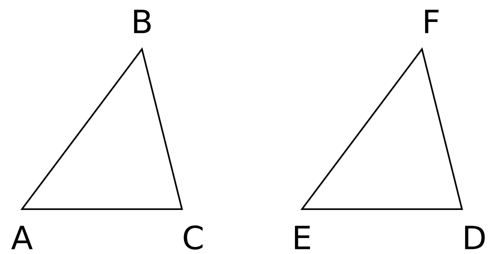
This proposition seems to have been introduced into the Elements merely for the purpose of establishing that which follows it. The demonstration is that form of argument which logicians call a *dilemma*, and a species of argument which seldom occurs in the Elements. If two triangles whose conterminous sides are equal could stand on the same side of the same base, the vertex of the one must necessarily either fall within the other or without it, or on one of the sides of it: accordingly, it is successively proved in the demonstration, that to suppose it in any of these positions would lead to a contradiction in terms. It is not supposed that the vertex of the one could fall on the vertex of the other; for that would be supposing the two triangles to be one and the same, whereas they are, by hypothesis, different.

In the Greek text there is but one (the first) of the cases of this proposition given. It is however conjectured, that the second case must have been formerly in the text, because it is the only instance in which Euclid uses that part of the fifth proposition which proves the equality of the angles below the base. It is argued, that there must have been some reason for introducing into the fifth a principle which follows at once from the thirteenth; and that none can be assigned except the necessity of the principle in the second case of the seventh. The third case required to be mentioned only to preserve the complete logical form of the argument.

Proposition VIII. Theorem.

(68) *If two triangles (*A B C *and* E F D*) have two sides of the one respectively equal to two sides of the other (*A B *to* E F *and* C B *to* D F*), and also have the base (*A C*) equal to the base (*E D*), then the angles (*B *and* F*) contained by the equal sides are equal.*

For if the equal bases A C, E D be conceived to be placed one upon the other, so that the triangles shall lie at the same side of them, and that the



equal sides A B and E F, C B and D F be conterminous, the vertex B must fall on the vertex F; for to suppose them not coincident would contradict the seventh proposition. The sides B A and B C being therefore coincident with F E and F D, the angles B and F are equal.

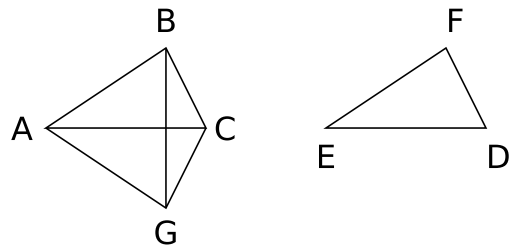
(69) It is evident that in this case all the angles and sides of the triangles are respectively equal each to each, and that the triangles themselves are equal. This appears immediately by the eighth axiom.

In order to remove from the threshold of the Elements a proposition so useless, and, to the younger students, so embarrassing as the seventh, it would be desirable that the eighth should be established independently of it. There are several ways in which this might be effected. The following proof seems liable to no objection, and establishes the eighth by the fifth.

Let the two equal bases be so applied one upon the other that the equal sides shall be conterminous, and that the triangles shall lie at opposite sides of them, and let a right line be conceived to be drawn joining the vertices.

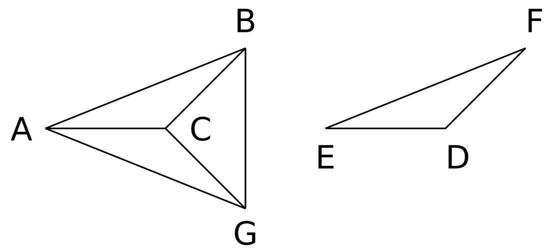
1° Let this line intersect the base.

Let the vertex F fall at G, the side E F in the position A G, and D F in the position C G. Hence B A and A G being equal, the angles G B A and B G A are equal (V). Also C B and C G being equal, the angles C G B and C B G are equal (V). Adding these equals to the former, the angles A B C and A G C are equal; that is, the angles E F D and A B C are equal.



2° Let the line G B fall outside the coincident bases.

The angles G B A and B G A, and also B G C and G B C are proved equal as before; and taking the latter from the former, the remainders, which are



the angles A G C and A B C, are equal, but A G C is the angle F.

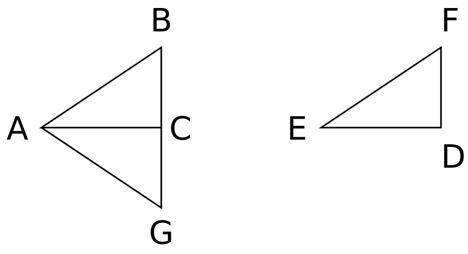
3° Let the line B G pass through either extremity of the base.

In this case it follows immediately (V) that the angles A B C and A G C are equal; for the lines B C and C G must coincide with B G, since each has two points upon it (52).

Hence in every case the angles B and F are equal.

This proposition is also sometimes demonstrated as follows.

Conceive the triangle E F D to be applied to A B C, as in Euclid's proof. Then because E F is equal to A B, the point F must be in the circumference of a circle described with A as centre, and A B as radius. And for the same reason, F must be on a circumference with the centre C, and the radius C B. The vertex must therefore be at the point where these circles meet. But the vertex B must be also at that point; wherefore &c.


B
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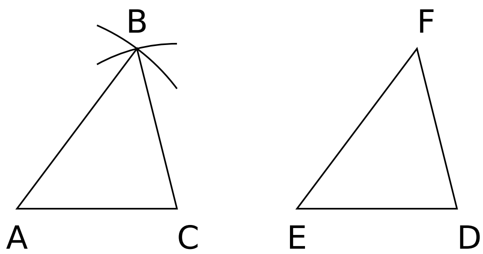
B
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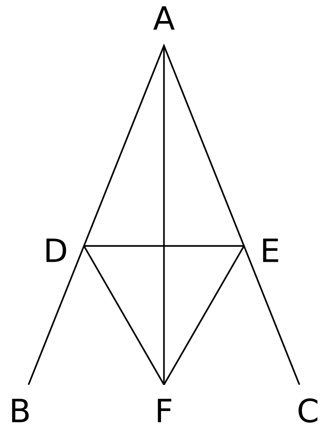
Proposition IX. Problem.

(70) *To bisect a given rectilinear angle (*B A C*).*

Solution.

Take any point D in the side A B, and from A C cut off A E equal to A D (III), draw D E, and upon it describe an equilateral triangle D F E (I) at the



side remote from A. The right line joining the points A and F bisects the given angle B A C.

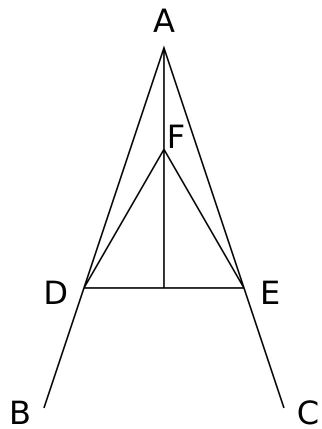
Demonstration.

Because the sides A D and A E are equal (const.), and the side A F is common to the triangles F A D and F A E, and the base F D is also equal to F E (const.); the angles D A F and E A F are equal (VIII), and therefore the right line bisects the given angle.

By this proposition an angle may be divided into 4. 8, 16 &c. equal parts, or, in general, into any number of equal parts which is expressed by a power of *two*.

It is necessary that the equilateral triangle be constructed on a different side of the joining line D E from that on which the given angle is placed, lest the vertex F of the equilateral triangle should happen to coincide with the vertex A of the given angle; in which case there would be no joining line F A, and therefore no solution. In these cases, however, in which the vertex of the equilateral triangle does not coincide with that of the given angle, the problem can be solved by constructing the equilateral triangle on the same side of the joining line D E with the given angle. Separate demonstrations are necessary for the two positions which the vertices may assume.

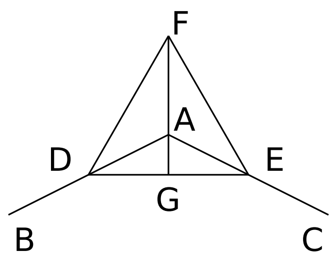
1. Let the vertex of the equilateral triangle fall within that of the given angle.



The demonstration already given will apply to this without any modification.

2. Let the vertex of the given angle fall within the equilateral triangle.

The line F A produced will in this case bisect the angle; for the three sides of the triangle D F A are respectively equal to those of the triangle E F A. Hence the angles D F A and E F A are equal (VIII). Also, in the triangles



D F G and E F G the sides D F and E F are equal, the side G F is common, and the angles D F G and E F G are equal; hence (IV) the bases D G and E G are equal, and also the angles D G A and E G A. Again, in the triangles D G A and E G A the sides D G and E G are equal, A G is common, and the angles at G are equal; hence (IV) the angles D A G and E A G are equal, and therefore the angle B A C is bisected by A G.

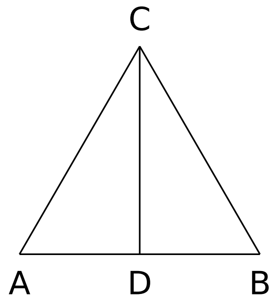
It is evident, that an isosceles triangle constructed on the joining line D E would equally answer the purpose of the solution.

Proposition X. Problem.

(71) *To bisect a given right line (*A B*).*

Solution.

Upon the given line A B describe an equilateral triangle A C B (I),



bisect the angle A C B by the right line C D (IX); this line bisects the given line in the point D.

Demonstration.

Because the sides A C and C B are equal (const.), and C D common to the triangles A C D and B C D, and the angles A C D and B C D also equal (const.); therefore (IV) the bases A D and D B are equal, and the right line A B is bisected in the point D.

In this and the following proposition an isosceles triangle would answer the purposes of the solution equally with an equilateral. In fact, in the demonstrations the triangle is contemplated merely as isosceles: for nothing is inferred from the equality of the base with the sides.

Proposition XI. Problem.

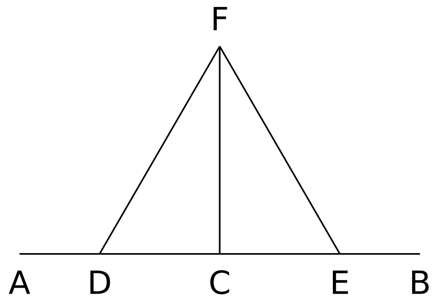
(72) *From a given point* (C) *in a given right line* (A B) *to draw a perpendicular to the given line.*

Solution.

In the given line take any point D and make C E equal to C D (III); upon D E describe an equilateral triangle D F E (I); draw F C, and it is perpendicular to the given line.

Demonstration.

Because the sides D F and D C are equal to the sides E F and E C (const.), and C F is common to the triangles D F C and E F C, therefore (VIII) the



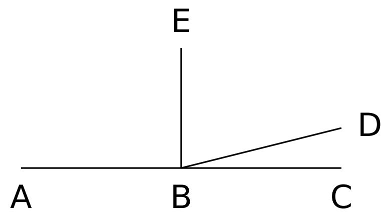
angles opposite to the equal sides D F and E F are equal, and therefore F C is perpendicular to the given right line A B at the point C.

Cor.—By help of this problem it may be demonstrated, that two straight lines cannot have a common segment.

It it be possible, let the two straight lines A B C, A B D have the segment A B common to both of them. From the point B draw B E at right angles to A B; and because A B C is a straight line, the angle C B E is equal to the angle E B A; in the same manner, because A B D is a straight line, the angle D B E is equal to the angle E B A; wherefore the angle D B E is equal to the angle C B E, the less to the greater, which is impossible; therefore the two straight lines cannot have a common segment.

If the given point be at the extremity of the given right line, it must be produced, in order to draw the perpendicular by this construction.

In a succeeding article, the student will find a method of drawing a perpendicular through the extremity of a line *without producing it.*



The corollary to this proposition is useless, and is omitted in some editions.

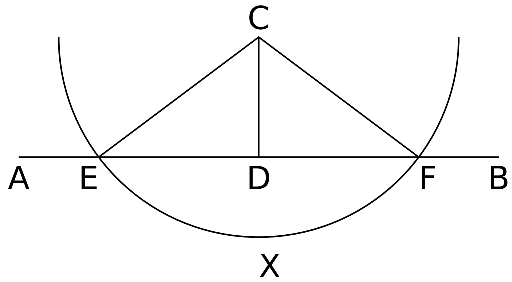
It is equivalent to proving that a right line cannot be produced through its extremity in more than one direction, or that it has but one production.

Proposition XII. Problem.

(73) *To draw a perpendicular to a given indefinite right line (A B). from a point (C) given without it.*

Solution.

Take any point X on the other side of the given line, and from the centre C with the radius C X describe a circle cutting the given line in E and F. Bisect E F in D (X), and draw from the given point to the point of bisection



the right line C D; this line is the required perpendicular.

Demonstration.

For draw C E and C F, and in the triangles E D C and F D C the sides E C and F C, and E D and F D, are equal (const.) and C D common; therefore (VIII) the angles E D C and F D C opposite to the equal sides E C and F C are equal, and therefore D C is perpendicular to the line A B (11).

In this proposition it is necessary that the right line A B be indefinite in length, for otherwise it might happen that the circle described with the centre C and the radius C X might not intersect it in two points, which is essential to the solution of the problem.

It is assumed in the solution of this problem, that the circle will intersect the right line in two points. The centre of the circle being on one side of the given right line, and a part of the circumference (X) on the other, it is not difficult to perceive that a part of the circumference must also be also on the same side of the given line with the centre, and since the circle is a continued line it must cross the right line twice. The properties of the circle form the subject of the third book, and those which are assumed here will be established in that part of the Elements.

The following questions will afford the student useful exercise in the application of the geometrical principles which have been established in the last twelve propositions.

(74) *In an isosceles triangle the right line which bisects the vertical angle also bisects the base, and is perpendicular to the base.*

For the two triangles into which it divides the isosceles, there are two sides (those of the isosceles) equal, and a side (the bisector) common, and the angles included by those sides equal, being the parts of the bisected angle; hence (IV) the remaining sides and angles are respectively equal; that is, the parts into which the base is divided by the bisector are equal, and the angle which the bisector makes with the base are equal. Therefore it bisects the base, and is perpendicular to it.

It is clear that the isosceles triangle itself is bisected by the bisector of its vertical angle, since the two triangles are equal.

(75) It follows also, that *in an isosceles triangle the line which is drawn from the vertex to the middle point of the base bisects the vertical angle, and is perpendicular to the line.*

For in this case the triangle is divided into two triangles, which have their three sides respectively equal each to each, and the property is established by (VIII)

(76) *If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.*

For in this case in the two triangles into which the whole is divided by the perpendicular, there are two sides (the parts of the base) equal, one side (the perpendicular) common, and the included angles equal, being right. Hence (IV) the sides of the triangle are equal.

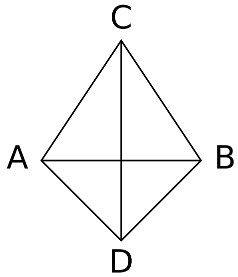
(77) *To find a point which is equidistance from the three vertical points of a triangle* A B C.

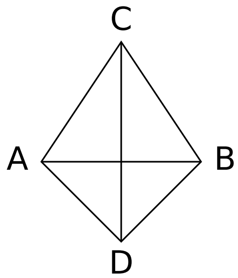
Bisect the sides A B and B C at D and E (X), through the points D and E draw perpendiculars, and produce them until they meet at F. The point F is at equal distances from A, B and C.

For draw F A, F B, F C. B F A is isosceles by (76), and for the same reason B F C is isosceles. Hence it is evident that F A, F C, and F B are equal.

(78) Cor.—Hence F is the centre, and F A the radius of a circle circumscribed about the triangle.

(79) *In a quadrilateral formed by two isosceles triangles* A C B *and* A D B *constructed on different sides of the same base, the diagonals intersect at right angles, and that which is the common base of the isosceles triangles is bisected by the other.*





For in the triangles C A D and C B D the three sides are equal each to each, and therefore (VIII) the angles A C E and B C E are equal. The truth of the proposition therefore follows from (74)

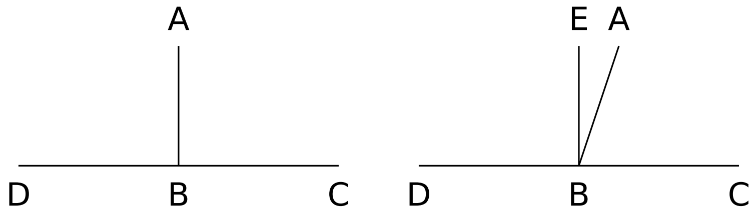
(80) Hence it follows that *the diagonals of a lozenge bisect each other at right angles.*

(81) It follows from (76) that *if the diagonals of a quadrilateral bisect each other at right angles it is a lozenge.*

Proposition XIII. Theorem.

(82) *When a right line* (A B) *standing upon another* (D C) *makes angles with it, they are either two right angles, or together equal to two right angles.*

If the right line A B is perpendicular to D C, the angles A B C and A B D are right (11). If not, draw B E perpendicular to D C (XI), and it is evident



that the angles C B A and A B D together are equal to the angles C B E and E B D, and therefore to two right angles.

The words 'makes angles with it,' are introduced to exclude the case in which the line A B is at the extremity of B C.

(83) From this proposition it appears, that if several right lines stand on the same right line at the same point, and make angles with it, all the angles taken together are equal to two right angles.

Also if two right lines intersecting one another make angles, these angles taken together are equal to four right angles.

The lines which bisect the adjacent angles A B C and A B D are at right angles; for the angle under these lines is evidently half the sum of the angles A B C and A B D.

If several right lines diverge from the same point, the angles into which they divide the surrounding space are together equal to four right angles.

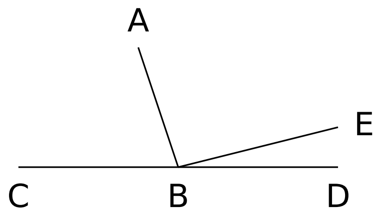
(84) When two angles as A B C and A B D are togther equal to two right angles, they are said to be *supplemental*, and one is called the *supplement* of the other.

(85) If two angles as C B A and E B A are together equal to a right angle, they are said to be *complemental*, one one is said to be the *complement* of the other.

Proposition XIV. Theorem.

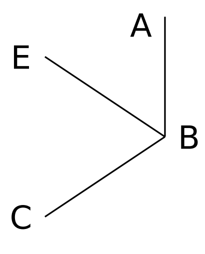
(86) *If two right lines (*C B *and* B D*) meeting another right line (*A B*) at the same point (*B*), and at opposite sides, make angles with it which are together equal to two right angles, those right angles (*C B *and* B D*) form one continued right line.*

For if possible, let B E and not B D be the continuation of the right line C B, then the angles C B A and A B E are are equal to two right angles



(XIII), but C B A and A B D are also equal to two right angles, by hypothesis, therefore C B A and A B D taken together are equal to C B A and A B E; take away from these equal quantities C B A which is common to both, and A B E shall be equal to A B D, a part to the whole, which is absurd; therefore B E is not the continuation of C B, and in the same manner it can be proved, that no other line except B D is the continuation of it, therefore B D forms with B C one continued right line.

In the enunciation of this proposition, the student should be cautious not to overlook the condition that the two right lines C B and B E forming angles, which are together equal to two right angles, with B A lie *at opposite sides* of B A. They might form angles together equal to two right angles with

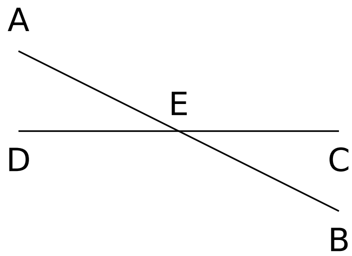


B A, yet not lie in the same continued line, if as in this figure they lay at *the same side of it.* It is assumed in this proposition that the line C B has a production. This is however granted by Postulate 2.

Proposition XV. Theorem.

(87) *If two right lines (*A B *and* C D*) intersect one another, the vertical angles are equal (*C E A *to* B E D, *and* C E B *to* A E D*).*

Because the right line C E stands upon the right line A B, the angle A E C together with the angle C E B is equal to two right angles (XIII); and because



the right line B E stands on the right line C D, the angle C E B together with the angle B E D is equal to two right angles (XIII); therefore A E C and C E B together are equal to C E B and B E D; take away the common angle C E B, and the remaining angle A E C is equal to B E D.

This proof may shortly be expressed by saying, that opposite angles are equal, because they have a common *supplement* (84).

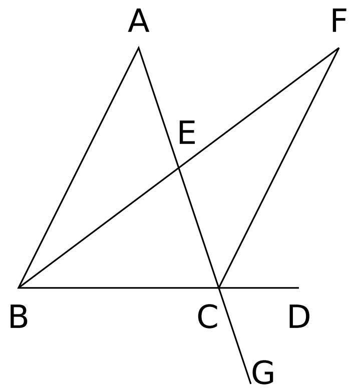
It is evident that angles which have a common supplement or complement (85) are equal, and that if they be equal, their supplements and complements must also be equal.

(88) The *converse* of this proposition may easily be proved, scil. If four lines meet at a point, and the angles vertically opposite be equal, each alternate pair of lines will be in the same right line. For if C E A be equal to B E D, and also C E B to A E D, it follows that C E A and C E B together are equal to B E D and A E D together. But all the four are together equal to four right angles (83), and therefore C E A and C E B are together equal to two right angles, therefore (XIV) A E and A B are in one continued line. In like manner it may be proved, that C E and D E are in one line.

Proposition XVI. Theorem.

(89) *If one side (*B C*) of a triangle (*B A C*) be produced, the external angle (*A C D*) is greater than either of the internal opposite angles (*A *or* B.*)*

For bisect the side A C in E (X), draw B E and produce it until E F be equal to B E (III), and join F C.



The triangles C E F and A E B have the sides C E and E F equal to the sides A E and E B (const.), and the angle C E F equal to A E B (XV), therefore the angles E C F and A are equal (IV), and therefore A C D is greater than A. In like manner it can be shown, that if A C be produced, the external angle B C G is greater than the angle B, and therefore that the angle A C D, which is equal to B C G (XV), is greater than the angle B.

(90) Cor. 1.—Hence it follows, that each angle of a triangle is less than the supplement of either of the other angles (84). For the external angle is the supplement of the adjacent internal angle (XIII).

(91) Cor. 2.—If one angle of a triangle be right or obtuse, the others must be acute. For the supplement of a right or obtuse angle is right or acute (82), and each of the other angles must be less than this supplement, and must therefore be acute.

(92) Cor. 3.—More than one perpendicular cannot be drawn from the same point to the same right line. For if two lines be supposed to be drawn, one of which is perpendicular, they will form a triangle having one right angle. The other angles must therefore be acute (91), and therefore the other line is not perpendicular.

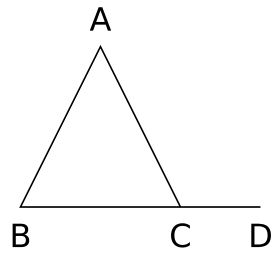
(93) Cor. 4.—If from any point a right line be drawn to a given right line, making with it an acute and obtuse angle, and from the same point a perpendicular be drawn, the perpendicular must fall at the side of the acute angle. For otherwise a triangle would be formed having a right and an obtuse angle, which cannot be (91).

(94) Cor. 5.—The equal angles of an isosceles triangle must be both acute.

Proposition XVII. Theorem.

(95) *Any two angles of a triangle (*B A C*) are together less than two right angles.*

Produce any side B C, then the angle A C D is greater than either of the angles A or B (XVI), therefore A C B together with either A or B is less than



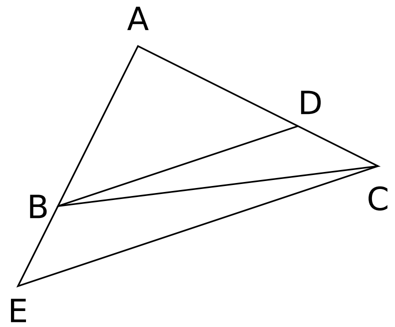
the same angle A C B together with A C D; that is, less than two right angles (VIII). In the same manner, if C B be produced from the point B, it can be demonstrated that the angle A B C together the angle A is less than two right angles; therefore any two angles of the triangle are less than two right angles.

This proposition and the sixteenth are included in the thirty-second. which proves that the three angles are together equal to two right angles.

Proposition XVIII. Theorem.

(96) *In any triangle (*B A C*) if one side (*A C*) be greater than another (*A B*), the angle opposite to the greater side is greater than the angle opposite to the less.*

From the greater side A C cut off the part A D equal to the less (III), and conterminous with it, and join B D.



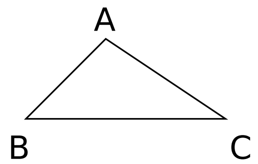
The triangle B A D being isosceles (V), the angles A B D and A D B are equal; but A D B is greater than the internal angle A C B (XVI): therefore A B D is greater than A C B, and therefore A B C is greater than A C B: but A B C is opposite the greater side A C, and A C B is opposite the less A B.

This proposition might also be proved by producing the lesser side A B, and taking A E equal to the greater side. In this case the angle A E C is equal to A C E (V), and therefore greater than A C B. But A B C is greater than A E C (XVI), and therefore A B C is greater than A C B

Proposition XIX. Theorem.

(97) *If in any triangle (*B A C*) one angle (*B*) be greater than another (*C*), the side (*A C*) which is opposite the greater angle is greater than the side* A B*, which is opposite to the less.*

For the side A C is either equal, or less, or greater than A B. It is not equal to A B, because the angle B would then be equal to C (V), which is contrary to the hypothesis.



It is not less than A B, because the angle B would then be less than C (XVIII), which is also contrary to the hypothesis.

Since therefore the side A C is neither equal to not less than A B, it is greater than it.

This proposition holds the same relation to the sixth, as the preceding does to the fifth. The four might be thus combined: one angle of a triangle is greater or less than another, or equal to it, according as the side opposed to the one is greater or less than, or equal to the side opposed to the other, and *vice versa*.

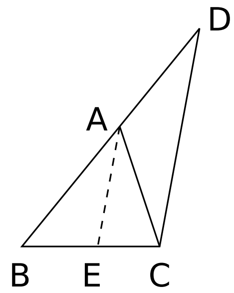
The student generally feels it difficult to remember which of the two, the eighteenth or nineteenth, is proved by construction, and which indirectly. By referring them to the fifth and sixth the difficulty will be removed.

Proposition XX. Theorem.

(98) *Any two sides (*A B *and* A C*) of a triangle (*B A C*) taken together, are greater than the third side (*B C*).*

Let the side B A be produced, and let A D be cut off equal to A C (III), and let D C be drawn.

Since A D and A C are equal, the angles D and A C D are equal (V). Hence



the angle B C D is greater than the angle D, and therefore the side B D in the triangle B C D is greater than B C (XIX). But B D is equal to B A and A C taken together, since A D was assumed equal to A C. Therefore B A and A C taken together are greater than B C.

This proposition is sometimes proved by bisecting the angle A. Let A E bisect it. The angle B E A is greater than E A C, and the angle C E A is greater than E A B (XVI); and since the parts of the angle A are equal, it follows, that each of the angles E is greater than each of the parts of A; and thence, by (XIX), it follows that B A is greater than B E, and A C greater than C E, and therefore that the sum of the former is greater than the sum of the latter.

The proposition might likewise be proved by drawing a perpendicular from the angle A on the side B C; but these methods seem inferior in clearness and brevity to that of Euclid.

Some geometers, among whom may be reckoned Archimedes, ridicule this proposition as being self evident, and contend that it should therefore be one of the axioms. That a truth is considered self evident is, however, not a sufficient reason why it should be adopted as a geometrical axiom (57).

(99) It follows immediately from this proposition, that the difference of any two sides of a triangle is less than the remaining side. For the sides A C and B C taken together are greater than A B; let the side A C be taken from

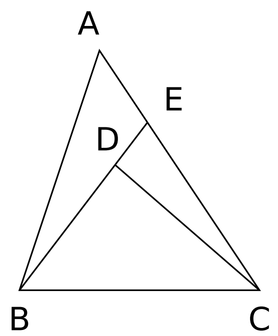
both, and we shall have the side B C greater than the remainder upon taking A C from A B; that is, then the difference between A B and A C.

In this proof we assume something more than is expressed in the fifth axiom. For we take for granted, that if one quantity (a) be greater than another (b), and that equals be taken from both, the remainder of the former (a) will be grater than the remainder of the latter (b). This is a principle which is frequently used, though not directly expressed in the axiom (55).

Proposition XXI. Theorem.

(100) *The sum of two right lines (*D B *and* D C*) drawn to a point (*D*) within a triangle (*B A C*) from the extremities of any side (*B C*), is less than the sum of the other two sides of the triangle (*A B *and* A C*), but the lines contain a greater angle.*

Produce B D to E. The sum of the sides B A and A E of the triangle B A E is greater than the third side B E (XX); add E C to each, and the sum



of the sides B A and A C is greater than the sum of B E and E C, but the sum of the sides D E and E C of the triangle D E C is greater than the third side D C (XX); add B D to each, and the sum of B E and E C is greater than the sum of B D and D C, but the sum of B A and A C is greater than that of B E and E C; therefore the sum of B A and A C is greater than that of B D and D C.

Because the external angle B D C is greater than the internal D E C (XVI), and for the same reason D E C is greater than A, the angle B D C is greater than the angle A.

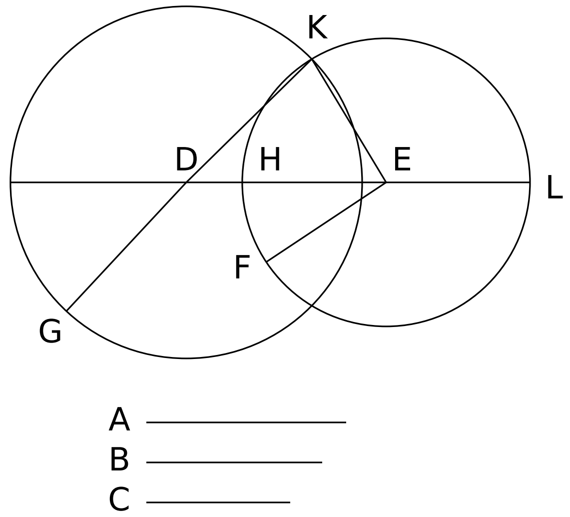
*** By the thirty-second proposition it will follow, that the angle B D C exceeds the angle A by the sum of the angles A B D and A C D. For the angle B D C is equal to the sum of D E C and D C E; and, again, the angle D E C is equal to the sum of the angles A and A B E. Therefore the angle B D C is equal to the sum of A, and the angles A B D and A C D.

Proposition XXII. Problem.

(101) *Given three right lines (*A, B *and* C*) the sum of any two of which is greater than the third, to construct a triangle whose sides shall be respectively equal to the given lines.*

Solution.

From any point D draw the right line D E equal to one of the given lines A (II), and from the same point draw D G equal to another of the given



lines B, and from the point E draw E F equal to C. From the centre D with the radius D G describe a circle, and from the centre E with the radius E F describe another circle, and from a point K of intersection of these circles draw K D and K E.

Demonstration.

It is evident, that the sides D E, D K and K E of the triangle D K E are equal to the given right lines A, B and C.

$^*{}_*{}^*$ In this solution Euclid assumes that the two circles will have at least one point of intersection. To prove this, it is only necessary to show that a part of one of the circles will be within, and another part without the other (58).

Since D E and E K or E L are together greater than D K, it follows, that D L is greater than the radius of the circle K G, and therefore the point L is outside the circle. Also, since D K and E K are together greater than D E, if the equals E K and E H be taken from both, D H is less than D K, that is, D H is less than the radius of the circle, and therefore the point H is within it. Since the point H is within the circle and L without it, the one circle must intersect the other.

It is evident, that if the sum of the lines B and C were equal to the line A, the points H and K would coincide; for then the sum of D K and K E would equal D E. Also, if the sum of A and C were equal to B, the points K and L would coincide; for then D K would be equal to E K and D E, or to L D. It will hereafter appear, that in the former case the circles would touch externally, and in the latter internally.

If the line A were greater than the sum of B and C, it is easy to perceive that the circle would not meet, one being wholly outside the other; and if B were greater than the sum of A and C, they would not meet, one being wholly within the other.

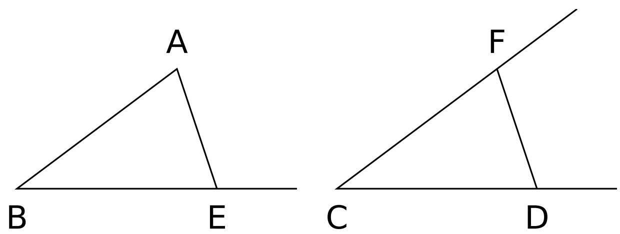
If the three right lines A B C be equal, this proposition becomes equivalent to the first, and the solution will be found to agree exactly with that of the first.

Proposition XXIII. Problem.

(102) *At a given point (*B*) in a given right line (*B E*) to make an angle equal to a given angle (*C*).*

Solution.

In the sides of the given angle take any points D and F; join D F, and construct a triangle E B A which shall be equilateral with the triangle D C F, and whose sides A B and E B meeting at the given point B shall be equal to



F C and D C of the given angle C (XXII). The angle E B A is equal to the given angle D C F.

Demonstration.

For as the triangles D C F and E B A have all their sides respectively equal, the angles F C D and A B E opposite the equal sides D F and E A are equal (VIII).

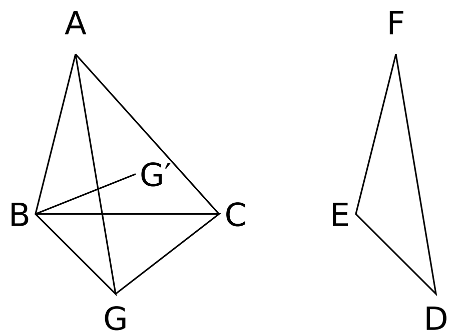
It is evident that the eleventh proposition is a particular case of this

Proposition XXIV. Theorem.

(103) *If two triangles (*E F D, B A C*) have two sides of the one respectively equal to two sides of the other (*F E *to* A B *and* F D *to* A C*), and if one of the angles (*B A C*) contained by the equal sides be greater than the other (*E F D*), the side (*B C*) which is opposite to the greater angle is greater than the side (*E D*) which is opposite to the less angle.*

From the point A draw the right line A G, making with the side A B, which is not the greater, an angle B A G equal to the angle E F D (XXIII). Make A G equal to F D (III), and draw B G and G C.

In the triangles B A G and E F D the sides B A and A G are equal respectively to E F and F D, and the included angles are equal (const.), and therefore B G is equal to E D. Also, since A G is equal to F D by const.,



and A C is equal to it by hyp., A G is equal to A C, therefore the triangle G A C is isosceles, and therefore the angles A C G and A G C are equal (V); but the angle B G C is greater than A G C, therefore greater than A C G, and therefore greater than B C G; then in the triangle B G C the angle B G C is greater than B C G, therefore the side B C is greater than B G (XIX), but B G is equal to E D, and therefore B C is greater than E D.

In this demonstration it is assumed by Euclid, that the points A and G will be on different sides of B C, or, in other words, that A H is less than A G or A C. This may be proved thus:—The side A C not being less than A B, the angle A B C cannot be less than the angle A C B (XVIII). But the angle A B C must be less than the angle A H C (XVI); therefore the angle A C B is less than A H C, and therefore A H less than A C or A G (XIX).

In the construction for this proposition Euclid has omitted the words 'with the side which is not the greater.' Without these it would not follow that the point G would fall below the base B C, and it would be necessary to give demonstrations for the cases in which the point G falls on, or above the base B C. On the other hand, if these words be inserted, it is necessary in order to give validity to the demonstration, to prove as above, that the point G falls below the base.

If the words 'with the side not the greater' be not inserted, the two omitted cases may be proved as follows:

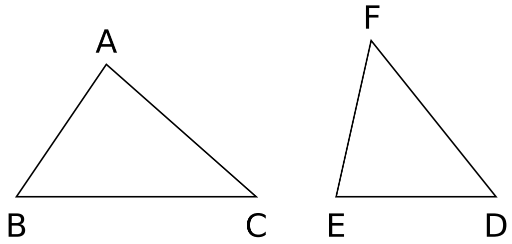
If the point G fall on the base B C, it is evident that B G is less than B C (51).

If G fall above the base B C, let it be at G′. The sum of the lines B G′ and A G′ is less than the sum of A C and C B (XXI). The equals A C and A G′ being taken away, there will remain B G′ less than B C.

Proposition XXV. Theorem.

(104) *If two triangles (*B A C *and* E F D*) have two sides of the one respectively equal to two of the other (*B A *to* E F *and* A C *to* F D*), and if the third side of the one (*B C*) be greater than the third side (*E D*) of the other, the angle (*A*) opposite the greater side is greater than the angle (*F*), which is opposite to the less.*

The angle A is either equal to the angle F, or less than it, or greater than it.



It is not equal; for if it were, the side B C would be equal to the side E D (IV), which is contrary to the hypothesis.

It is not less; for if it were, the side B C would be less than the side E D (XXIV), which is contrary to the hypothesis.

Since therefore the angle A is neither equal to, nor less than F, it must be greater.

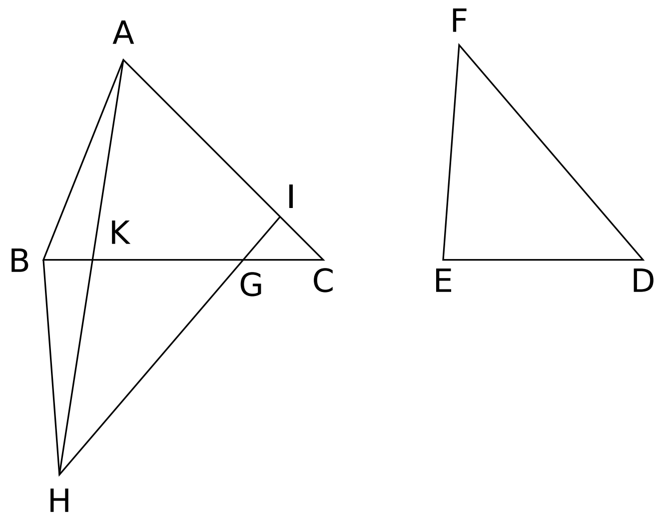
This proposition might be proved directly thus: On the greater side B C take B G equal to the lesser side E D, and on B G construct a triangle B H G equilateral with E F D. Join A H and produce H G to I.

The angle H will then be equal to the angle F.

1° Let B G be greater than B K.

Since B A and B H are equal, the angles B A H and B H A are equal (V). Also since H G is equal to A C, it is greater than A I, and therefore H I is greater than A I, and therefore the angle H A I is greater than the angle A H I (XVIII). Hence, if the equal triangles B H A and B A H be added to these, the angle B A C will be found greater than the angle B H G, which is equal to F.

2° If B G be not greater than B K, it is evident that the angle H is less than the angle A.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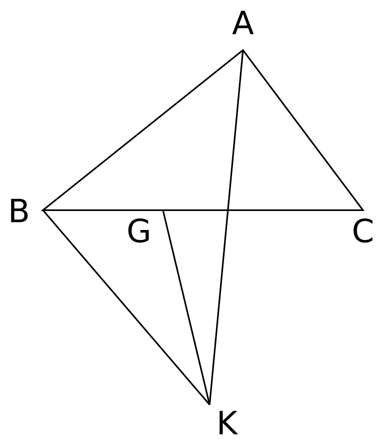
A
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The twenty-fourth and twenty-fifth propositions are analogous to the fourth and eighth, in the same manner as the eighteenth and nineteenth are to the fifth and sixth . The four might be announced together thus:

If two triangles have two sides of the one respectively equal to two sides of the other, the remaining side of the one will be greater or less than, or equal to the remaining side of the other, according as the angle opposed to it in the one is greater or less than, or equal to the angle opposed to it in the other, or *vice versa*.

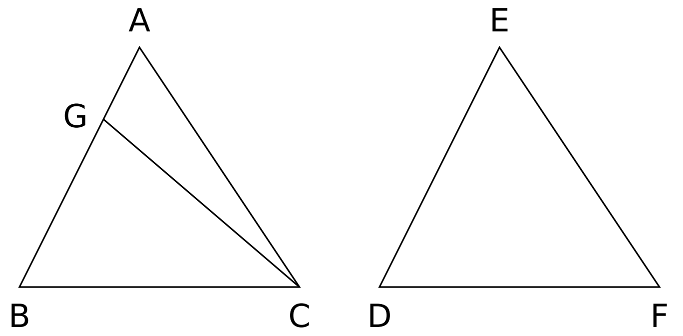
In fact, these principles amount to this, that if two lines of given lengths be placed so that one pair of extremities coincide, and so that in their initial position the lesser line is placed upon the greater, the distance between the extremities will then be the difference of the lines. If they be opened as to form a gradually increasing angle, the line joining their extremities will gradually increase, until the angle they include becomes equal to two right angles, when they will be in one continued line, and the line joining their extremities is their sum. Thus the major and minor limits of this line is the *sum* and *difference* of the given lines. This evidently includes the twentieth proposition.

Proposition XXVI. Theorem.

(105) *If two triangles (*B A C*,* D E F*) have two angles of the one respectively equal to two angles of the other (*B *to* D *and* C *to* F*), and a side of the one equal to a side of the other similarly placed with respect to the equal angles, the remaining sides and angles are respectively equal to one another.*

First let the equal sides be B C and D F, which lie between the equal angles; then the side B A is equal to the side D E.

For if it be possible, let one of them B A be greater than the other; make B G equal to D E, and join C G.

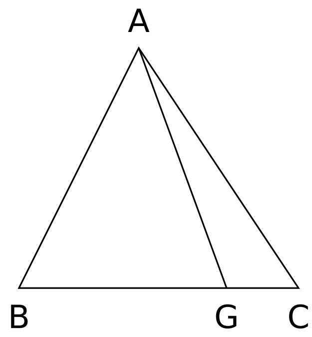


In the triangles G B C, E D F the sides G B, B C are respectively equal to the sides E D, D F (const.), and the angle B is equal to the angle D (hyp.), therefore the angles B C G and D F E are equal (IV); but the angle B C A is also equal to D F E (hyp.) therefore the angle B C G is equal to B C A (51), which is absurd: neither of the sides B A and D E therefore is greater than the other, and therefore they are equal, and also B C and D F are equal (IV), and the angles B and D; therefore the side A C is equal to the side E F, as also the angle A to the angle E (IV).

Next, let the equal sides be B A and D E, which are opposite to the equal angles C and F, and the sides B C and D F, shall also be equal.

For if it be possible, let one of them B C be greater than the other; make B G equal to D F, and join A G.

In the triangles A B G, E D F, the sides A B, B G are respectively equal to the sides E D, D F (const.), and the angle B is equal to the angle D (hyp.); therefore the angles A G B and E F D are equal (IV); but the angle C is also equal to E F D, therefore A G B and C are equal, which is absurd (XVI). Neither of the sides B C and D F is therefore greater than the other, and



they are consequently equal. But B A and D E are also equal, as also the angles B and D; therefore the side A C is equal to the side E F, and also the angle A to the angle E (IV).

It is evident that the triangles themselves are equal in every respect.

⁎ (106) COR. 1.—From this proposition and the principles previously established, it easily follows, that a line being drawn from the vertex of a triangle to the base, if any two of the following equalities be given (except the first two), the others may be inferred.

1° The equality of the sides of the triangle.

2° The equality of the angles at the base.

3° The equality of the angles under the line drawn, and the base.

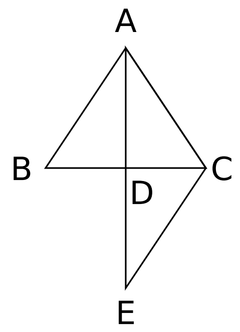
4° The equality of the angles under the line drawn, and the sides.

5° The equality of the segments of the base.

Some of the cases of this investigation have already been proved (74), (75), (76). The others present no difficulty, except in the case where the fourth and fifth equalities are given to infer the others. This case may be proved as follows.

If the line A D which bisects the vertical angle (A) of a triangle also bisect the base B C, the triangle will be isosceles; for produce A D so that D E shall be equal to A D, and join E C. In the triangles D C E and A D B the angles vertically opposed at D are equal, and also the sides which contain them; therefore (IV) the angles B A D and D E C are equal, and also the sides A B and E C. But the angle B A D is equal to D A C (hyp.); and therefore D A C is equal to the angle E, therefore (VI) the sides A C and E C are equal. But A B and E C have already been proved equal, and therefore A B and A C are equal.

⁎ (107) The twenty-sixth proposition furnishes the third criterion which has been established in the Elements for the equality of two trian-



gles. It may be observed, that in a triangle there are six quantities which may enter into consideration, and in which two triangles may agree or differ; viz. the three sides and the three angles. We can in most cases infer the equality of two triangles in every respect, if they agree in any three of those six quantities *which are independent of each other.* To this, however, there are certain exceptions, as will appear by the following general investigation of the question.

When two triangles agree in three of the six quantities already mentioned, these three must be some of the six following combinations:

1° Two sides and the angle between them.

2° Two angles and the side between them.

3° Two sides, and the angle opposed to one of them.

4° Two angles, and the side opposed to one of them.

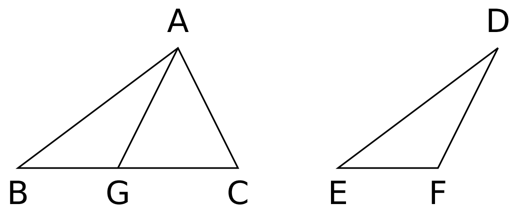
5° The three sides.

6° The three angles.

The first case has been established in the fourth, and the second and fourth in the twenty-sixth proposition. The fifth case has been established by the eighth, and in the sixth case the triangles are not necessarily equal. In this case, however, the three data are not independent, for it will appear by the thirty-second proposition, that any one angle of a triangle can be inferred from the other two.

The third is therefore the only case which remains to be investigated.

*** (108) 3° *To determine under what circumstance two triangles having two sides equal each to each, and the angles opposed to one pair of equal sides equal, shall be equal in all respects.* Let the sides A B and B C be equal to D E and E F, and the angle A be equal to the angle D. If the two angles B and E be equal, it is evident that the triangles are in every respect equal by (IV), and that C and F are equal. But if B and E be not equal, let one B be greater than the other E; and from B let a line B G be drawn, making



the angle A B G equal to the angle E. In the triangles A B G and D E F, the angles A and A B G are equal respectively to D and E, and the side A B is equal to D E, therefore (XXVI) the triangles are in every respect equal; and the side B G is equal to E F, and the angle B G A equal to the angle F. But since E F is equal to B C, B G is equal to B C, and therefore (V) B G C is equal to B C G, and therefore C and B G A or F are supplemental.

(109) Hence, *if two triangles have two sides in the one respectively equal to two sides in the other, and the angles opposed to one pair of equal sides equal, the angles opposed to the other equal sides will be either equal or supplemental.*

*** (110) Hence it follows, that if two triangles have two sides respectively equal each to each, and the angles opposed to one pair of equal sides equal, the remaining angles will be equal, and therefore the triangles will be in every respect equal, if there be any circumstance from which it may be inferred that the angles opposed to the other pair of equal sides are of the same species.

(Angles are said to be of the same species when they are both acute, both obtuse, or both right).

For in this case, if they be not right they cannot be supplemental, and must therefore be equal (109), in which case the triangles will be in every respect equal, by (XXVI).

If they be both right, the triangles will be equal by (108); because in that case G and C being right angles, B G must coincide with B C, and the triangle B G A with B C A; but the triangle B G A is equal to E F D, therefore &c.

*** (111) There are several circumstances which may determine the angles opposed to the other pair of equal sides to be of the same species, and therefore which will determine the equality of the triangles; amongst which are the following:

If one of the two angles opposed to the other pair of equal side be right; for a right angle is its own supplement.

If the angles which are given equal be obtuse or right; for then the other angles must be all acute (91), and therefore of the same species.

If the angles which are included by the equal sides be both right or obtuse; for then the remaining angles must be both acute.

If the equal sides opposed to angles which are not given equal be less than the other sides, these angles must be both acute (XVIII).

In all these cases it may be inferred, that the triangles are in every respect equal.

It will appear by prop. 38, that if two triangles have two sides respectively equal, and the included angles supplemental, their areas are equal.

(The *area* of a figure is the quantity of surface within its perimeter).

(112) If several right lines be drawn from a point to a given right line.

1° The shortest is that which is perpendicular to it.

2° Those equally inclined to the perpendicular are equal, and *vice versa*.

3° Those which meet the right line at equal distances from the perpendicular are equal, and *vice versa*.

4° Those which make greater angles with the perpendicular are greater, and *vice versa*.

5° Those which meet the line at greater distances from the perpendicular are greater, and *vice versa*.

6° More than two equal right lines cannot be drawn from the same point to the same right line.

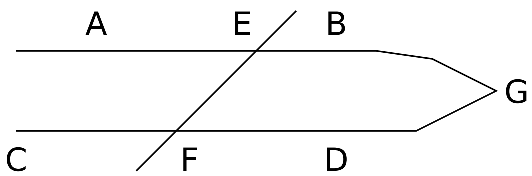
The student will find no difficulty in establishing these principles.

*** (113) If any number of isosceles triangles be constructed upon the same base, their vertices will be all placed upon the right line, which is perpendicular to the base, and passes through its middle point. This is a very obvious and simple example of a species of theorem which frequently occurs in geometrical investigations. This perpendicular is said to be the *locus* of the vertex of isosceles triangles standing on the same base.

Proposition XXVII. Theorem.

*(114) If a line (*E F*) intersect two right lines (*A B *and* C D*), and make the alternate angles equal to each other (*A E F *to* E F D*), these right lines are parallel.*

For, if it be possible, let those lines not be parallel but meet in G; the external angle A E F of the triangle E G F is greater than the internal E F G (XVI); but it is also equal to it (by hyp.), which is absurd; therefore A B

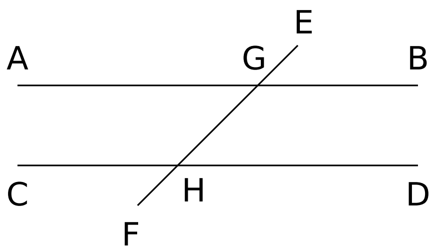


and C D do not meet at the side B D; and in the same manner it can be demonstrated, that they do not meet at the side A C; since, then, the right lines do not meet on either side they are parallel.

Proposition XXVIII. Theorem.

(115) *If a line (*E F*) intersect two right lines (*A B *and* C D*), and make the external angle equal to the internal and opposite angle on the same side of the line (*E G A *to* G H C*, and* E G B *to* G H D*); or make the internal angles at the same side (*A G H *and* C H G *or* B G H *and* D H G*) equal together to two right angles, the two right lines are parallel to one another.*

First, let the angles E G A and G H C be equal; and since the angle E G A is equal to B G H (XV), the angles G H C and B G H are equal; but they



are the alternate angles, therefore the right lines A B and C D are parallel (XXVII).

In the same manner the proposition can be demonstrated, if the angles E G B and G H D were given equal.

Next, let the angles A G H and C H G taken together be equal to two right angles; since the angles G H D and G H C taken together are also equal to two right angles (XIII). the angles A G H and C H G taken together are equal to the angles G H D and C H G taken together; take away the common angle C H G and the remaining angle A G H is equal to G H D; but they are the alternate angles, and therefore the right lines A B and C D are parallel (XXVII). In the same manner the proposition can be demonstrated, if the angles B G H and D H G were given equal to two right angles.

By this proposition it appears, that if the line G B makes the angle B G H equal to the supplement of G H D (84), the line G B will be parallel to H D. In the twelfth axiom (54) it is *assumed*, that if a line make an angle with G H less than the supplement of G H D, that line will *not* be parallel to H D, and will therefore meet it, if produced. The principle, therefore, which is really assumed is, that two right lines which intersect each other cannot be both parallel to the same right line, a principle which seems to be nearly self-evident.

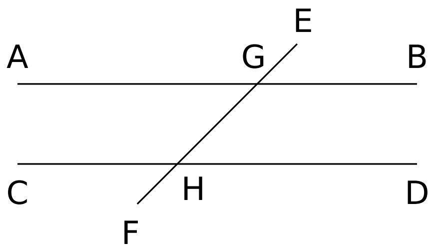
If it be granted that the two right lines which make with the third, G H, angles less than two right angles be not parallel, it is plain that they must meet on that side of G H on which the angles are less than two right angles; for the line passing through G, which makes a less angle than B G H, with G H on the side B D, will make a greater angle than A G H with G H on the side A C; and therefore that part of the line which lies on the side A C will lie above A G, and therefore can never meet H C.

Various attempts have been made to supercede the necessity of assuming the twelfth axiom; but all that we have ever seen are attended with still greater objections. Neither does it seem ot us, that the principle which is really assumed as explained above can reasonably be objected against. See Appendix, II.

Proposition XXIX. Theorem.

(116) *If a right line (*E F*) intersect two parallel right lines (*A B *and* C D*), it makes the alternate angles equal (*A G H *to* G H D*, and* C H G *to* H G B*); and the external angle equal to the internal and opposite upon the same side (*E G A *to* G H C*, and* E G B *to* G H D*); and also the two internal angles at the same side (*A G H *and* C H G, B G H *and* D H G*) together equal to two right angles.*

1° The alternate angles A G H and G H D are equal; for if it be possible, let one of them A G H be greater than the other, and adding the angle B G H to both, A G H and B G H together are greater than B G H and G H D; but



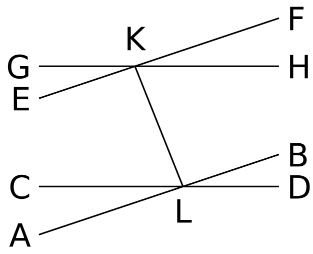
A G H and B G H together are equal to two right angles (XIII), therefore B G H and G H D are less than two right angles, and therefore the lines A B and C D, if produced, would meet at the side B D (Axiom 12); but they are parallel (hyp.), and therefore cannot meet, which is absurd. Therefore neither of the angles A G H and G H D is greater than the other; they are therefore equal.

In the same manner it can be demonstrated, that the angles B G H and G H C are equal.

2° The external angle E G B is equal to the internal G H D; for the angle E G B is equal to the angle A G H (XV); and A G H is equal to the alternate angle G H D (first part); therefore E G B is equal to G H D. In the same manner it can be demonstrated, that E G A and G H C are equal.

3° The internal angles at the same side B G H and G H D together are equal to two right angles; for since the alternate angles G H D and A G H are equal (first part), if the angle B G H be added to both, B G H and G H D together are equal to B G H and A G H and therefore are equal to two right angles (XIII). In the same manner it can be demonstrated, that the angles A G H and G H C together are equal to two right angles.

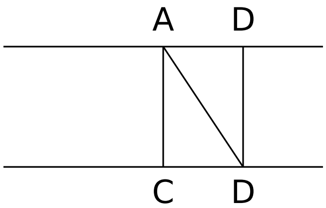
(117) Cor. 1.—If two right lines which intersect each other (A B, C D) be parallel respectively to two others (E F, G H), the angles included by those lines will be equal.



Let the line I K be drawn joining the points of intersection. The angles C I K and I K H are equal, being alternate; and the angles A I K and I K F are equal, for the same reason. Taking the former from the latter, the angles A I C and H K F remain equal. It is evident that their supplements C I B and G K F are also equal.

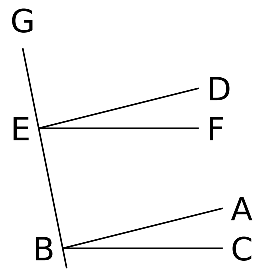
(118) Cor. 2.—If a line be perpendicular to one of two parallel lines, it will be also perpendicular to the other; for the alternate angles must be equal.

(119) Cor. 3.—The parts of all perpendiculars to two parallel lines intercepted between them are equal.



For let A B be drawn. The angles B A C and A B D are equal, being alternate; and the angles B A D and A B C are equal, for the same reason; the side A B being common to the two triangles, the sides A C and B D must be equal (XXVI).

(120) Cor. 4.—If two angles be equal (A B C and D E F), and the sides A B and D E be parallel, and the other sides B C and E F lie at the same side of them, they will also be parallel; for draw B E. Since A B and D E

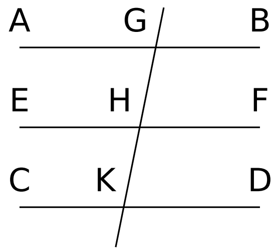


are parallel, the angles G B A and G E D are equal. But, by hypothesis, the angles A B C and D E F are equal; adding these to the former, the angles G B C and G E F are equal. Hence the lines B C and E F are parallel.

PROPOSITION XXX. THEOREM.

(121) *If two right lines (*A B, C D*) be parallel to the same right line (*E F*), they are parallel to each other.*

Let the right line G K intersect them; the angle A G H is equal to the angle G H F (XXIX); and also the angle H K D is equal to G H F (XXIX);



therefore A G H is equal to G K D; and therefore the right lines A B and C D are parallel.

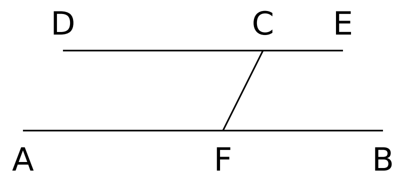
(122) COR.—Hence two parallels to the same line cannot pass through the same point. This is, in fact, equivalent to the twelfth axiom (115).

Proposition XXXI. Problem.

(123) *Through a given point* (C) *to draw a right line parallel to a given right line* (A B)*.*

Solution.

In the line A B take any point F, join C F, and at the point C and with the right line C F make the angle F C E equal to A F C (XXIII), but at the opposite side of the line C F; the line D E is parallel to A B.



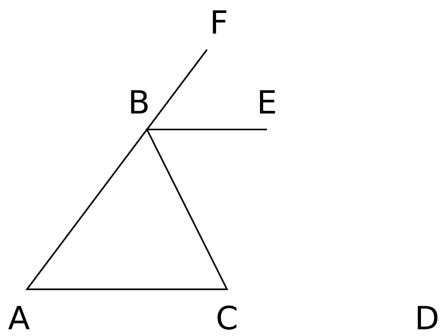
Demonstration.

For the right line F C intersecting the lines D E and A B makes the alternate angles E C F and A F C equal, and therefore the lines are parallel (XXVII).

Proposition XXXII. Theorem.

(124) *If any side (A B) of a triangle (A B C) be produced, the external angle (F B C) is equal to the sum of the two internal and opposite angles (A and C); and the three internal angles of every triangle taken together are equal to two right angles.*

Through B draw B E parallel to A C (XXXI.) The angle F B E is equal to the internal angle A (XXIX), and the angle E B C is equal to the alternate C (XXIX); therefore the whole external angle F B C is equal to the two internal angles A and C.



The angle A B C with F B C is equal to two right angles (XIII); but F B C is equal to the two angles A and C (first part); therefore the angle A B C together with the angles A and C is equal to two right angles. See Appendix, II.

(125) Cor. 1.—If one angle of a triangle be right, the sum of the other two is equal to a right angle.

(126) Cor. 2.—If one angle of a triangle be equal to the sum of the other two angles, that angle is a right angle.

(127) Cor. 3.—An obtuse angle of a triangle is greater and an acute angle less than the sum of the other two angles.

(128) Cor. 4.—If one angle of a triangle be greater than the sum of the other two it must be obtuse; and if it be less than the sum of the other two it must be acute.

(129) Cor. 5.—If two triangles have two angles in the one respectively equal to two angles in the other, the remaining angles must be also equal.

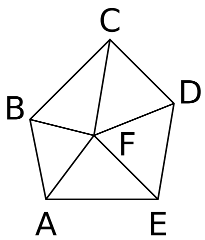
(130) Cor. 6.—Isosceles triangles having equal vertical angles must also have equal base angles.

(131) Cor. 7.—Each base angle of an isosceles triangle is equal to half the external vertical angle.

(132) Cor. 8.—The line which bisects the external vertical angle of an isosceles triangle is parallel to the base, and *vice versa*.

(133) Cor. 9.—In a right-angled isosceles triangle each base angle is equal to half a right angle.

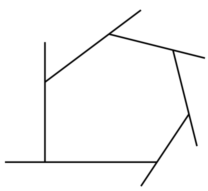
(134) Cor. 10.—All the internal angles of any rectilinear figure A B C D E, together with four right angles, are equal to twice as many right angles as the figure has sides.



Take any point F within the figure, and draw the right lines F A, F B, F C, F D, and F E. There are formed as many triangle as the figure has sides, and therefore all their angles taken together are equal to twice as many right angles as the figure has sides (XXXII); but the angles at the point F are equal to four right angles (83); and therefore the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has sides.

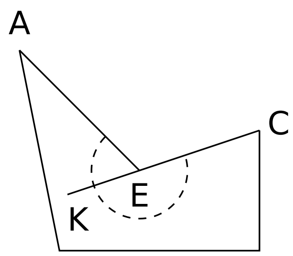
This is the first corollary in the Elements, and the following is the second.

(135) Cor. 11.—The external angles of any rectilinear figure are together equal to four right angles: for each external angle, with the internal adjacent to it, is equal to two right angles (XIII); therefore all the external



angles with all the internal are equal to twice as many right angles as the figure has sides; but the internal angles, together with four right angles, are equal to twice as many right angles as the figure has sides (134). Take from both, the internal angles and the internal remain equal to four right angles.

*** This corollary is only true of what are called *convex figures*; that is, of figures in which every internal angle is less than two right angles. Some figures, however, have angles which are called *reentrant* angles, and which are greater than two right angles. Thus in this figure the angle A B C exceeds two



right angles, by the figure K B A, formed by the side B A with the production of the side B C. This angle K B A, is that which in ordinary cases is the external angle, but which in the present instance constitutes a part of the internal angle, and in this case there is no external angle. The angle which is considered as the reentrant angle, and one of the internal angles of the figure is marked with the dotted curve in the figure. See (14).

*** (136) A figure which has no reentrant angle is called a *convex figure.*

It should be observed, that the first corollary applies to all rectilinear figures, whether convex or not, but the second only to convex figures.

*** (137) If a figure be not convex each reentrant angle exceeds two right angles by a certain excess, and has no adjacent external angle, while each ordinary angle, together with its adjacent external angle, is equal to two right angles. Hence it follows, that the sum of all the angles internal and external, including the reentrant angles, is equal to twice as many right angles as the figure has sides, together with the excess of every reentrant angle above two right angles. But (134) the sum of the internal angles alone is equal to twice as many right angles as the figure has sides, deducting four; hence the sum of the external angles must be equal to those four right angles, together with the excess of every reentrant angle above two right angles.

The sum of the external angles of every convex figure must be the same; and, however numerous the sides and angles be, this sum can never exceed four right angles.

If every pair of alternate sides of a convex figure be produced to meet, the sum of the angles so formed will be equal to $2n - 8$ right angles. This may be proved by showing that each of these angles with two of the external angles is equal to two right angles.

_ (138) Cor. 12.—The sum of the internal angles of a figure is equal to a number of right angles expressed by twice the number of sides, deducing four; also as each reentrant angle must be greater than two right angles, the sum of the reentrant angles must be greater than twice as many right angles as there are reentrant angles. Hence it follows, that twice the number of sides deducting four, must be greater than twice the number of reentrant angles, and therefore that the number of sides deducting two, must be greater than the number of reentrant angles; from which it appears, that the number of reentrant angles in a figure must always be at least three less than the number of sides. There must be therefore at least three angles in every figure, which are each less than two right angles.

_ (139) Cor. 13.—A triangle cannot therefore have any reentrant angle, which also follows immediately from considering that the three angles are together equal to two right angles, while a single reentrant angle would be greater than two right angles.

_ (140) Cor. 14.—No equiangular figure can have a reentrant angle, for if one angle were reentrant all should be so, which cannot be (138).

_ (141) Cor. 15.—If the number of sides in an equiangular figure be given, the magnitude of its angles can be determined. Since it can have no reentrant angle, the sum of its external angles is equal to four right angles; the magnitude of each external angle is therefore determined by dividing four right angles by the number of sides. This being deducted from two right angles, the remainder will be the magnitude of each angle. Thus the fraction whose numerator is 4, and those denominator is the number of sides, expresses the part of a right angle which is equal to the external angle of the figure, and if this fraction be deducted from the number 2, the remainder wil express the internal angle in parts of a right angle. In the notation of arithmetic, if n be the number of sides, the external angle is the $\dfrac{4}{n}^{\text{th}}$ and the internal angle the $\left(2-\dfrac{4}{n}\right)^{\text{th}}$ of a right angle.

_ (142) Cor. 16.—The sum of the angles of every figure is equal to an even number of right angles. For twice the number of sides is necessarily even, and the even number four being subducted leaves an even remainder. Hence it appears, that no figure can be constructed the sum of whose angles is equal to 3, 5, or 7 right angles, &c.

_ (143) Cor. 17.—If the number of right angles to which the sum of the angles of any figure is equal be given, the number of sides may be found.

For since the number of right angles increased by four is equal to twice the number of sides, it follows, that half the number of right angles increased by two is equal to the number of sides.

⁎ (144) Cor. 18.—If all the angles of a figure be right, it must be a quadrilateral, and therefore a right angled parallelogram. For (141) the magnitude of each external angle is determined in parts of a right angle by dividing 4 by the number of sides; in the present case each external angle must be a right angle, and therefore 4 divided by the number of sides must be 1, and therefore the number of sides must be four. Each of the four angles being right, every adjacent pair is equal to two right angles, and therefore the opposite sides of the figure are parallel.

⁎ (145) Cor. 19.—The angle of an equilateral triangle is equal to one third of two right angles, or two thirds of a right angle.

That one third of two right angles is equal to two thirds of one right angle, easily appears from considering that as three thirds of a right angle is equal to one right angle, six thirds will be equal to two right angles, and one third of this is two thirds of one right angle.

⁎ (146) Cor. 20.—To trisect a right angle. Construct any equilateral triangle and draw a line (XXIII), cutting off from the given angle an angle equal to an angle of the equilateral triangle. This angle being two thirds of the whole, if it be bisected, the whole right angle will be trisected.

By the combination of bisection and trisection a right angle may be divided into 2, 3, 4, 6, 8, &c equal parts.

N.B. The general problem to trisect *any angle* is one which has never been solved by plane Geometry.

⁎ (147) Cor. 21.—The multisection of a right angle may be extended by means of the angles of the regular polygons.

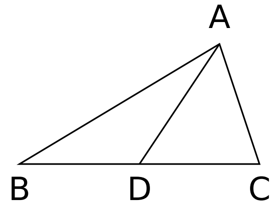
In a regular pentagon the external angle is four fifths of a right angle; the complement of this angle being the fifth of a right angle solves the problem to divide a right angle into five equal parts.

In a regular heptagon the external angle is four sevenths of a right angle, which being divided into four equal parts (IX) gives the seventh of a right angle, and solves the problem to divide a right angle into seven equal parts.

Thus in general the problem of the multisection of a right angle is resolved to that of the construction of the regular polygons, and *vice versa*. On this subject the student is referred to the fourth book of the Elements.

⁎ (148) Cor. 22.—The vertical angle A of a triangle is right, acute or obtuse, according as the line A D which bisects the base B C is equal to, greater or less than half the base B D.

1. If the line A D be equal to half the base B D, the triangles A D B and A D C will be isosceles, therefore the angles B A D and C A D will be

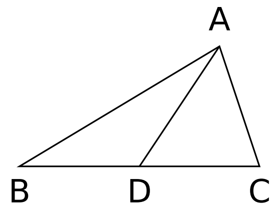


respectively equal to the angles B and C. The angle A is therefore equal to the sum of B and C, and is therefore (126) a right angle.

2. If A D be greater than B D or D C, the angles B A D and C A D are respectively less than the angles B and C, and therefore the angle A is less than the sum of B and C, and is therefore (128) acute.

3. If A D be less than B D or D C, the angles B A D and C A D are respectively greater than B and C, and therefore the angle A is greater than the sum of B and C, and is therefore (128) obtuse.

⁎ (149) Cor. 23.—The line drawn from the vertex A of a triangle bisecting the base B C is equal to, greater or less than half the base, according as the angle A is right, acute, or obtuse.



1. Let the angle A be right. Draw A D so that the angle B A D shall be equal to the angle B. The line A D will then bisect B C, and be equal to half of it.

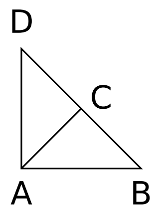
For the angles B and C are together equal to the angle A (125), and since B is equal to B A D, C must be equal to C A D. Hence it follows, (VI) that B D A and C D A are isosceles triangles, and that B D and C D are equal to A D and to each other.

2. Let A be acute, and draw A D bisecting B C. The line A D must be greater than B D or D C; for if it were equal to them the angle A would be right, and if it were less it would be obtuse (148).

3. Let A be obtuse, and draw A D bisecting B C. The line A D must be less than each of the parts B D, D C; for if it were equal to them the angle A

would be right, and if it were greater the angle A would be acute (148).

⁎ (150) Cor. 24.—To draw a perpendicular to a given right line through its extremity without producing it.

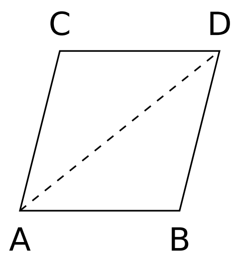


Take a part A B from the extremity A, and construct on it an equilateral triangle A C B. Produce B C so that C D shall be equal to A C, and draw D A. This will be the perpendicular required. For since A C bisects B D, and is equal to half of it, the angle D A B is right (148).

Proposition XXXIII. Theorem.

(150) *Right lines (*A C *and* B D*) which join the adjacent extremities of two equal and parallel right lines (*A B *and* C D*) are themselves equal and parallel.*

Draw the diagonal A D, and in the triangles C D A and B A D the sides C D and B A are equal (by hyp); A D is common to both triangles, and the

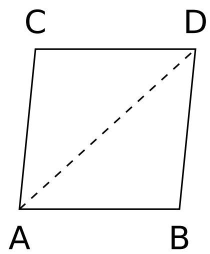


angle C D A is equal to the alternate B A D (XXIX); therefore the lines A C and B D are equal, and also the angles C A D and B D A; therefore the right line A D cutting the right lines A C and B D makes the alternate angles equal, and therefore (XXVII) the right lines A C and B D are parallel.

Proposition XXXIV. Theorem.

(151) *The opposite sides (*A B *and* C D, A C *and* B D*) of a parallelogram (*A D*) are equal to one another, as are also the opposite angles (*A *and* D, C *and* B*), and the parallelogram itself is bisected by its diagonal (*A D*).*

For in the triangles C D A, B A D, the alternate angles C D A and B A D, C A D and B D A are equal to one another (XXIX), and the side A D between the equal angles is common to both triangles; therefore the sides C D and



C A are equal to A B and B D (XXVI), and the triangle C D A is equal to the triangle B A D, and the angles A C D and A B D are also equal; and since the angle A C D with C A B is equal to two right angles (XXIX), and A B D with C D B is equal to two right angles, take the equals A C D and A B D from both, and the remainders C A B and C D B are equal.

(152) Cor. 1.—If two parallelograms have an angle in the one equal to an angle in the other, all the angles must be equal each to each. For the opposite angles are equal by this proposition, and the adjacent angles are equal, being their supplements.

(153) Cor. 2.—If one angle of a parallelogram be right, all its angles are right; for the opposite angle is right by (151), and the adjacent angles are right, being the supplements of a right angle.

(154) Both diagonals A D, B C being drawn, it may, with a few exceptions, be proved that a quadrilateral figure which has any *two* of the following properties will also have the others:

1° The parallelism of A B and C D.
2° The parallelism of A C and B D.
3° The equality of A B and C D.
4° The equality of A C and B D.
5° The equality of the angles A and D.
6° The equality of the angles B and C.

7° The bisection of A D by B C.

8° The bisection of B C by A D.

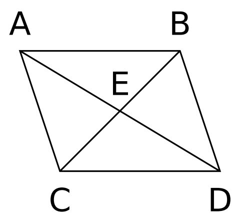
9° The bisection of the area by A D.

10° The bisection of the area by B C.

These ten data combined in pairs will give 45 distinct pairs; with each of these pairs it may be required to establish any of the eight other properties, and thus 360 questions respecting such quadrilaterals may be raised. These questions will furnish the student with a useful geometrical exercise. Some of the most remarkable cases are among the following corollaries:

The 9th and 10th data require the aid of subsequent propositions.

(155) Cor. 3.—*The diagonals of a parallelogram bisect each other.*

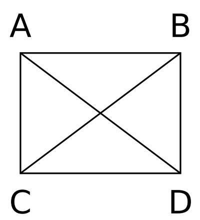


For since the sides A C and B D are equal, and also the angles C A E and B D E, as well as A C E and D B E, the sides (XXVI) C E and B E, and also A E and E D are equal.

(156) Cor. 4.—*If the diagonals of a quadrilateral bisect each other, it will be a parallelogram.*

For since A E and E C are respectively equal to D E and E B, and the angles A E C and D E B (XV) are also equal, the angles A C E and D B E are equal (IV); and, therefore, the lines A C and B D are parallel, and, in like manner, it may be proved that A B and C D are parallel.

(157) Cor. 5.—*In a right angled parallelogram the diagonals are equal.*



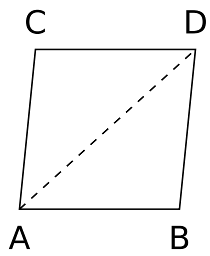
For the adjacent angles A and B are equal, and the opposite sides A C and B D are equal, and the side A B is common to the two triangles C A B and A B D, and therefore (IV) the diagonals A D and C B are equal.

If the diagonals of a parallelogram be equal, it will be right angled.

For in that case the three sides of the triangle C A B are respectively equal to those of D B A, and therefore (VIII) the angles A and B are equal. But they are supplemental, and therefore each is a right angle.

*** (158) The converses of the different parts of the 34th proposition are true, and may be established thus:

If the opposise sides of a quadrilateral be equal it is a parallelogram.



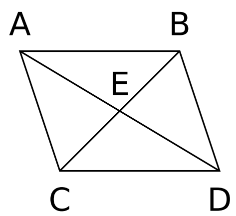
For draw A D. The sides of the triangles A C D and A B D are respectively equal, and therefore (VIII) the angles C A D and A D B are equal, and also the angles C D A and D A B. Hence the sides A C and B D, and also the sides A B and C D are parallel.

Hence the lozenge is a parallelogram, and a square has all its angles right.

If the opposite angles of a quadrilateral be equal, it will be a parallelogram.

For all the angles together are equal to four right angles (134); and since the opposite angles are equal, the adjacent angles are equal to half the sum of all the angles, that is, to two right angles, and therefore (XXVIII) the opposite sides are parallel.

If each of the diagonals bisect the quadrilateral, it will be a parallelogram.



This principle requires the aid of the 39th proposition to establish it. The triangles C A D C B D are equal, each being half of the whole area, therefore (XXXIX) the lines A B and C D are parallel. In the same manner D A B and D C B are equal, and therefore A C and B D are parallel.

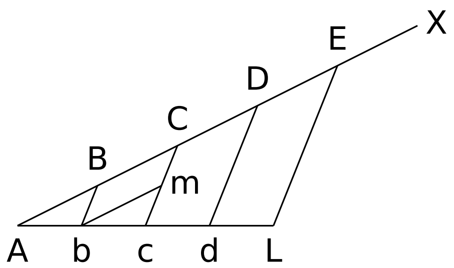
*** (159) *The diagonals of a lozenge bisect its angles.*

For each diagonal divides the lozenge into two isosceles triangles whose sides and angles are respectively equal.

*** (160) *If the diagonals of a quadrilateral bisect its angles, it will be a lozenge.*

For each diagonal in that case divides the figure into two triangles, having a common base placed between equal angles, and therefore (VI) the conterminous sides of the figure are equal.

*** (161) *To divide a finite right line* A L *into any given number of equal parts.*



From the extremity A draw any right line A X of indefinite length, and take upon it any part A B. Assume B C, C D, D E, &c. successively equal to A B (III), and continue in this manner until a number of parts be assumed on A X equal in number to the parts into which it is required to divide A L. Join the extremity of the last part E with the extremity L, and through B C D &c. draw parallels to E L. These parallels will divide A L into the required number of equal parts.

It is evident that the number of parts is the required number.

But these parts are also equal. For through b draw $b\ m$ parallel to A E, and $b\ c$ is a parallelogram; therefore $b\ m$ is equal to B C or to A B. Also the angle A is equal to the angle $c\ b\ m$ and A b B to $b\ c\ m$ Hence (XXVI) A b and $b\ c$ are equal. In like manner it may be proved, that $b\ c$ and $c\ d$ are equal, and so on.

(162) Parallelograms whose sides and angles are equal are themselves equal. For the triangles into which they are divided by their diagonals have

two sides and the included angles respectively equal, and are therefore (IV) equal, and therefore their doubles, the parallelograms, are equal.

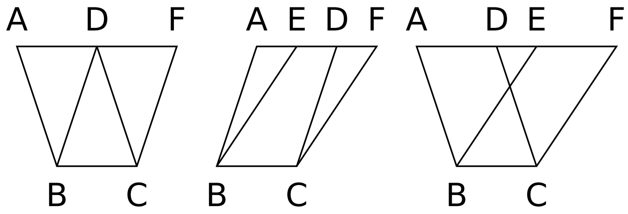
(163) Hence the squares of equal lines are equal.

(164) Also equal squares have equal sides. For the diagonals being drawn, the right angled isosceles triangles into which they divide the squares are equal; the sides of those triangles must be equal, for if not let parts be cut off from the greater equal to the less, and their extremities being joined, an isosceles right angled triangle will be found equal to the isosceles right angled triangle whose base is the diagonal of the other square (IV), and therefore equal to half of the other square, and also equal to half of the square a part of which it is; thus a part of the half square is equal to the half square itself, which is absurd.

Proposition XXXV. Theorem.

(165) *Parallelograms on the same base (*B C*) and between the same parallels are equal.*

For the angles B A F and C D F and also B E A and C F D are equal (XXIX), and the sides A B and D C are also equal (XXXIV), and therefore



(XXVI) the triangles B A E and C D F are equal. These being successively taken from the whole quadrilateral B A F C, leave the remainders, which are the parallelograms B D and B F, equal.

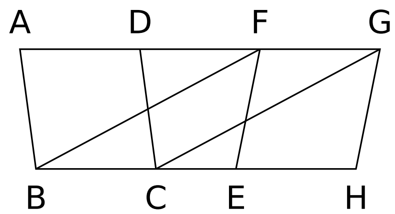
We have in this proof departed from Euclid in order to avoid the subdivision of the proposition into cases. The equality which is expressed in this and the succeeding propositions is merely equality of *area*, and not of sides or angles. The mere equality of area is expressed by *Legendre* by the word *equivalent*, while the term *equal* is reserved for equality in all respects. We have not thought this of sufficient importance however to justify any alteration in the text.

Proposition XXXVI. Theorem.

(166) *Parallelograms (*B D *and* E G*) on equal bases and between the same parallels are equal.*

Draw the right lines B F and C G.

Because the lines B C and F G are equal to the same E H (XXXIV), they are equal to one another; but they are also parallel, therefore B F and C G



which join their extremities are parallel (XXXIII), and B G is a parallelogram; therefore equal to both B D and E G (XXXV), and therefore the parallelograms B D and E G are equal.

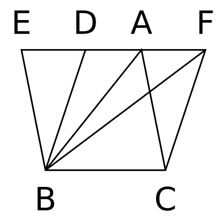
It is here supposed that the equal bases are placed in the same right line.

(167) Cor.—If two opposite sides of a parallelogram be divided into the same number of equal parts, and the corresponding points of division be joined by right lines, these right lines will severally divide the parallelogram into as many equal parallelograms.

Proposition XXXVII. Theorem.

(168) *Triangles (*B A C *and* B F C*) on the same base and between the same parallels are equal.*

Through the point B draw B E parallel to C A, and draw B D parallel to C F, and produce A F to meet these lines at E and D. The figures B E A C



and B D F C are parallelograms on the same base B C and between the same parallels, and therefore, (XXXV) equal; and the triangles B A C and B F C are their halves (XXXIV), and therefore also equal.

Proposition XXXVIII. Theorem.

(169) Triangles on equal bases and between the same parallels are equal.

For by the same construction as in the last proposition they are shown to be the halves of parallelograms on equal bases and between the same parallels.

(170) Cor. 1.—Hence a right line drawn from the vertex of a triangle bisecting the base bisects the area.

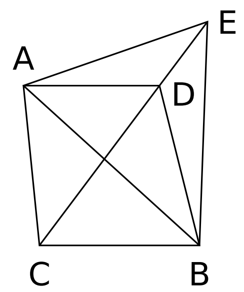
This proves that if two triangles have two sides respectively equal, and the included angles supplemental, the areas will be equal; for the two triangles into which the bisector of the base divides the triangle are thus related.

(171) Cor. 2.—In general, if the base of a triangle be divided into any number of equal parts (161) lines drawn from the vertex to the several points of division will divide the area of the triangle into as many equal parts.

Proposition XXXIX. Theorem.

(172) *Equal triangles (*B A C *and* B D C*) on the same base and on the same side of it are between the same parallels.*

For if the right line A D which joins the vertices of the triangles be not parallel to B C, draw through the point A a right line A E parallel to B C,



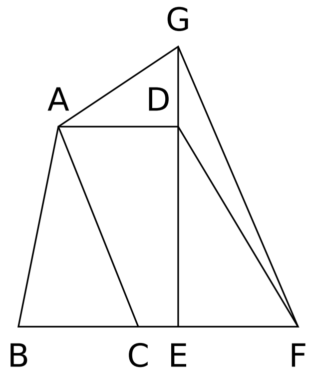
cutting a side B D of the triangle B D C or the side produced in a point E different from the vertex, and draw C E.

Because the right lines A E and B C are parallel, the triangle B E C is equal to B A C (XXXVII); but B D C is also equal to B A C (hyp.), therefore B E C and B D C are equal; a part equal to the whole, which is absurd. Therefore the line A E is not parallel to B C; and in the same manner it can be demonstrated, that no other line except A D is parallel to it; therefore A D is parallel to B C.

Proposition XL. Theorem.

(173) *Equal triangles (*B A C *and* E D F*) on equal bases and on the same side, are between the same parallels.*

For if the right line A D which joins the vertices of the two triangles be not parallel to B F, draw through the point A the right line A G parallel to B F, cutting a side D E of the triangle E D F, or the side produced in a point G different from the vertex, and join F G.



Because the right line A G is parallel to B F, and B C and E F are equal, the triangle G E F is equal to B A C (XXXVIII); but E D F is also equal to B A C (hyp.), therefore E G F and E D F are equal; a part equal to the whole, which is absurd. Therefore A G is not parallel to B F, and in the same manner it can be demonstrated, that no other line except A D is parallel to B F, therefore A D is parallel to B F.

From this and the preceding propositions may be deduced the following corollaries.

(174) Cor. 1.—Perpendiculars being drawn through the extremities of the base of a given parallelogram or triangle, and produced to meet the opposite side of the parallelogram or a parallel to the base of the triangle through its vertex, will include a right angled parallelogram which shall be equal to the given prallelogram; and if the diagonal of this right angled parallelogram be drawn, it will cut off a right angled triangle having the same base with the given triangle and equal to it. Hence any parallelogram or triangle is equal to a right angled parallelogram or triangle having an equal base and altitude.

(175) Cor. 2.—Parallelograms and triangles whose bases and altitudes are respectively equal are equal in area.

(176) Cor. 3.—Equal parallelograms and triangles on equal bases have equal altitudes.

(177) Cor. 4.—Equal parallelograms and triangles in equal altitudes have equal bases.

(178) Cor. 5.—If two parallelograms or triangles have equal altitudes, and the base of one be double the base of the other, the area of one will be also double the area of the other. Also if they have equal bases and the altitude of one be double the altitude of the other, the area of the one will be double the area of the other.

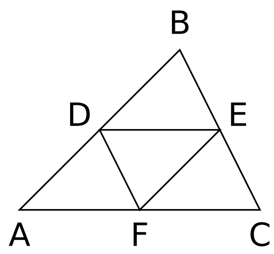
(179) Cor. 6.—The line joining the points of bisection fo the sides of a triangle is parallel to the base.

For if lines be drawn from the extremities of the bse to the points of bisection they will each bisect the area (170) of the triangle; therefore the triangles having the base of the given triangle as a common base and their vertices at the middle points of the sides, are equal, and therefore between the same parallel.

(180) Cor. 7.—*A parallel to the base of a triangle through the point of bisection of one side will bisect the other side.*

For by the last Cor. the line joining the points of bisection of the sides is parallel to the base, and two parallels to the same line cannot pass through the same point.

(181) Cor. 8.—*The lines which join the middle points* D E F *of the*



three sides of a triangle divide it into four triangles which are equal in every respect.

(182) Cor. 9.—*The line joining the points of bisection of each pair of sides is equal to half of the third side.*

*** (183) Cor. 10.—If two conterminous sides of a parallelogram be divided each into any number of equal parts, and through the several points of division of each side parallels be drawn to the other side, the whole parallelogram will be divided into a number of equal parallelograms, and this number is found by multiplying the number of parts in one side by the number of parts in the other. This is evident from considering, that by the parallels through the points of division of one side the whole parallelogram is resolved into as many equal parallelograms as there are parts in the side through the points of which the parallels are drawn; and the parallels through the points of division of the other side resolve each of these component parallelograms into as many equal parallelograms as there are parts in the other side. Thus the total number of parallelograms into which the entire is divided, is the product of the number of parts in each side.

*** (184) Cor. 11.—The square on a line is four times the square of its half.

*** (185) Cor. 12.—If the sides of a right angled parallelogram be divided into any number of equal parts, and such that the parts of one side shall have the same magnitude as those of the other, the whole parallelogram will be equal to the square of one of the parts into which the sides are divided, multiplied by the product of the number of parts in each side. Thus, if the base of the parallelogram be six feet and the altitude be eight feet, the area will be one square foot multiplied by the product of six and eight or forty-eight square feet. In this sense the area of such a parallelogram is said to be found by multiplying its base by its altitude.

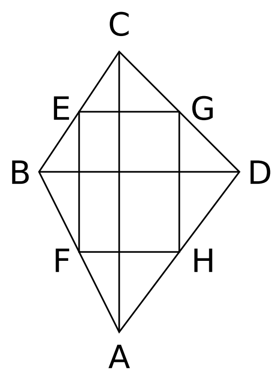
*** (186) Cor. 13.—Also, since the area of any parallelogram is equal to that of a right-angled parallelogram having the same base and altitude, and that of a triangle is equal to half that area, it follwos that the area of a parallelogram is the product of its base and its altitude, and that of a triangle is equal to half that product.

The phrase 'the product of two lines,' or 'multiplying one line by another, is only an abridged manner of expressing the multiplication of the *number of parts* in one of the lines by the *number of parts* in the other. Multiplication is an operation which can only be effected, properly speaking, by a *number* and not by a *line*

*** (187) Cor. 14.—The area of a square is found numerically by multiplying the number of equal parts in the side of the square by itself. Thus a square whose side is twelve inches contains in its area 144 square inches. Hence, in arithmetic, when a number is multiplied by itself the product is called its square. Thus 9, 16, 25, &c. are the *squares* of 3, 4, 5, &c.; and 3, 4, 5,&c. are called the *square roots* of the numbers 9, 16, 25, &c. Thus *square* and *square root* are correlative terms.

*** (188) Cor. 15.—*If the four sides of a quadrilateral* A B C D *be bisected, and the middle points* E F H G *of each pair of conterminous sides joined by right lines, those joining lines will form a parallelogram* E F H G *whose area is equal to half that of the quadrilateral.*

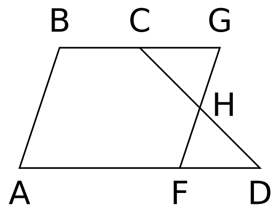
Draw C A and B D. The lines E F and G H are parallel to C A (179), and equal to half of C A (182). Therefore E F and G H are equal and parallel,



and therefore (XXIII) E F H G is a parallelogram. But E B F is one-fourth of C B A and G H D one fourth of C D A (181), and therefore E B F and G D H are together one-fourth of the whole figure. In like manner E C G and F A H are together one-fourth of the whole, and therefore F B E, E C G, G D H, and H A F are together one-half of the whole figure, and therefore E F H G is equal to half the figure.

*** (189) Cor. 16.—*A trapezium is equal to a parallelogram in the same altitude, and whose base is half the sum of the parallel bases.*

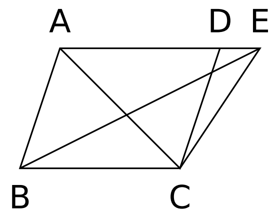
Let C D be bisected at H, and through H draw G F parallel to A B.



Since C G and F D are parallel, the angles G C H and G are respectively equal to D, and H F D (XXIX) and C H is equal to H D, therefore (XXVI) C G is equal to F D, and the triangle C H G to the triangle D H F. Therefore A F and B G are together equal to A D and B C, and the parallelogram A G to the trapezium A C; and since A F and B G are equal, A F is half the sum of A D and B C.

Proposition XLI. Theorem.

*(190) If a parallelogram (*B D*) and a triangle (*B E C*) have the same base and be between the same parallels, the parallelogram is double of the triangle.*



Draw C A. The triangle B E C is equal to the triangle B A C (XXXVII); but B D is double of the triangle B A C (XXXIV), therefore B D is also double of the triangle B E C.

(191) This proposition may be generalized thus: *If a parallelogram and triangle have equal bases and altitudes, the parallelogram is double the triangle* (175).

(192) Also, *If a parallelogram and a triangle have equal altitudes, and the base of the triangle be double the base of the parallelogram, the parallelogram and triangle will be equal* (178).

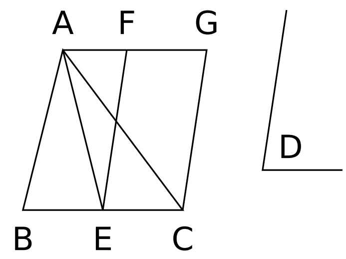
(193) *If a parallelogram and triangle have equal bases, and the altitude of the triangle be double the altitude of the parallelogram, they will be equal.*

Proposition XLII. Problem.

(194) *To construct a parallelogram equal to a given triangle (*B A C*) and having an angle equal to a given one (*D*).*

Solution.

Through the point A draw the right line A F parallel to B C, bisect B C the base of the triangle in E, and at the point E, and with the right line C E



make the angle C E F equal to the given one D; through C draw C G parallel to E F until it meet the line A F in G. C F is the required parallelogram.

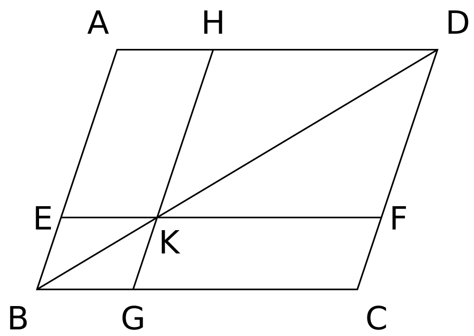
Demonstration.

Because E C is parallel to A G (const.), and E F parallel to C G, E G is a parallelogram, and has the angle C E F equal to the given one D (const.); and it is equal to the triangle B A C, because it is between the same parallels and on half of the base of the triangle (192).

Proposition XLIII. Theorem.

(195) *In a parallelogram (*A C*) the complements (*A K *and* K C*) of the parallelograms about the diagonal (*E G *and* H F*) are equal.*

Draw the diagonal B D, and through any point in it K draw the right lines F E and G H parallel to B C and B A; then E G and H F are the parallelograms



about the diagonal, and A K and K C their complements.

Because the triangles B A D and B C D are equal (XXXIV), and the triangles B G K, K F D are equal to B E K, K H D (XXXIV); take away the equals B G K and K E B, D F K and K H D from the equals B C D and B A D, and the remainders, namely, the complements A K and K C, are equal.

(196) Each parallelogram about the diagonal of a lozenge is itself a lozenge equiangular with the whole. For since A B and A D are equal, A B D and A D B are equal (V). But E K B and A D B are equal (XXIX), therefore E K B and E B K are equal, therefore E K and E B are equal, and therefore E G is a lozenge. It is evidently equiangular with the whole.

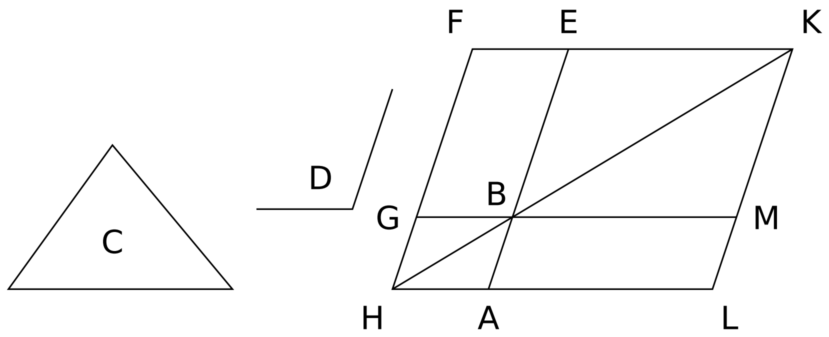
(197) It is evident that the parallelograms about the diagonal and also their complements, are equiangular with the whole parallelogram; for each has an angle in common with it (152).

Proposition XLIV. Problem.

(198) *To a given right line (A B) to apply a parallelogram which shall be equal to a given triangle (C), and have one of its angles equal to a given angle (D).*

Solution.

Construct the parallelogram B E F G equal to the given triangle C, and having the angle B equal to D, and so that B E be in the same right line with A B; and produce F G, and through A draw A H parallel to B G, and



join H B. Then because H L and F K are parallel the angles L H F and F are together equal to two right angles, and therefore B H F and F are together less than two right angles, and therefore H B and F E being produced will meet as at K. Produce H A and G B to meet K L parallel to H F, and the parallelogram A M will be that which is required.

Demonstration.

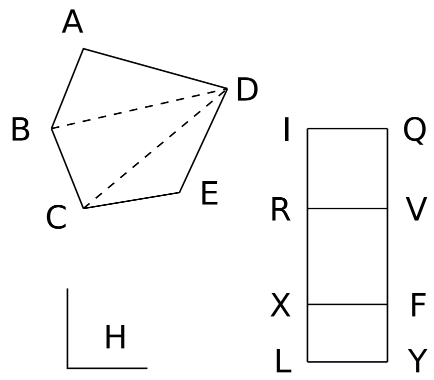
It is evidently constructed on the given line A B; also in the parallelogram F L, the parallelograms A M and G E are equal (XLIII); but G E is equal to C (const.), therefore A M is equal to C. The angle E B G is equal to A B M (XV), but also to D (const.), therefore A B M is equal to D. Hence A M is the parallelogram required.

Proposition XLV. Problem.

(199) *To construct a parallelogram equal to a given rectilinear figure (*A B C E D*), and having an angle equal to a given one (*H*).*

Solution.

Resolve the given rectilinear figure into triangles; construct a parallelogram R Q equal to the triangle B D A (XLIV), and having an angle I equal to a given angle H;



on a side of it, R V, construct the parallelogram X V equal to the triangle C B D, and having an angle equal to the given one (XLIV), and so on construct parallelograms equal to the several triangles into which the figure is resolved. L Q is a parallelogram equal to the given rectilinear figure, and having an angle I equal to the given angle H.

Demonstration.

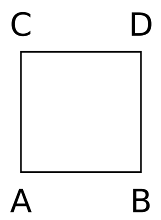
Because R V and I Q are parallel the angle V R I together with I is equal to two right angles (XXIX); but V R X is equal to I (const.), therefore V R I with V R X is equal to two right angles, and therefore I R and R X form one right line (XIV); in the same manner it can be demonstrated, that R X and X L form one right line, therefore I L is a right line, and because Q V is parallel to I R the angle Q V R together with V R I is equal to two right angles (XXIX); but I R is parallel to V F, and therefore I R V is equal to F V R (XXIX), and therefore Q V R together with F V R is equal to two right angles, and Q V and F V form one right line (XIV); in the same manner it can be demonstrated of V F and F Y, therefore Q Y is a right line and also is parallel to I L; and because L Y and R V are parallel to the same line X F, I Y is parallel to R V (XXX); but I Q and R V are parallel, therefore L Y is parallel to I Q, and therefore L Q is a parallelogram, and it has the angle I equal to the given angle H, and is equal to the given rectilinear figure A B C E D.

(200) Cor.—Hence a parallelogram can be applied to a given right line and in a given angle equal to a given rectilinear figure, by applying to the given line a parallelogram equal to the first triangle.

Proposition XLVI. Problem.

(201) *On a given right line (A B) to describe a square.*

Solution.



From either extremity of the given right line A B draw a line A C perpendicular (XI), and equal to it (III); through C draw C D parallel to A B (XXXI), and through B draw B D parallel to A C; A D is the required square.

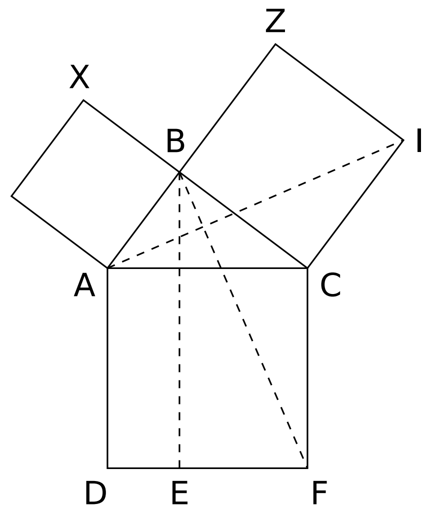
Demonstration.

Because A D is a parallelogram (const.), and the angle A a right angle, the angles C, D, and B are also right (153); and because A C is equal to A B (const.), and the sides C D and D B are equal to A B and A C (XXXIV), the four sides A B, A C, C D, D B are equal, therefore A D is a square.

Proposition XLVII. Theorem.

(202) *In a right angled triangle (*A B C*) the square on the hypotenuse (*A C*) is equal to the sum of the squares of the sides (*A B *and* C B*).*

On the sides A B, A C, and B C describe the squares A X, A F, and B I, and draw B E parallel to either C F or A D, and join B F and A I.

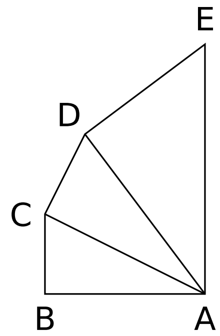


Because the angles I C B and A C F are equal, if B C A be added to both, the angles I C A and B C F are equal, and the sides I C, C A are equal to the sides B C, C F, therefore the triangles I C A and B C F are equal (IV); by A Z is parallel to C I, therefore the parallelogram C Z is double of the triangle I C A, as they are upon the same base C I, and between the same parallels (XLI); and the parallelogram C E is double of the triangle B C F, as they are upon the same base C F, and between the same parallels (XLI); therefore the parallelograms C Z and C E, being double of the equal triangles I C A and B C F, are equal to one another. In the same manner it can be demonstrated, that A X and A E are equal, therefore the whole D A C F is equal to the sum of C Z and A X.

*** (203) Cor. 1.—Hence if the sides of a right angled triangle be given in numbers, its hypotenuse may be found; for let the squares of the sides be added together, and the square root of their sum will be the hypotenuse (187).

*** (204) Cor. 2.—If the hypotenuse and one side be given in numbers, the other side may be found; for let the square of the side be subtracted from that of the hypotenuse, and the remainder is equal to the square of the other side. The square root of this remainder will therefore be equal to the other side.

(205) Cor. 3.—Given any number of right lines, to find a line whose square is equal to the sum of their squares.



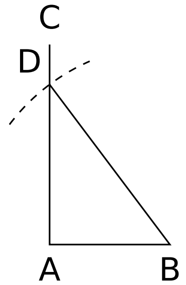
Draw two lines A B and B C as right angles, and equal to the first two of the given lines, and draw A C. Draw C D equal to the third and perpendicular to A C, and draw A D. Draw D E equal to the fourth and perpendicular to A D, and draw A E, and so on. The square of the line A E will be equal to the sum of the squares of A B, B C, C D, &c., which are respectively equal to the given lines.

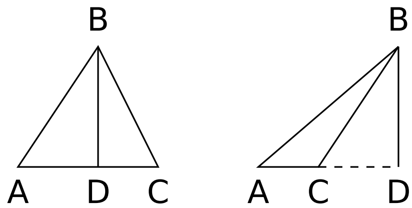
For the sum of the squares of A B and B C is equal to the square of A C. The sum of the squares of A C and C D, or the sum of the squares of A B, B C, C D is equal to the square of A D, and so on; the sum of the squares of all the lines is equal to the square of A E.

(206) Cor. 4.—To find a right line whose square is equal to the difference of the squares of two given right lines.

Through one extremity A of the lesser line A B draw an indefinite perpendicular A C; and from the other extremity B inflect on A C a line equal to the greater of the given lines (60); which is always possible, since the line so inflected is greater than B A, which is the shortest line which can be drawn from B to A C. The square of the intercept A D will be equal to the difference of the squares of B D and B A, or of the given lines.

(207) Cor. 5.—If a perpendicular (B D) be drawn from the vertex of a triangle to the base, the difference of the squares of the sides (A B and C B) is equal to the difference between the squares of the segments (A D and C D). For the square of A B is equal to the sum of the squares of A D and B D, and the square of C B is equal to the sum of the squares of C D and B D. The





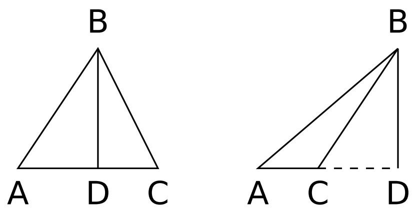
latter being taken from the former, the remainders, which are the difference of the squares of the sides A B and C B, and the difference of the squares of the segments A D and C D, are equal

(208) To understand this corollary perfectly, it is necessary to attend to the meaning of the term *segments*. When a line is cut at any point, the intercepts between the point of section and its extremities are called its *segments*. When the point of section lies between the extremities of the line it is said to be cut *internally*; but when, as sometimes happens, it is not the line itself but its production that is cut, and therefore the point of section lies beyond one of its extremities, it is said to be cut *externally*. By due attention to the definition of *segments* given above, it will be perceived that when a line is cut *internally*, the line is the *sum* of its own *segments*: but when cut *externally*, it is their *difference*.

The case of a perpendicular from the vertex on the base of a triangle offers an example of both species of section. If the perpendicular fall within the triangle, the base is cut internally by it; but if it fall outside, it is cut externally. In both cases the preceding corollary applies, and it is established by the same proof. the *segments* are in each case the intercepts A D and C D between the perpendicular and the extremities of the base.

(209) Cor. 6.—If a perpendicular be drawn from the vertex B to the

base, the sums of the squares of the sides and alternate segments are equal.



For the sum of the squares of A B and B C is equal to the sum of the squares A B, B D and C D, since the square of B C is equal to the sum of the squares of B D and D C. For a similar reason, the sum of the squares of A B and B C is equal to the sum of the squares of A D, D B and B C. Hence the sum of the squares of A B, B D and D C is equal to that of A D, B D and B C. Taking the square of B D from both, the sum of the squares of A B and C D is equal to that of B C and A D.

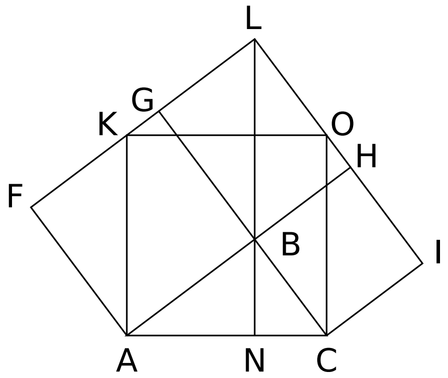
Whether we consider the 47th proposition with reference to the peculiar and beautiful relation established in it, or to its innumerable uses in every department of mathematical science, or to its fertility in the consequences derivable from it, it must certainly be esteemed the most celebrated and important in the whole of the elements, if not in the whole range of mathematical science. It is by the influence of this proposition, and that which establishes the similitude of equiangular triangles (in the sixth book), that Geometry has been brought under the domininon of Algebra, and it is upon these same principles that the whole science of Trigonometry is founded.

The XXXIId and XLVIIth propositions are said to have been discovered by Pythagoras, and extraordinary accounts are given of his exultation upon his first perception of their truth. It is however supposed by some that Pythagoras acquired a knowledge of them in Egypt, and was the first to make them known in Greece.

Besides the demonstration in the Elements there are others by which this celebrated proposition is sometimes established, and which, in a principle of such importance, it may be gratifying to the student to know.

*** (210) 1° Having constructed squares on the sides A B, B C on opposite sides of them from the triangle, produce I H and F G to meet at L. Through A and C draw perpendiculars to the hypotenuse, and join K O.

In the triangles A F K and A B C, the angles F and B are equal, being both right, and F A K and B A C are equal, having a common complement K A B, and the sides F A and F B are equal. Hence A K and A C are equal, and in like manner C O and A C are equal. Hence A O is an equilateral



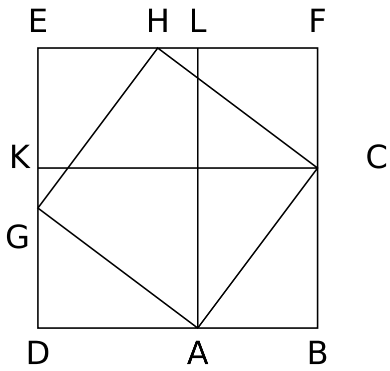
parallelogram, and the angle at A being right, it is a square. The triangle L G B is, in every respect, equal to B C A, since B G is equal to B A, and L G is equal to B H or B C, and the angle at G is equal to the right angle B. Hence it is also equal in every respect to the triangle K F A Since, then, the angles G L B and F K A are equal, K A is parallel to B L, and therefore A L is a parallelogram. The square A G and the parallelogram A L are equal, being on the same base A B, and between the same parallels (XXXV); and for the same reason the parallelograms A L and K N are equal, A K being their common base. Therefore the square A G is equal to the parallelogram K N

In like manner the square C H is equal to the parallelogram O N, and therefore the squares A G and C H are together equal to A O.

*** (211) 2° Draw A G perpendicular and equal to A C, and produce B A, and draw G D perpendicular to it. In the same manner draw C H perpendicular and equal to C A, and produce B C and draw H F perpendicular to it. Produce F H and D G to meet in E, and draw G H.

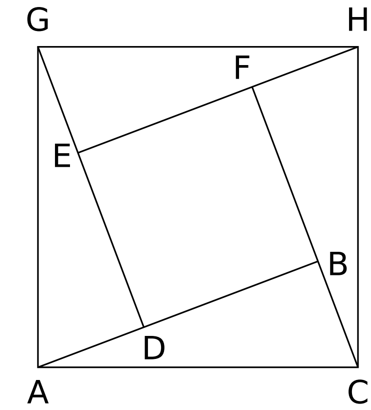
The triangles G D A and H F C are equal in every respect to A B C (XXVI). Hence F C, G D and A B are equal, and also H F, D A and B C, and the angles in each triangle opposed to these sides are equal. Also, since G A and H C are equal to A C, and therefore to each other, and the angles at A and C are right, A H is a square (XXXIII). Since G H is equal to A C, and the angles at G and H are right, it follows that the triangle G E H is in all respects equal to A B C (XXVI), in the same manner as for the triangles G D A and H F C.

Through C and A draw the lines C K and A L parallel to B D and B F. Since C B and A I are equal and also C B and A D, it follows that A K is the square of B C, and in like manner that C L is the square of A B. The parallelograms B I and K L have bases and altitudes equal to those of the triangle A B C, and are therefore each equal to twice the triangle, and together



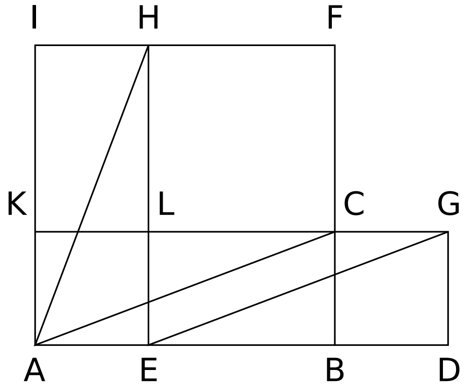
equal to four times the triangle. Hence B I and K L are together equal to A B C, C F H, H E G and G D A together. Taking the former and the latter successively from the whole figure, the remainders are in the one case the squares D I and C L of the sides B C and B A, and in the latter the square A H of the hypotenuse. Therefore, &c.

(212) 3° On the hypotenuse A C construct the square A H, and draw G D and H E parallel to C B and A B, and produce these lines to meet in F, E and D. The triangles A B C, A D G, G E H and H F C are proved in



every respect equal (XXVI). It is evident, that the angles D, E, F, B are all right. But also since D G and A B are equal, and also G E and A D, taking the latter from the former D E and D B remain equal. Hence B E is a square on the difference D B of the sides; and therefore the square of A C is divided into four triangles, in all respects equal to A B C and the square B E of the difference of the sides.

Now let squares B G and B I be constructed on the sides, and taking A E on the greater side equal to B C the less, and draw E H parallel to B C, and produce G C to K. Draw G E and A H



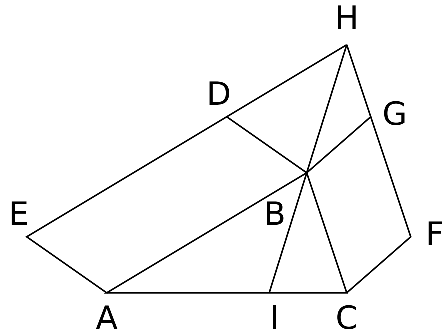
The part B E is the difference of the sides A B and B C. And since B F is equal to A B, F C is also the difference of the sides, wherefore F L is the square of this difference. Also since A E and B D are equal A B and D E are equal, therefore the parallelogram D L is double the triangle A B C. The sides and angles of the parallelogram A H are equal respectively to those of D L, and therefore these two parallelograms together are equal to four times the triangle A B C. Hence the squares A F and B G may be divided into four triangles G D E, G L E, A E H and A I H in all respects equal to the triangle A B C, and the square C H of the difference of the sides. But by the former construction the square of the hypotenuse was shown to be divisible into the same parts. Therefore, &c.

The peculiarity of this proof is, that it shows that the squares of the sides may be so dissected that they may be laid upon the square of the hypotenuse so as exactly to cover it, and *vice versa*, that the square of the hypotenuse may be so dissected as to exactly cover the squares of the sides.

(213) The forty-seventh proposition is included as a case of the following more general one taken from the mathematical collections of *Pappus*, an eminent Greek Geometer of the fourth century.

*In any triangle (*A B C*) parallelograms* A E *and* C G *being described on the sides, and their sides* D E *and* F G *being produced to meet at* H, *and* H B I *being drawn, the parallelogram on* A C *whose sides are equal and parallel to* B H *is equal to* A E *and* C G *together.*

For draw A K and C L parallel to B H, to meet D H and F H in K and L. Since A H is a parallelogram, A K is equal to B H, and for a similar reason



C L is equal to B H. Hence C L and A K are equal and parallel, and therefore (XXXIII) A L is a parallelogram. The parallelograms A E and A H are equal, being on the same base A B, and between the same parallels, and also A H and K I whose common base is A K. Hence the parallelograms A E and K I are equal. In like manner the parallelograms C G and L I are equal, and therefore A E and C G are together equal to A L.

This proof is applied to the forty-seventh in (210).

(214) The forty-seventh proposition is also a particular case of the following more general one:

*In any triangle (*A B C*) squares being constructed on the sides (*A B *and* B C*) and on the base; and perpendiculars (*A D F *and* C E G*) being drawn from the extremities of the base to the sides, the parallelograms* A G *and* C F *formed by the segments* C D, A E*, with the sides of the squares, will be together equal to the square of the base* A C*.*

For draw A H and B I; and also B K perpendicular to A C.

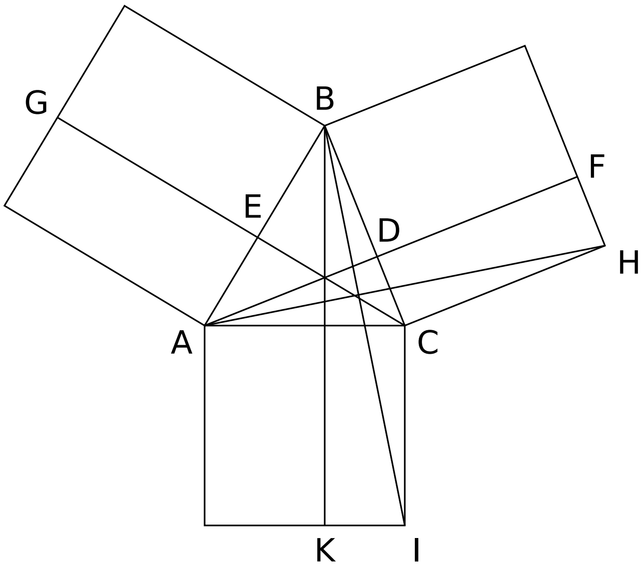
The parallelograms K C and C F are proved equal, exactly as C E and C Z are proved equal in the demonstration of the XLVIIth. And in like manner it follows, that A K and A G are equal, and therefore the square on A C is equal to the parallelograms A G and C F together.

If the triangle be right angled at B, the lines G E and D F will coincide with the sides of the squares. and the proposition will become the XLVIIth.

(215) If B be acute the perpendiculars A D and C E will fall within the triangle, and the parallelograms A G and C F are less than the squares of the sides; but if B be obtuse the perpendiculars fall outside the triangle, and the parallelograms A G and C F are greater than the squares of the sides.

Hence the forty-seventh proposition may be extended thus:

The square of the base of a triangle is less than, equal to, or greater than the sum of the squares of the sides, according as the vertical angle is less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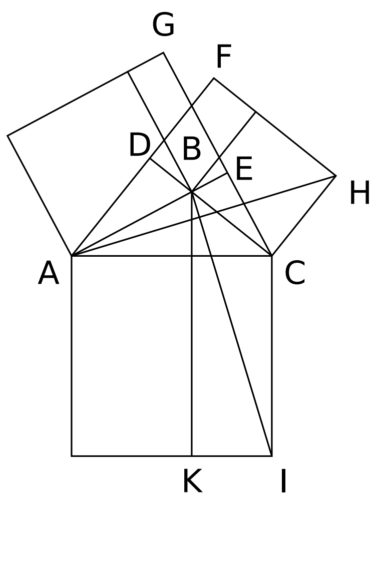
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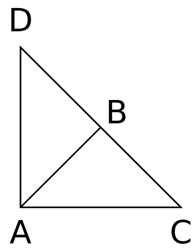
than, equal to, or greater than a right angle..

Proposition XLVIII. Theorem.

(216) *If the square on one side* (A C) *of a triangle* (A B C) *be equal to the sum of the squares of the other two sides* (A B *and* B C)*, the angle* (A B C) *opposite to that side is a right angle.*

From the point B draw B D perpendicular (XI) to one of the sides A B, and equal to the other B C (III), and join A D.



The square of A D is equal to the squares of A B and B D (XLVII), or two the squares of A B and B C which is equal to B D (const.); but the squares of A B and B C are together equal to the square of A C (hyp.), therefore the squares of A D and A C are equal, and therefore the lines themselves are equal; but also D B and B C are equal, and the side A B is common to both triangles, therefore the triangles A B C and A B D are mutually equilateral, and therefore also mutually equiangular, and therefore the angle A B C is equal to the angle A B D; but A B D is a right angle, therefore A B C is also a right angle.

This proposition may be extended thus:

The vertical angle of a triangle is less than, equal to, or greater than a right angle, according as the square of the base is less than, equal to, or greater than the sum of the squares of the sides.

For from B draw B D perpendicular to A B and equal to B C, and join A D.

The square of A D is equal to the squares of A B and B D or B C. The line A C is less than, equal to, or greater than A D, according as the square of the line A C is less than, equal to, or greater than the squares of the sides A B and B C. But the angle B is less than, equal to, greater than a right angle, according as the side A C is less than, equal to, or greater than A D (XXV, VIII); therefore &c.