MA2321—Analysis in Several Variables
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David R. Wilkins

9.6. Partial Derivatives and Continuous Differentiability

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ denote the standard basis of $\mathbb{R}^m$, defined so that

$$(z_1, z_2, \ldots, z_m) = \sum_{j=1}^{m} z_j \mathbf{e}_j$$

for all $(z_1, z_2, \ldots, z_m) \in \mathbb{R}^m$. Similarly let $\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \ldots, \overline{\mathbf{e}}_n$ denote the standard basis of $\mathbb{R}^n$, defined so that

$$(w_1, w_2, \ldots, w_n) = \sum_{j=1}^{n} w_j \overline{\mathbf{e}}_j$$

for all $(w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$.

Let X be an open set in $\mathbb{R}^m$, and let $\varphi\colon X \to \mathbb{R}^n$ be a function from X to $\mathbb{R}^n$ which is differentiable at some point $\mathbf{p}$ of X. Then the partial derivative of the ith component f_i of the function φ with respect to the jth coordinate function x_j at a point $\mathbf{p}$ of X is determined by the formula

$$\left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{p}} = \overline{\mathbf{e}}_i . (D\varphi)_{\mathbf{p}} \mathbf{e}_j .$$

Definition

Let X be an open set in $\mathbb{R}^m$. A function $\varphi\colon X \to \mathbb{R}^n$ is *continuously differentiable* if the function sending each point $\mathbf{x}$ of X to the derivative $(D\varphi)$ of φ at the point $\mathbf{x}$ is a continuous function from X to the vector space $L(\mathbb{R}^m, \mathbb{R}^n)$ of linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$.

A function of several real variables is said to be "C^1" if and only if it is continuously differentiable.

Lemma 9.9

Let X be an open set in $\mathbb{R}^m$. and let $\varphi\colon X \to \mathbb{R}^n$ be a continuously differentiable function on X. Then the first order partial derivatives of the components of φ exist and are continuous throughout X.

Proof

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the basis vectors that determine the standard basis of $\mathbb{R}^m$ and let $\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \ldots, \overline{\mathbf{e}}_n$ be the basis vectors that determine the standard basis of $\mathbb{R}_n$. Then the partial derivative of the ith component f_i of the function φ with respect to the jth coordinate function x_j at a point $\mathbf{p}$ of X is determined by the formula

$$\left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{p}} = \overline{\mathbf{e}}_i.(D\varphi)_{\mathbf{p}}\mathbf{e}_j.$$

It follows that if $(D\varphi)_{\mathbf{p}}$ is a continuous function of $\mathbf{p}$ then so are the partial derivatives of φ. ∎

9.7. Functions with Continuous Partial Derivatives

Proposition 9.10

Let X be an open set in $\mathbb{R}^m$, let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X, where $\mathbf{p} = (p_1, p_2, \ldots, p_m)$. Suppose that the partial derivatives of the components of φ with respect to the Cartesian coordinates exist and are continuous throughout X. Suppose also that the partial derivatives of the components of φ are all equal to zero at the point $\mathbf{p}$. Then, given any positive real number ε, there exists a positive real number δ such that, for all points $\mathbf{u}$ and $\mathbf{v}$ of $H(\mathbf{p}, \delta)$, $\mathbf{u} \in X$, $\mathbf{v} \in X$ and

$$|f(\mathbf{u}) - f(\mathbf{v})| \leq \varepsilon |\mathbf{u} - \mathbf{v}|,$$

where

$$H(\mathbf{p}, \delta) = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : |x_j - p_j| < \delta \text{ for } j = 1, 2, \ldots, m\}.$$

Proof

Let us denote the jth partial derivative $\dfrac{\partial f_i}{\partial x_j}$ of the ith component f_i of φ by $\partial_j f_i$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then $\partial_j f_i$ is a continuous function on f.

Let some positive real number ε be given. Then there exists a positive real number δ that is small enough to ensure that $\mathbf{x} \in X$ and

$$|(\partial_j f_i)(x_1, x_2, \ldots, x_m)| \leq \varepsilon/\sqrt{mn}$$

for all points $\mathbf{x}$ of $H(\mathbf{p}, \delta)$.

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ denote the standard basis of $\mathbb{R}^m$, defined so that

$$(z_1, z_2, \ldots, z_m) = \sum_{j=1}^{m} z_j \mathbf{e}_j$$

for all $(z_1, z_2, \ldots, z_m) \in \mathbb{R}^m$. Let $\mathbf{u}$ and $\mathbf{v}$ be points of $H(\mathbf{p}, \delta)$, and let points $\mathbf{q}_j$ be defined for $j = 0, 1, 2, \ldots, m$ so that $\mathbf{q}_0 = \mathbf{v}$ and

$$\mathbf{q}_j = \mathbf{q}_{j-1} + (u_j - v_j)\mathbf{e}_j$$

for $j = 1, 2, \ldots, n$. Then $\mathbf{q}_m = \mathbf{u}$ and $\mathbf{q}_j \in H(\mathbf{p}, \delta)$ for $j = 1, 2, \ldots, m$.

Now, for each integer j between 1 and m, the points $\mathbf{q}_j$ and $\mathbf{q}_{j-1}$ differ only in the jth coordinate. Applying the Mean Value Theorem of single-variable calculus (Theorem 7.2), we find that, given any pair of integers i and j, where $1 \leq i \leq n$ and $1 \leq j \leq m$, there exists some real number θ satisfying $0 < \theta < 1$ such that

$$f_i(\mathbf{q}_j) - f_i(\mathbf{q}_{j-1}) = (u_j - v_j)(\partial_j f_i)\Big((1-\theta)\mathbf{q}_{j-1} + \theta\mathbf{q}_j\Big).$$

It follows that

$$|f_i(\mathbf{q}_j) - f_i(\mathbf{q}_{j-1})| \leq \frac{\varepsilon}{\sqrt{mn}}|u_j - v_j|$$

for $j = 1, 2, \ldots, m$. It then follows that

$$|f_i(\mathbf{u}) - f_i(\mathbf{v})| \leq \sum_{j=1}^{m} |f_i(\mathbf{q}_j) - f_i(\mathbf{q}_{j-1})| \leq \frac{\varepsilon}{\sqrt{mn}} \sum_{j=1}^{m} |u_j - v_j|.$$

On applying Schwarz's Inequality (Lemma 2.1), we find that

$$\left(\sum_{j=1}^{m} |u_j - v_j| \right)^2 \leq m \sum_{j=1}^{m} (u_j - v_j)^2 = m \, |\mathbf{u} - \mathbf{v}|^2.$$

It follows that

$$\sum_{j=1}^{m} |u_j - v_j| \leq \sqrt{m} \, |\mathbf{u} - \mathbf{v}|$$

and therefore

$$|f_i(\mathbf{u}) - f_i(\mathbf{v})| \leq \frac{\varepsilon}{\sqrt{n}} |\mathbf{u} - \mathbf{v}|.$$

It follows that

$$|\varphi(\mathbf{u}) - \varphi(\mathbf{v})|^2 = \sum_{i=1}^{n} (f_i(\mathbf{u}) - f_i(\mathbf{v}))^2 \leq \varepsilon^2 |\mathbf{u} - \mathbf{v}|^2$$

for all points $\mathbf{u}$ and $\mathbf{v}$ of $H(\mathbf{p}, \delta)$. The result follows. ∎

Remark

The essential strategy underlying the proof of Proposition 9.10 can be presented, in the two-dimensional case, as follows. Consider a city laid out on a gridiron pattern, where all streets run either from north to south, or from east to west. To get from one street intersection to another, it is always possible to find a route that does not involve both northward and southward legs, and does not involve both eastward and westward legs. (Thus to get from one street intersection to another that lies to the northeast, one can choose a route that involves only travelling northwards or travelling eastwards along city streets.) Suppose that all streets have a maximum gradient equal to m. Then the height difference between any two intersections is bounded above by $\sqrt{2}md$, where d is the direct distance between those street intersections.

Corollary 9.11

Let $\varphi\colon X \to \mathbb{R}^n$ be a continuously differentiable function defined over an open set X in $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. Let M be a positive real number satisfying $M > \|(D\varphi)_\mathbf{p}\|_{\mathrm{op}}$, where $\|(D\varphi)_\mathbf{p}\|_{\mathrm{op}}$ denotes the operator norm of the derivative of φ at $\mathbf{p}$. Then there exists a positive real number δ such that

$$|\varphi(\mathbf{u}) - \varphi(\mathbf{v})| \leq M|\mathbf{u} - \mathbf{v}|$$

for all points $\mathbf{u}$ and $\mathbf{v}$ of X that satisfy $|\mathbf{u} - \mathbf{p}| < \delta$ and $|\mathbf{v} - \mathbf{p}| < \delta$.

Proof

Let $M_0 = \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}}$ and let $\varepsilon = M - M_0$. Let $\psi\colon X \to \mathbb{R}^n$ be defined such that

$$\psi(\mathbf{u}) = \varphi(\mathbf{u}) - (D\varphi)_{\mathbf{p}}\mathbf{u}$$

for all $\mathbf{u} \in X$. Then $(D\psi)_{\mathbf{p}} = (D\varphi)_{\mathbf{p}} - (D\varphi)_{\mathbf{p}} = 0$. It follows from Proposition 9.10 that there exists a positive real number δ such that

$$|\psi(\mathbf{u}) - \psi(\mathbf{v})| \leq \varepsilon|\mathbf{u} - \mathbf{v}|$$

for all points $\mathbf{u}$ and $\mathbf{v}$ of X that satisfy $|\mathbf{u} - \mathbf{p}| < \delta$ and $|\mathbf{v} - \mathbf{p}| < \delta$. Then

$$\begin{aligned}
|\varphi(\mathbf{u}) - \varphi(\mathbf{v})| &= |\psi(\mathbf{u}) - \psi(\mathbf{v}) + (D\varphi)_{\mathbf{p}}(\mathbf{u} - \mathbf{v})| \\
&\leq |\psi(\mathbf{u}) - \psi(\mathbf{v})| + |(D\varphi)_{\mathbf{p}}(\mathbf{u} - \mathbf{v})| \\
&\leq \varepsilon|\mathbf{u} - \mathbf{v}| + M_0|\mathbf{u} - \mathbf{v}| = M|\mathbf{u} - \mathbf{v}|
\end{aligned}$$

for all points $\mathbf{u}$ and $\mathbf{v}$ of X that satisfy $|\mathbf{u} - \mathbf{p}| < \delta$ and $|\mathbf{v} - \mathbf{p}| < \delta$, as required. ∎

Corollary 9.11 ensures that continuously differentiable functions of several real variables are *locally Lipschitz continuous*. This means that they satisfy a Lipschitz condition in some sufficiently small neighbourhood of any given point. This in turn ensures that standard theorems concerning the existence and uniqueness of ordinary differential equations can be applied to systems of ordinary differential equations specified in terms of continuously differentiable functions.

Theorem 9.12

Let X be an open subset of $\mathbb{R}^m$ and let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$. Suppose that the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \dots & \dfrac{\partial f_1}{\partial x_m} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \dots & \dfrac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \dots & \dfrac{\partial f_m}{\partial x_m} \end{pmatrix}$$

exists at every point of X, where f_i denotes the ith component of φ for $i = 1, 2, \dots, n$. Suppose also that the coefficients of this Jacobian matrix are continuous functions on X. Then φ is differentiable at every point of X, and the derivative of φ at each point is represented by the Jacobian matrix.

Proof

Let $\mathbf{p} \in X$, and, for each integer i between 1 and n, let $g_i \colon X \to \mathbb{R}$ be defined such that

$$g_i(\mathbf{x}) = f_i(\mathbf{x}) - \sum_{j=1}^{m} J_{i,j}(x_j - p_j)$$

for all $\mathbf{x} \in X$, where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and

$$J_{i,j} = (\partial_j f_i)(\mathbf{p}) = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{p}}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. The partial derivatives $\partial_j g_i$ of the function g_i are then determined by those of f_i so that

$$(\partial_j g_i)(\mathbf{x}) = (\partial_j f_i)(\mathbf{x}) - J_{i,j}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. It follows that $(\partial_j g_i)(\mathbf{p}) = 0$ for $j = 1, 2, \ldots, m$.

Let $\psi\colon X \to \mathbb{R}^n$ be defined so that $\psi(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \ldots, g_n(\mathbf{x}))$ for all $\mathbf{x} \in X$. Then the partial derivatives of the function $\psi\colon X \to \mathbb{R}^n$ are all equal to zero at the point $\mathbf{p}$.

Let some positive real number ε be given. It follows from Proposition 9.10 that there exists some positive real number δ such that

$$|\psi(\mathbf{x}) - \psi(\mathbf{p})| \leq \varepsilon|\mathbf{x} - \mathbf{p}|$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - J(\mathbf{x} - \mathbf{p})| \leq \varepsilon|\mathbf{x} - \mathbf{p}|$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where J denotes the Jacobian matrix of φ at the point $\mathbf{p}$ (i.e., the matrix whose coefficient in the ith row and jth column of the matrix is equal to the value of the partial derivative

$$\frac{\partial f_i}{\partial x_j}$$

at the point $\mathbf{p}$).

It follows from this that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|} |\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - J(\mathbf{x}-\mathbf{p})| = 0,$$

and thus the function φ is differentiable at $\mathbf{p}$. Moreover the matrix representing the derivative $(D\varphi)_{\mathbf{p}}$ of φ at the point $\mathbf{p}$ is the Jacobian matrix at that point, as required. ∎

Corollary 9.13

Let X be an open set in $\mathbb{R}^m$. A function $\varphi\colon X \to \mathbb{R}^n$ is continuously differentiable if and only if the first order partial derivatives of the components of φ exist and are continuous throughout X.

Proof

The result follows directly on combining the results of Lemma 9.9 and Theorem 9.12. ∎

9.8. Summary of Differentiability Results

We now summarize the main conclusions regarding differentiability of functions of several real variables. They are as follows.

(i) A function $\varphi\colon X \to \mathbb{R}^n$ defined on an open subset X of $\mathbb{R}^m$ is said to be *differentiable* at a point $\mathbf{p}$ of X if and only if there exists a linear transformation $(D\varphi)_\mathbf{p}\colon \mathbb{R}^m \to \mathbb{R}^n$ with the property that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|} \left(\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - (D\varphi)_\mathbf{p}(\mathbf{x}-\mathbf{p})\right) = \mathbf{0}.$$

The linear transformation $(D\varphi)_\mathbf{p}$ (if it exists) is unique and is known as the *derivative* (or *total derivative*) of φ at $\mathbf{p}$.

(ii) If the function $\varphi\colon X \to \mathbb{R}^n$ is differentiable at a point $\mathbf{p}$ of X then the derivative $(D\varphi)_\mathbf{p}$ of φ at $\mathbf{p}$ is represented by the Jacobian matrix of the function φ at $\mathbf{p}$ whose entries are the first order partial derivatives of the components of φ.

(iii) There exist functions $\varphi\colon X \to \mathbb{R}^n$ whose first order partial derivatives are well-defined at a particular point of X but which are not differentiable at that point. Indeed there exist such functions whose first order partial derivatives exist throughout their domain, though the functions themselves are not even continuous. *Thus in order to show that a function is differentiable at a particular point, it is not sufficient to show that the first order partial derivatives of the function exist at that point.*

(iv) However if the first order partial derivatives of the components of a function $\varphi\colon X \to \mathbb{R}^n$ exist *and are continuous* throughout some neighbourhood of a given point then the function is differentiable at that point. (However the converse does not hold: there exist functions which are differentiable whose first order partial derivatives are not continuous.)

(v) Linear transformations are everywhere differentiable.

(vi) A function $\varphi\colon X \to \mathbb{R}^n$ is differentiable if and only if its components are differentiable functions on X (where X is an open set in $\mathbb{R}^m$).

(vii) Given two differentiable functions from X to $\mathbb{R}$, where X is an open set in $\mathbb{R}^m$, the sum, difference and product of these functions are also differentiable.

(viii) (The Chain Rule). The composition of two differentiable functions is differentiable, and the derivative of the composition of the functions at any point is the composition of the derivatives of the functions.