MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2018 Section 4: Limits and Continuity for Functions of Several Variables

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4. Limits and Continuity for Functions of Several Variables

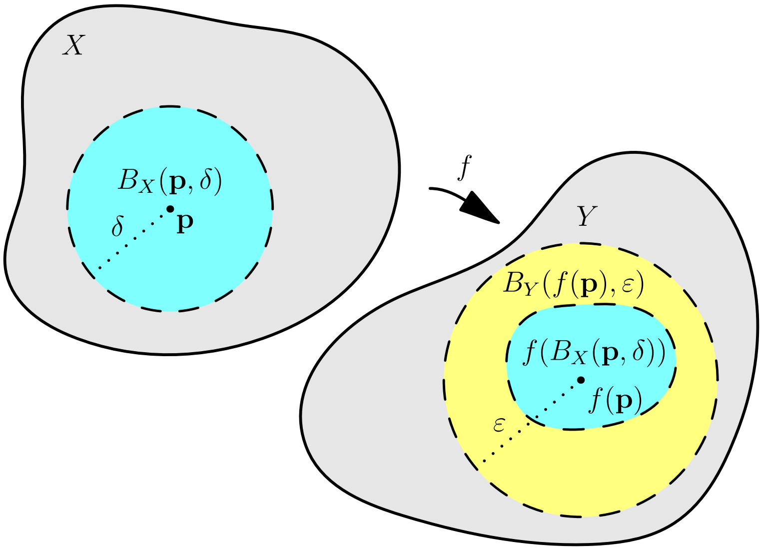
4.1. Continuity of Functions of Several Real Variables

Definition

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.*

The function $f\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.


X
$B_X(\mathbf{p}, \delta)$
δ
$\mathbf{p}$
f
Y
$B_Y(f(\mathbf{p}), \varepsilon)$
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
ε

Lemma 4.1

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof

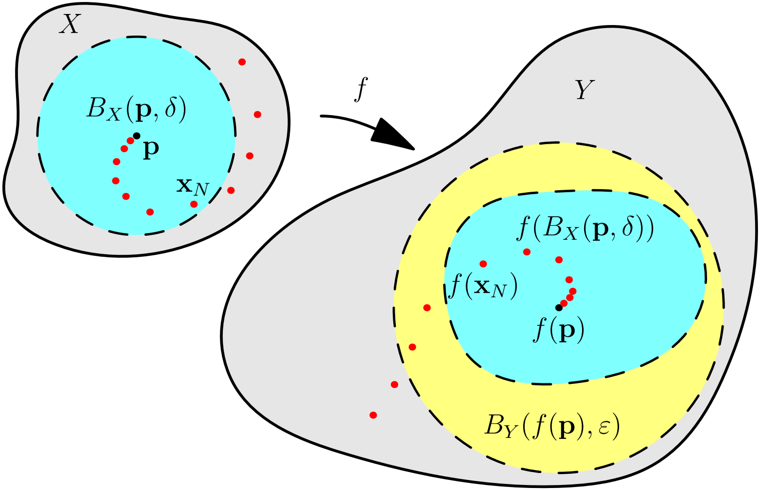
Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.2

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

Proof

Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎


X
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
$\mathbf{x}_N$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{x}_N)$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 4.3

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f\colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

Proof

Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 4.1. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.4

The functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x,y) = x + y$ and $m(x,y) = xy$ are continuous.

Proof

Let $(u,v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x,y) is any point of $\mathbb{R}^2$ whose distance from (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence

$$|s(x,y) - s(u,v)| = |x+y-u-v| \leq |x-u| + |y-v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ be given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta \leq (1 + |u| + |v|)\delta \leq \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 4.5

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

Proof

Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 4.3, Lemma 4.4 and Lemma 4.1 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. $\blacksquare$

Example

Consider the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 4.5. It then follows from Proposition 4.3 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Lemma 4.6

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

Proof

Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

4.2. Limits of Functions of Several Real Variables

Definition

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.*

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Proposition 4.7

Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f\colon X \to \mathbb{R}^n$ has the property that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof

Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$. Let i be an integer between 1 and n, and let some positive real number ε be given. Then there exists some positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - q_i| \leq |f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ then $\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$ for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < |f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 4.8

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ and $g: X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof

Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (\mathbf{q} + \mathbf{r})| &\leq |f(\mathbf{x}) - \mathbf{q}| + |g(\mathbf{x}) - \mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r},$$

as required. ■

Definition

Let $f: X \to \mathbb{R}^n$ be a function mapping some subset X of m-dimensional Euclidean space $\mathbb{R}^m$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. We say that $f(\mathbf{x})$ *remains bounded* as $\mathbf{x}$ tends to $\mathbf{p}$ in X if strictly positive constants C and δ can be determined so that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Proposition 4.9

Let $f: X \to \mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} \Big(h(\mathbf{x}) f(\mathbf{x}) \Big) = \mathbf{0}.$$

Proof

Let some strictly positive real number ε be given. Now $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined so that $|h(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|h(\mathbf{x})| \leq C$ and $|f(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.10

Let $f: X \to \mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x}) = 0$. Suppose also that $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x}) f(\mathbf{x})) = \mathbf{0}.$$

Proof

Let some strictly positive real number ε be given. Now $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|h(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| \leq C$ and $|h(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.11

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) \cdot g(\mathbf{x}) \Big) = 0.$$

Proof

Let some strictly positive real number ε be given. Now $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|g(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| < \varepsilon_0$ and $|g(\mathbf{x})| \leq C$. It then follows from Schwarz's Inequality (Proposition 2.1) that

$$|f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{x})|\,|g(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.12

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of $\mathbb{R}^n$ and let s be a real number. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = s.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})f(\mathbf{x}) = s\mathbf{q}.$$

Proof

The functions f and h satisfy the equation

$$h(\mathbf{x})f(\mathbf{x}) = h(\mathbf{x})\Big(f(\mathbf{x}) - \mathbf{q}\Big) + (h(\mathbf{x}) - s)\mathbf{q} + s\mathbf{q},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) - \mathbf{q}\Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Big(h(\mathbf{x}) - s\Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that $|h(\mathbf{x}) - s| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|h(\mathbf{x})| < |s| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) = \mathbf{0}$$

(see Proposition 4.10).

Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x}) - s)\,\mathbf{q} = \mathbf{0}.$$

It follows that

$$\begin{aligned} & \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) \\ = \quad & \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) + \lim_{\mathbf{x}\to\mathbf{p}} \left(\Big(h(\mathbf{x}) - s\Big)\mathbf{q}\right) + s\mathbf{q} \\ = \quad & \mathbf{0} + s\mathbf{q}, \end{aligned}$$

as required. ∎

Lemma 4.13

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f\colon X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g\colon Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof

Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}),$$

as required. ∎

Proposition 4.14

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) = \mathbf{q} \cdot \mathbf{r}.$$

Proof

The functions f and g satisfy the equation

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = (f(\mathbf{x}) - \mathbf{q}) \cdot g(\mathbf{x}) + \mathbf{q} \cdot (g(\mathbf{x}) - \mathbf{r}) + \mathbf{q} \cdot \mathbf{r},$$

where

$$\lim_{\mathbf{x} \to \mathbf{p}} \Bigl(f(\mathbf{x}) - \mathbf{q} \Bigr) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{p}} \Bigl(g(\mathbf{x}) - \mathbf{r} \Bigr) = \mathbf{0}.$$

Moreover there exists a strictly positive constant δ_0 such that $|g(\mathbf{x}) - \mathbf{r}| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|g(\mathbf{x})| < |\mathbf{r}| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} \Bigl(\Bigl(f(\mathbf{x}) - \mathbf{q} \Bigr) \cdot g(\mathbf{x}) \Bigr) = 0$$

(see Proposition 4.11).

Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q}\cdot\big(g(\mathbf{x})-\mathbf{r}\big)\Big) = 0.$$

It follows that

$$\begin{aligned} &\lim_{\mathbf{x}\to\mathbf{p}} \left(f(\mathbf{x})\cdot g(\mathbf{x})\right) \\ = \;& \lim_{\mathbf{x}\to\mathbf{p}} \Big(\big(f(\mathbf{x})-\mathbf{q}\big)\cdot g(\mathbf{x})\Big) + \lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q}\cdot\big(g(\mathbf{x})-\mathbf{r}\big)\Big) + \mathbf{q}\cdot\mathbf{r} \\ = \;& \mathbf{q}\cdot\mathbf{r}, \end{aligned}$$

as required. ∎

Proposition 4.15

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x}))$ and $\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x}))$, and moreover

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

First Proof

It follows from Proposition 4.8 (applied in the case when the target space is one-dimensional) that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

It follows from Proposition 4.12 (applied in the case when the target space is one-dimensional), or alternatively from Proposition 4.14, that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Let $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the reciprocal function defined so that $e(t) = 1/t$ for all non-zero real numbers t. Then the reciprocal function e is continuous. Applying the result of Lemma 4.13, we find that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{g(\mathbf{x})} = \lim_{\mathbf{x}\to\mathbf{p}} e(g(\mathbf{x})) = e\left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\right) = \frac{1}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})},$$

as required. ∎

Second Proof

Let $q = \lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})$ and $r = \lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})$, and let $h \colon X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x}) = (q, r)$$

(see Proposition 4.7).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u,v) = u + v$ and $m(u,v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 4.4). Also $f + g = s \circ h$ and $f \cdot g = m \circ f$. It follows from this that

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(h(\mathbf{x})) \\
&= s\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = s(q,r) = q + r,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(h(\mathbf{x})) \\
&= m\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = m(q,r) = qr
\end{aligned}$$

(see Lemma 4.13).

Also

$$\lim_{\mathbf{x}\to\mathbf{p}} (-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = q - r.$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $e(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. ∎

Proposition 4.16

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to \mathbb{R}^k$ be functions satisfying $f(X) \subset Y$. Let $\mathbf{p}$ *be a limit point of X in $\mathbb{R}^m$, let* $\mathbf{q}$ *be a limit point of Y in $\mathbb{R}^n$ let* $\mathbf{r}$ *be a point of $\mathbb{R}^k$. Suppose that the following three conditions are satisfied:*

(i) $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$;

(ii) $\lim\limits_{\mathbf{y}\to\mathbf{q}} g(\mathbf{y}) = \mathbf{r}$;

(iii) *there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$.*

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = \mathbf{r}.$$

Proof

Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{y} \in Y$ satisfies $0 < |\mathbf{y} - \mathbf{q}| < \eta$. There then exists some positive real number δ_1 such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Also there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and $0 < |f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. But this then ensures that $|g(f(\mathbf{x})) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

Proposition 4.17

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.

Proof

The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x} - \mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

4.3. Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

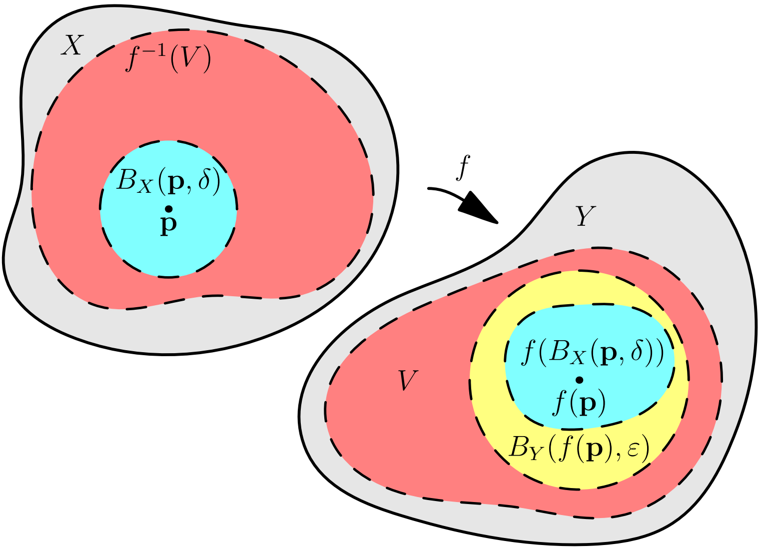
Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 4.18

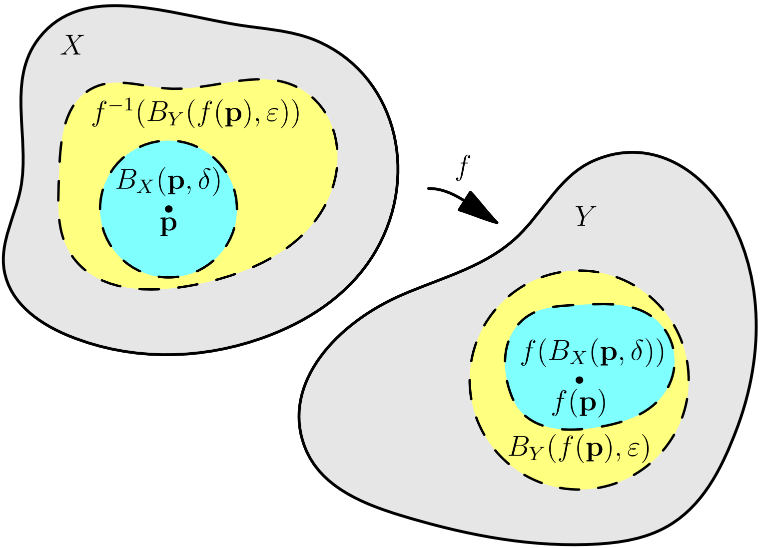
Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

Proof

Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.


X
$f^{-1}(V)$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
V
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Conversely suppose that $f: X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 3.1, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎


X
$f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

4.4. Limits and Neighbourhoods

Definition

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. A subset N of X is said to be a *neighbourhood* of $\mathbf{p}$ in X if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N.$$

Lemma 4.19

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X that is not an isolated point of X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in X such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$.

Proof

This result follows directly from the definitions of limits and neighbourhoods. ∎

Remark

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a limit point of X that does not belong to X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in $X \cup \{\mathbf{p}\}$ such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$. Thus the result of Lemma 4.19 can be extended so as to apply to limits of functions taken at limit points of the domain that do not belong to the domain of the function.

4.5. The Multidimensional Extreme Value Theorem

Proposition 4.20

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a point $\mathbf{w}$ of X such that $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$.

Proof

Let $g: X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the function mapping each $\mathbf{x} \in X$ to $|f(\mathbf{x})|$ is continuous (see Lemma 4.6) and quotients of continuous functions are continuous where they are defined (see Lemma 4.5). It follows that the function $g: X \to \mathbb{R}$ is continuous.

Let

$$m = \inf\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$g(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{w}$ of $\mathbb{R}^n$.

Now the point $\mathbf{w}$ belongs to X because X is closed (see Lemma 3.7). Also

$$m \leq g(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $g(\mathbf{x}_{k_j}) \to m$ as $j \to +\infty$. It then follows from Lemma 4.2 that

$$g(\mathbf{w}) = g\left(\lim_{j\to+\infty} \mathbf{x}_{k_j}\right) = \lim_{j\to+\infty} g(\mathbf{x}_{k_j}) = m.$$

Then $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$, and therefore $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$, as required. ■

Theorem 4.21 (The Multidimensional Extreme Value Theorem)

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f : X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof

It follows from Proposition 4.20 that the function f is bounded on X. It follows that there exists a real number C large enough to ensure that $f(\mathbf{x}) + C > 0$ for all $\mathbf{x} \in X$. It then follows from Proposition 4.20 that there exists some point $\mathbf{v}$ of X such that

$$f(\mathbf{x}) + C \leq f(\mathbf{v}) + C.$$

for all $\mathbf{x} \in X$. But then $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$. Applying this result with f replaced by $-f$, we deduce that there exists some $\mathbf{u} \in X$ such that $-f(\mathbf{x}) \leq -f(\mathbf{u})$ for all $\mathbf{x} \in X$. The result follows. ∎

4.6. Uniform Continuity for Functions of Several Real Variables

Definition

Let X be a subset of $\mathbb{R}^m$. A function $f : X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 4.22

Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f : X \to \mathbb{R}^n$ is uniformly continuous.

Proof

Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 2.6). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since

$$\lim_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0.$$

But then the sequences

$$f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$$

and

$$f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$$

would both converge to $f(\mathbf{p})$, since f is continuous (Lemma 4.2), and thus

$$\lim_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that

$$|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$$

for all j. We conclude therefore that there must exist some positive real number δ such that such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

4.7. Norms on Vector Spaces

Definition

A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

The Euclidean norm $|.|$ is a norm on $\mathbb{R}^n$ defined so that

$$|(x_1, x_2, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

for all $(x_1, x_2, \ldots, x_n)$. There are other useful norms on $\mathbb{R}^n$. These include the norms $\|.\|_1$ and $\|.\|_{\mathrm{sup}}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\mathrm{sup}} = \mathrm{maximum}(|x_1|, |x_2|, \ldots, |x_n|)$$

for all $(x_1, x_2, \ldots, x_n)$.

Definition

Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 4.23

If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.

Proof

let $\|.\|_*$ and $\|.\|_{**}$ be norms on a real vector space X that are both equivalent to a norm $\|.\|$ on X. Then there exist constants c_*, c_{**}, C_* and C_{**}, where $0 < c_* \leq C_*$ and $0 < c_{**} \leq C_{**}$, such that

$$c_* \|x\| \leq \|x\|_* \leq C_* \|x\|$$

and

$$c_{**} \|x\| \leq \|x\|_{**} \leq C_{**} \|x\|$$

for all $x \in X$. But then

$$\frac{c_{**}}{C_*} \|x\|_* \leq \|x\|_{**} \leq \frac{C_{**}}{c_*} \|x\|_*.$$

for all $x \in X$, and thus the norms $\|.\|_*$ and $\|.\|_{**}$ are equivalent to one another. The result follows. ∎

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 4.24

Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive real number C with the property that $\|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots,$$

$$\mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ be a point of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

Using Schwarz's Inequality, we see that

$$\begin{aligned} \|\mathbf{x}\| &= \left\| \sum_{j=1}^{n} x_j \mathbf{e}_j \right\| \leq \sum_{j=1}^{n} |x_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x}|, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and

$$|\mathbf{x}| = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The result follows. ∎

Lemma 4.25

Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive constant C such that

$$\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\| \leq C|\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

It follows that

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

and

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|,$$

and therefore

$$\Big| \|\mathbf{y}\| - \|\mathbf{x}\| \Big| \leq \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The result therefore follows from Lemma 4.24. ∎

Theorem 4.26

Any two norms on $\mathbb{R}^n$ are equivalent.

Proof

Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now it follows from Lemma 4.25 that the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous. Also S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. It therefore follows from the Extreme Value Theorem (Theorem 4.21) that there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. $\blacksquare$