MA2321—Analysis in Several Variables
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Section 2: Convergence in Euclidean Spaces

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2. Convergence in Euclidean Spaces

2.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda \mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 2.1

(Schwarz's Inequality) Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.

Proof

We note that $|\lambda \mathbf{x} + \mu \mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda \mathbf{x} + \mu \mathbf{y}|^2 = (\lambda \mathbf{x} + \mu \mathbf{y}).(\lambda \mathbf{x} + \mu \mathbf{y}) = \lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2.$$

Therefore $\lambda^2 |\mathbf{x}|^2 + 2\lambda\mu \mathbf{x} \cdot \mathbf{y} + \mu^2 |\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4 |\mathbf{x}|^2 - 2|\mathbf{y}|^2 (\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2 |\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right) |\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2 |\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Proposition 2.2

(Triangle Inequality) Let $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$. *Then* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Proof

Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Proposition 2.2) that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact that the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2. Convergence of Sequences in Euclidean Spaces

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.*

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 2.3

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.

Proof

Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}}\right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. $\blacksquare$

2.3. Limit Points of Subsets of Euclidean Spaces

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^n$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Lemma 2.4

Let X be a subset of n-dimensional Euclidean space $\mathbf{R}^n$. A point $\mathbf{p}$ is a limit point of the set X if and only if, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set.

Proof

Suppose that, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set. Then, for each positive real number δ, the set thus determined by δ must consist of more than just the single point $\mathbf{p}$, and therefore there exists $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus $\mathbf{p}$ is a limit point of the set X.

Now let $\mathbf{p}$ be an arbitrary point of $\mathbb{R}^n$. Suppose that there exists some positive real number δ_0 for which the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

is finite. If this set does not contain any points of X distinct from the point $\mathbf{p}$ then $\mathbf{p}$ is not a limit point of the set X. Otherwise let δ be the minimum value of $|\mathbf{x} - \mathbf{p}|$ as $\mathbf{x}$ ranges over all points of the finite set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

that are distinct from $\mathbf{p}$. Then $\delta > 0$, and $|\mathbf{x} - \mathbf{p}| \geq \delta$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Thus the point $\mathbf{p}$ is not a limit point of the set X. The result follows. $\blacksquare$

Lemma 2.5

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$ and let $\mathbf{p} \in \mathbb{R}^n$. Then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

Proof

Suppose that $\mathbf{p}$ is a limit point of X. Then, for each positive integer j, there exists a point $\mathbf{x}_j$ of X for which $0 < |\mathbf{x}_j - \mathbf{p}| < 1/j$. The points $\mathbf{x}_j$ satisfying this condition then constitute an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, that converge to the point $\mathbf{p}$.

Conversely suppose that $\mathbf{p}$ is some point of $\mathbb{R}^n$ that is the limit of some infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X that are all distinct from the point $\mathbf{p}$. Let some positive number δ be given. The definition of convergence ensures that there exists a positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. Moreover $|\mathbf{x}_j - \mathbf{p}| > 0$ for all positive integers j. Thus $0 < |\mathbf{x}_j - \mathbf{p}| < \delta$ when the positive integer j is sufficiently large. Thus the point $\mathbf{p}$ is a limit point of the set X, as required. ∎

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{p}$ of X is said to be an *isolated point* of X if it is not a limit point of X.

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in X$. It follows immediately from the definition of isolated points that the point $\mathbf{p}$ is an isolated point of the set X if and only if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} = \{\mathbf{p}\}.$$

2.4. The Multidimensional Bolzano-Weierstrass Theorem

We introduce some terminology and notation for discussing convergence along subsequences of bounded sequences of points in Euclidean spaces. This will be useful in proving the multi-dimensional version of the Bolzano-Weierstrass Theorem.

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$, let J be an infinite subset of the set $\mathbb{N}$ of positive integers, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{p}$ is the *limit* of $\mathbf{x}_j$ as j *tends to infinity in the set* J, and write "$\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ in J" if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \in J$ and $j \geq N$.*

The one-dimensional version of the Bolzano-Weierstrass Theorem (Theorem 1.9) is equivalent to the following statement:

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, there exists an infinite subset J of the set $\mathbb{N}$ of positive integers and a real number p such that $x_j \to p$ as $j \to +\infty$ in J.*

Given an infinite subset J of $\mathbb{N}$, the elements of J can be labelled as $k_1, k_2, k_3, \ldots$, where $k_1 < k_2 < k_3 < \cdots$, so that k_1 is the smallest positive integer belonging of J, k_2 is the next smallest, etc. Therefore any standard result concerning convergence of sequences of points can be applied in the context of the convergence of subsequences of a given sequence of points.

The following result is therefore a direct consequence of the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9):

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given an infinite subset J of the set $\mathbb{N}$ of positive integers, there exists an infinite subset K of J and a real number p such that $x_j \to p$ as $j \to +\infty$ in K.*

The above statement in fact corresponds to the following assertion:—

Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ *of real numbers, and given any subsequence*

$$x_{k_1}, x_{k_2}, x_{k_3}, \cdots$$

of the given infinite sequence, there exists a convergent subsequence

$$x_{k_{m_1}}, x_{k_{m_2}}, x_{k_{m_3}}, \ldots$$

of the given subsequence. Moreover this convergent subsequence of the given subsequence is itself a convergent subsequence of the given infinite sequence, and it contains only members of the given subsequence of the given sequence.

The basic principle can be presented purely in words as follows:

> *Given a bounded sequence of real numbers, and given a subsequence of that original given sequence, there exists a convergent subsequence of the given subsequence. Moreover this subsequence of the subsequence is a convergent subsequence of the original given sequence.*

We employ this principle in the following proof of the Multidimensional Bolzano-Weierstrass Theorem.

Theorem 2.6 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

Proof

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$, and, for each positive integer j, and for each integer i between 1 and n, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then

$$\mathbf{x}_j = \Bigl((\mathbf{x}_j)_1, (\mathbf{x}_j)_2, \ldots, (\mathbf{x}_j)_n\Bigr).$$

for all positive integers j. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9) that there exists an infinite subset J_1 of the set $\mathbb{N}$ of positive integers and a real number p_1 such that $(\mathbf{x}_j)_1 \to p_1$ as $j \to +\infty$ in J_1.

Let k be an integer between 1 and $n-1$. Suppose that there exists an infinite subset J_k of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_k$ such that, for each integer i between 1 and k, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_k. It then follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_{k+1} of J_k and a real number p_{k+1}, such that $(\mathbf{x}_j)_{k+1} \to p_{k+1}$ as $j \to +\infty$ in J_{k+1}. Moreover the requirement that $J_{k+1} \subset J_k$ then ensures that, for each integer i between 1 and $k+1$, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_{k+1}. Repeated application of this result then ensures the existence of an infinite subset J_n of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_n$ such that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_n. Let

$$J_n = \{k_1, k_2, k_3, \ldots\},$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\lim\limits_{j \to +\infty} (\mathbf{x}_{k_j})_i = p_i$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 2.3 that $\lim\limits_{j \to +\infty} \mathbf{x}_{k_j} = \mathbf{p}$. The result follows. ∎

2.5. Cauchy Sequences in Euclidean Spaces

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if the following condition is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.*

Lemma 2.7

Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.

Proof

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 2.8

(Cauchy's Criterion for Convergence) An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.

Proof

First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 2.7. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎