MA2321—Analysis in Several Variables
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David R. Wilkins

11.4. The Implicit Function Theorem

Theorem 11.6

Let X be an open set in $\mathbb{R}^n$, let $f_1, f_2, \ldots, f_m$ be a continuously differentiable real-valued functions on X, where $m < n$, let

$$M = \{\mathbf{x} \in X : f_i(\mathbf{x}) = 0 \text{ for } i = 1, 2, \ldots, m\},$$

and let $\mathbf{p}$ be a point of M.

Suppose that $f_1, f_2, \ldots, f_m$ are zero at $\mathbf{p}$ and that the matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_m} \end{pmatrix}$$

is invertible at the point $\mathbf{p}$. Then there exists an open neighbourhood U of $\mathbf{p}$ and continuously differentiable functions $h_1, h_2, \ldots, h_m$ of $n - m$ real variables, defined around $(p_{m+1}, \ldots, p_n)$ in $\mathbb{R}^{n-m}$, such that

$$\begin{aligned} M \cap U &= \{(x_1, x_2, \ldots, x_n) \in U : \\ &\qquad x_i = h_i(x_{m+1}, \ldots, x_n) \text{ for } i = 1, 2, \ldots, m\}. \end{aligned}$$

Proof

Let $\varphi\colon X \to \mathbb{R}^n$ be the continuously differentiable function defined such that

$$\varphi(\mathbf{x}) = \Bigl(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}), x_{m+1}, \ldots, x_n\Bigr)$$

for all $\mathbf{x} \in X$. (Thus the ith Cartesian component of the function φ is equal to f_i for $i \leq m$, but is equal to x_i for $m < i \leq n$.) Let J be the Jacobian matrix of φ at the point $\mathbf{p}$, and let $J_{i,j}$ denote the coefficient in the ith row and jth column of J. Then

$$J_{i,j} = \frac{\partial f_i}{\partial x_j}$$

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Also $J_{i,i} = 1$ if $i > m$, and $J_{i,j} = 0$ if $i > m$ and $j \neq i$.

The matrix J can therefore be represented in block form as

$$J = \left(\begin{array}{c|c} J_0 & A \\ \hline 0 & I_{n-m} \end{array} \right),$$

where J_0 is the leading $m \times m$ minor of the matrix J, A is an $m \times (n-m)$ minor of the matrix J and I_{n-m} is the identity $(n-m) \times (n-m)$ matrix. It follows from standard properties of determinants that $\det J = \det J_0$. Moreover the hypotheses of the theorem require that $\det J_0 \neq 0$. Therefore $\det J \neq 0$. The derivative $(D\varphi)_{\mathbf{p}}$ of φ at the point $\mathbf{p}$ is represented by the Jacobian matrix J. It follows that $(D\varphi)_{\mathbf{p}} \colon \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation.

The Inverse Function Theorem (Theorem 11.5) now ensures the existence of a local inverse $\mu\colon W \to X$ for the function φ around $\mathbf{p}$. The range $\mu(W)$ of this local inverse is then an open set in X containing the point $\mathbf{p}$, and $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$.

Let $\mathbf{y}$ be a point of W, and let $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Then $\mathbf{y} = \varphi(\mu(\mathbf{y}))$, and therefore $y_i = f_i(\mu(\mathbf{y}))$ for $i = 1, 2, \ldots, m$, and y_i is equal to the ith component of $\mu(\mathbf{y})$ when $m < i \leq n$.

Now $\mathbf{p} \in \mu(W)$. Therefore there exists some point $\mathbf{q}$ of W satisfying $\mu(\mathbf{q}) = \mathbf{p}$. Now $\mathbf{p} \in M$, and therefore $f_i(\mathbf{p}) = 0$ for $i = 1, 2, \ldots, m$. But $q_i = f_i(\mu(\mathbf{q})) = f_i(\mathbf{p})$ when $1 \leq i \leq m$. It follows that $q_i = 0$ when $1 \leq i \leq m$. Also $q_i = p_i$ when $i > m$.

Let g_i denote the ith Cartesian component of the continuously differentiable map $\mu\colon W \to \mathbb{R}^n$ for $i = 1, 2, \ldots, n$. Then $g_i\colon W \to \mathbb{R}$ is a continuously differentiable real-valued function on W for $i = 1, 2, \ldots, n$. If $(y_1, y_2, \ldots, y_n) \in W$ then

$$(y_1, y_2, \ldots, y_n) = \varphi(\mu(y_1, y_2, \ldots, y_n)).$$

It then follows from the definition of the map φ that y_i is the ith Cartesian component of $\mu(y_1, y_2, \ldots, y_n)$ when $i > m$, and thus

$$y_i = g_i(y_1, y_2, \ldots, y_n) \quad \text{when} \quad i > m.$$

Now $\mu(W)$ is an open set, and $\mathbf{p} \in \mu(W)$. It follows that there exists some positive real number δ such that $H(\mathbf{p}, \delta) \subset \mu(W)$. where

$$\begin{aligned} H(\mathbf{p}, \delta) &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ &\qquad p_i - \delta < x_i < p_i + \delta \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Let

$$\begin{aligned} D &= \{(z_1, z_2, \ldots, z_{n-m}) \in \mathbb{R}^{n-m} : p_{m+j} - \delta < z_j < p_{m+j} + \delta \\ &\qquad \text{for } j = 1, 2, \ldots, n-m\}, \end{aligned}$$

and let $h_i \colon D \to \mathbb{R}$ be defined so that

$$h_i(z_1, z_2, \ldots, z_{n-m}) = g_i(0, 0, \ldots, 0, z_1, z_2, \ldots, z_{n-m})$$

for $i = 1, 2, \ldots, m$.

Let $\mathbf{x} \in H(\mathbf{p}, \delta)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then $\mathbf{x} \in \mu(W)$. It follows from Lemma 11.1 that

$$\begin{aligned} (x_1, x_2, \ldots, x_n) &= \mu(\varphi(\mathbf{x})) \\ &= \mu\Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}), x_{m+1}, \ldots, x_n\Big). \end{aligned}$$

On equating Cartesian components we find that

$$x_i = g_i\Big(f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}), x_{m+1}, \ldots, x_n\Big).$$

for $i = 1, 2, \ldots, n$.

In particular, if $\mathbf{x} \in H(\mathbf{p}, \delta) \cap M$ then

$$f_1(\mathbf{x}) = f_2(\mathbf{x}) = \cdots = f_m(\mathbf{x}) = 0,$$

and therefore

$$\begin{aligned} x_i &= g_i\Big(0, 0, \ldots, 0, x_{m+1}, \ldots, x_n\Big) \\ &= h_i\Big(x_{m+1}, \ldots, x_n\Big). \end{aligned}$$

for $i = 1, 2, \ldots, m$. It follows that

$$\begin{aligned} M \cap H(\mathbf{p}, \delta) \quad &\subset \quad \{(x_1, x_2, \ldots, x_n) \in H(\mathbf{p}, \delta) : \\ &\qquad\qquad x_i = h_i(x_{m+1}, \ldots, x_n) \text{ for } i = 1, 2, \ldots, m\}. \end{aligned}$$

Now let $\mathbf{x}$ be a point of $H(\mathbf{x}, \delta)$ whose Cartesian components $x_1, x_2, \ldots, x_n$ satisfy the equations

$$x_i = h_i(x_{m+1}, \ldots, x_n)$$

for $i = 1, 2, \ldots, m$. Then

$$x_i = g_i(0, 0, \ldots, 0, x_{m+1}, \ldots, x_n)$$

for $i = 1, 2, \ldots, m$. Now it was shown earlier that

$$y_i = g_i(y_1, y_2, \ldots, y_n)$$

for all $(y_1, y_2, \ldots, y_n) \in W$ when $i > m$. It follows from this that

$$x_i = g_i(0, 0, \ldots, 0, x_{m+1}, \ldots, x_n)$$

when $m < i \leq n$. The functions $g_1, g_2, \ldots, g_n$ are the Cartesian components of the map $\mu \colon W \to X$. We conclude therefore that

$$(x_1, x_2, \ldots, x_n) = \mu(0, 0, \ldots, 0, x_{m+1}, \ldots, x_n),$$

Applying the function φ to both sides of this equation we see that

$$\begin{aligned}\varphi(x_1, x_2, \ldots, x_n) &= \varphi(\mu(0, 0, \ldots, 0, x_{m+1}, \ldots, x_n)) \\ &= (0, 0, \ldots, 0, x_{m+1}, \ldots, x_n).\end{aligned}$$

It then follows from the definition of the map φ that

$$f_i(x_1, x_2, \ldots, x_n) = 0,$$

for $i = 1, 2, \ldots, m$. We have thus shown that if $\mathbf{x}$ is a point of $H(\mathbf{x}, \delta)$ whose Cartesian components $x_1, x_2, \ldots, x_n$ satisfy the equations

$$x_i = h_i(x_{m+1}, \ldots, x_n)$$

for $i = 1, 2, \ldots, m$ then $\mathbf{x} \in M$. The converse of this result was proved earlier. The proof of the theorem is therefore completed on taking $U = H(\mathbf{p}, \delta)$. ∎