MA2321—Analysis in Several Variables
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9.4. The Multidimensional Product Rule

Proposition 9.7 (Product Rule)

Let X be an open set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions mapping X into $\mathbb{R}$. Let $\mathbf{p}$ be a point of X. Suppose that f and g are differentiable at $\mathbf{p}$. Then the function $f \cdot g$ is differentiable at $\mathbf{p}$, and

$$D(f \cdot g)_{\mathbf{p}} = g(\mathbf{p})(Df)_{\mathbf{p}} + f(\mathbf{p})(Dg)_{\mathbf{p}}.$$

Proof

The functions f and g are differentiable at $\mathbf{p}$, and therefore there are well-defined functions $Q_1 \colon X \to \mathbb{R}$ and $Q_2 \colon X \to \mathbb{R}$, where

$$\lim_{\mathbf{x}\to\mathbf{p}} Q_1(\mathbf{x}) = 0 = Q_1(\mathbf{p}) \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} Q_2(\mathbf{x}) = 0 = Q_2(\mathbf{p}),$$

that are defined throughout X so as to ensure that

$$f(\mathbf{x}) = f(\mathbf{p}) + (Df)_{\mathbf{p}}\,(\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\,Q_1(\mathbf{x})$$

and

$$g(\mathbf{x}) = g(\mathbf{p}) + (Dg)_{\mathbf{p}}\,(\mathbf{x}-\mathbf{p}) + |\mathbf{x}-\mathbf{p}|\,Q_2(\mathbf{x})$$

for all $\mathbf{x} \in X$ (see Lemma 9.3).

Then

$$\begin{aligned} f(\mathbf{x})g(\mathbf{x}) &= f(\mathbf{p})g(\mathbf{p}) \\ &\quad + \Big(g(\mathbf{p})\,(Df)_{\mathbf{p}} + f(\mathbf{p})\,(Dg)_{\mathbf{p}}\Big)(\mathbf{x}-\mathbf{p}) \\ &\quad + |\mathbf{x}-\mathbf{p}|\, Q(\mathbf{x}) \end{aligned}$$

where

$$\begin{aligned} Q(\mathbf{x}) &= \frac{1}{|\mathbf{x}-\mathbf{p}|}(Df)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}) \times (Dg)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}) \\ &\quad + (g(\mathbf{p}) + (Dg)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}))Q_1(\mathbf{x}) \\ &\quad + (f(\mathbf{p}) + (Df)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}))Q_2(\mathbf{x}) \\ &\quad + |\mathbf{x}-\mathbf{p}|\, Q_1(\mathbf{x})Q_2(\mathbf{x}). \end{aligned}$$

Now

$$|(Df)_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| \leq \|(Df)_{\mathbf{p}}\|_{\mathrm{op}} |\mathbf{x} - \mathbf{p}|$$

where $\|(Df)_{\mathbf{p}}\|_{\mathrm{op}}$ denotes the operator norm of $(Df)_{\mathbf{p}}$ (see Lemma 8.1) Similarly

$$|(Dg)_{\mathbf{p}}(\mathbf{x} - \mathbf{p})| \leq \|(Dg)_{\mathbf{p}}\|_{\mathrm{op}} |\mathbf{x} - \mathbf{p}|.$$

It follows that

$$\begin{aligned} \left| \frac{1}{|\mathbf{x} - \mathbf{p}|} (Df)_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) \times (Dg)_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) \right| & \\ \leq \quad & \|(Df)_{\mathbf{p}}\|_{\mathrm{op}} \|(Dg)_{\mathbf{p}}\|_{\mathrm{op}} |\mathbf{x} - \mathbf{p}|, \end{aligned}$$

and therefore

$$\lim_{\mathbf{x} \to \mathbf{p}} \left(\frac{1}{|\mathbf{x} - \mathbf{p}|} (Df)_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) \times (Dg)_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) \right) = 0.$$

Next we note that

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{p}} \Big((g(\mathbf{p}) + (Dg)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}))\, Q_1(\mathbf{x}) \Big) \\ = \quad \lim_{\mathbf{x}\to\mathbf{p}} (g(\mathbf{p}) + (Dg)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})) \times \lim_{\mathbf{x}\to\mathbf{p}} Q_1(\mathbf{x}) = 0,\end{aligned}$$

because $\lim_{\mathbf{x}\to\mathbf{p}} Q_1(\mathbf{x}) = 0$.

Similarly

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{p}} \Big((f(\mathbf{p}) + (Df)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}))\, Q_2(\mathbf{x}) \Big) \\ = \quad \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{p}) + (Df)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})) \times \lim_{\mathbf{x}\to\mathbf{p}} Q_2(\mathbf{x}) = 0,\end{aligned}$$

because $\lim_{\mathbf{x}\to\mathbf{p}} Q_2(\mathbf{x}) = 0$.

The quantities $Q_1(\mathbf{x})$ and $Q_2(\mathbf{x})$ converge to zero and therefore remain bounded as $\mathbf{x}$ tends to $\mathbf{p}$. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} |\mathbf{x}-\mathbf{p}|\, Q_1(\mathbf{x}) Q_2(\mathbf{x}) = 0.$$

Putting these results together, we see that

$$\lim_{\mathbf{x}\to\mathbf{p}} Q(\mathbf{x}) = 0.$$

It follows from this that the function $f \cdot g$ is differentiable at $\mathbf{p}$, and

$$D(f \cdot g)_{\mathbf{p}} = g(\mathbf{p})(Df)_{\mathbf{p}} + f(\mathbf{p})(Dg)_{\mathbf{p}}$$

(see Lemma 9.3). This completes the proof. ∎

9.5. The Multidimensional Chain Rule

Proposition 9.8 (Chain Rule)

Let X be an open set in $\mathbb{R}^m$, and let $\varphi\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$. Let Y be an open set in $\mathbb{R}^n$ which contains $\varphi(X)$, and let $\psi\colon Y \to \mathbb{R}^k$ be a function mapping Y into $\mathbb{R}^k$. Let $\mathbf{p}$ be a point of X. Suppose that φ is differentiable at $\mathbf{p}$ and that ψ is differentiable at $\varphi(\mathbf{p})$. Then the composition $\psi \circ \varphi\colon \mathbb{R}^m \to \mathbb{R}^k$ (i.e., φ followed by ψ) is differentiable at $\mathbf{p}$. Moreover

$$D(\psi \circ \varphi)_{\mathbf{p}} = (D\psi)_{\varphi(\mathbf{p})} \circ (D\varphi)_{\mathbf{p}}.$$

Thus the derivative of the composition $\psi \circ \varphi$ of the functions at the given point is the composition of the derivatives of those functions at the appropriate points.

Proof

Let $\mathbf{q} = \varphi(\mathbf{p})$. The functions $\varphi\colon X \to \mathbb{R}^n$ and $\psi\colon Y \to \mathbb{R}^k$ are differentiable at $\mathbf{p}$ and $\mathbf{q}$ respectively, and therefore there are well-defined functions $\Omega_1\colon X \to \mathbb{R}^n$ and $\Omega_2\colon Y \to \mathbb{R}^k$ that are defined throughout X and Y respectively so as to ensure that

$$\lim_{\mathbf{x}\to\mathbf{p}} \Omega_1(\mathbf{x}) = \mathbf{0} = \Omega_1(\mathbf{p}), \quad \lim_{\mathbf{y}\to\mathbf{q}} \Omega_2(\mathbf{y}) = \mathbf{0} = \Omega_2(\mathbf{q})$$

for all $\mathbf{x} \in X$, and

$$\varphi(\mathbf{x}) = \varphi(\mathbf{p}) + (D\varphi)_{\mathbf{p}}\,(\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\,\Omega_1(\mathbf{x})$$

and

$$\psi(\mathbf{y}) = \psi(\mathbf{q}) + (D\psi)_{\mathbf{q}}\,(\mathbf{y} - \mathbf{q}) + |\mathbf{y} - \mathbf{q}|\,\Omega_2(\mathbf{y})$$

for all $\mathbf{y} \in Y$ (see Lemma 9.3).

Substituting $\varphi(\mathbf{x})$ and $\varphi(\mathbf{p})$ for $\mathbf{y}$ and $\mathbf{q}$ respectively, we find that

$$\begin{aligned}
\psi(\varphi(\mathbf{x})) &= \psi(\varphi(\mathbf{p})) + (D\psi)_{\mathbf{q}}(\varphi(\mathbf{x}) - \varphi(\mathbf{p})) \\
&\qquad + |\varphi(\mathbf{x}) - \varphi(\mathbf{p})|\, \Omega_2(\varphi(\mathbf{x})) \\
&= \psi(\varphi(\mathbf{p})) + (D\psi)_{\varphi(\mathbf{p})}((D\varphi)_{\mathbf{p}}(\mathbf{x} - \mathbf{p})) \\
&\qquad + |\mathbf{x} - \mathbf{p}|\, \Omega(\mathbf{x}),
\end{aligned}$$

where

$$\begin{aligned}
\Omega(\mathbf{x}) &= (D\psi)_{\varphi(\mathbf{p})}(\Omega_1(\mathbf{x})) \\
&\qquad + \left| \frac{1}{|\mathbf{x} - \mathbf{p}|}(D\varphi)_{\mathbf{p}}(\mathbf{x} - \mathbf{p}) + \Omega_1(\mathbf{x}) \right| \Omega_2(\varphi(\mathbf{x})).
\end{aligned}$$

Let

$$M(\mathbf{x}) = \left| \frac{1}{|\mathbf{x}-\mathbf{p}|} (D\varphi)_{\mathbf{p}}(\mathbf{x}-\mathbf{p}) + \Omega_1(\mathbf{x}) \right|$$

for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Then

$$0 \leq M(\mathbf{x}) \leq \frac{|(D\varphi)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})|}{|\mathbf{x}-\mathbf{p}|} + |\Omega_1(\mathbf{x})|$$

for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Moreover

$$|(D\varphi)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})| \leq \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}} |\mathbf{x}-\mathbf{p}|,$$

where $\|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}}$ denotes the operator norm of the linear operator $(D\varphi)_{\mathbf{p}}$ (see Lemma 8.1). It follows that

$$0 \leq M(x) \leq \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}} + |\Omega_1(\mathbf{x})|$$

for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. It follows from the continuity of the function Ω_1 at $\mathbf{p}$ that $M(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X.

Now

$$\Omega(\mathbf{x}) = (D\psi)_{\varphi(\mathbf{p})}(\Omega_1(\mathbf{x})) + M(\mathbf{x})\Omega_2(\varphi(\mathbf{x}))$$

Also the function $\varphi\colon X \to \mathbb{R}^n$ is continuous at $\mathbf{p}$ and the function $\Omega_2\colon Y \to \mathbb{R}^k$ is continuous at $\varphi(\mathbf{p})$. It follows that the composition function $\Omega_2 \circ \varphi$ is continuous at $\mathbf{p}$ (see Lemma 4.1), and therefore

$$\lim_{\mathbf{x}\to\mathbf{p}} \Omega_2(\varphi(\mathbf{x})) = \Omega_2(\varphi(\mathbf{p})) = \mathbf{0}.$$

We have already shown that $M(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (M(\mathbf{x})\Omega_2(\varphi(\mathbf{x})) = \mathbf{0}$$

(see Proposition 4.9).

Linear operators on finite-dimensional vector spaces are continuous. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (D\psi)_{\varphi(\mathbf{p})}(\Omega_1(\mathbf{x})) = (D\psi)_{\varphi(\mathbf{p})}\left(\lim_{\mathbf{x}\to\mathbf{p}} \Omega_1(\mathbf{x})\right) = \mathbf{0}.$$

It follows that

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{p}} \Omega(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{p}} (D\psi)_{\varphi(\mathbf{p})}(\Omega_1(\mathbf{x})) + \lim_{\mathbf{x}\to\mathbf{p}} (M(\mathbf{x})\Omega_2(\varphi(\mathbf{x}))) \\ &= \mathbf{0} = \Omega(\mathbf{p}).\end{aligned}$$

This result ensures that the composition function $\psi \circ \varphi$ is differentiable at $\mathbf{p}$, and that

$$D(\psi \circ \varphi)_{\mathbf{p}} = (D\psi)_{\varphi(\mathbf{p})} \circ (D\varphi)_{\mathbf{p}}$$

(see Lemma 9.3). The result follows. ∎

Example

Consider the function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(x,y) = \begin{cases} x^2 y^3 \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Now one can verify from the definition of differentiability that the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} t^2 \sin \dfrac{1}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \end{cases}$$

is differentiable everywhere on $\mathbb{R}$, though its derivative $h'\colon \mathbb{R} \to \mathbb{R}$ is not continuous at 0. Also the functions $(x,y) \mapsto x$ and $(x,y) \mapsto y$ are differentiable everywhere on $\mathbb{R}$ (by Lemma 9.2). Now $\varphi(x,y) = y^3 h(x)$. Using Proposition 9.6 and Proposition 9.8, we conclude that φ is differentiable everywhere on $\mathbb{R}^2$.

Let $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m)$ denote the standard basis of $\mathbb{R}^m$, where

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{e}_m = (0, 0, \ldots, 1).$$

Let us denote by $f_i \colon X \to \mathbb{R}$ the ith component of the map $\varphi \colon X \to \mathbb{R}^n$, where X is an open subset of $\mathbb{R}^m$. Thus

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$. The jth partial derivative of f_i at $\mathbf{p} \in X$ is then given by

$$\left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{p}} = \lim_{t \to 0} \frac{f_i(\mathbf{p} + t\mathbf{e}_j) - f_i(\mathbf{p})}{t}.$$

We see therefore that if φ is differentiable at $\mathbf{p}$ then

$$(D\varphi)_{\mathbf{p}} \mathbf{e}_j = \left(\frac{\partial f_1}{\partial x_j}, \frac{\partial f_2}{\partial x_j}, \ldots, \frac{\partial f_m}{\partial x_j} \right).$$

Thus the linear transformation $(D\varphi)_{\mathbf{p}}\colon \mathbb{R}^m \to \mathbb{R}^n$ is represented by the $n \times m$ matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_m} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_m} \end{pmatrix}$$

This matrix is known as the *Jacobian matrix* of φ at $\mathbf{p}$.

Example

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Note that this function is not continuous at $(0,0)$. Indeed $f(t,t) = 1/(4t^2)$ if $t \neq 0$ so that $f(t,t) \to +\infty$ as $t \to 0$, yet $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$, thus showing that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

cannot possibly exist. Because f is not continuous at $(0,0)$ we conclude from Lemma 9.4 that f cannot be differentiable at $(0,0)$. However it is easy to show that the partial derivatives

$$\frac{\partial f(x,y)}{\partial x} \text{ and } \frac{\partial f(x,y)}{\partial y}$$

exist everywhere on $\mathbb{R}^2$, even at $(0,0)$.

Indeed

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = 0, \qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = 0$$

on account of the fact that $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$.

Example

Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Given real numbers b and c, let $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ be defined so that $u_{b,c}(t) = g(bt, ct)$ for all $t \in \mathbb{R}$. If $b = 0$ or $c = 0$ then $u_{b,c}(t) = 0$ for all $t \in \mathbb{R}$. If $b \neq 0$ and $c \neq 0$ then

$$u_{b,c}(t) = \frac{bc^2t^3}{b^2t^2 + c^4t^4} = \frac{bc^2t}{b^2 + c^2t^2}.$$

We now show that the function $u_{b,c} \colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders. This is obvious when $b = 0$, and when $c = 0$. If b and c are both non-zero, and if the function $u_{b,c}$ has a derivative $u_{b,c}^{(k)}(t)$ of order k that can be represented in the form

$$u_{b,c}^{(k)}(t) = p_k(t)(b^2 + c^2 t^2)^{-k-1},$$

where $p_k(t)$ is a polynomial of degree at most $k + 1$, then it follows from standard single-variable calculus that the function $u_{b,c}$ has a derivative $u_{b,c}^{(k+1)}(t)$ of order $k + 1$ that can be represented in the form

$$u_{b,c}^{(k+1)}(t) = p_{k+1}(t)(b^2 + c^2 t^2)^{-k-2},$$

where $p_{k+1}(t)$ is the polynomial of degree at most $k + 2$ determined by the formula

$$p_{k+1}(t) = p_k'(t)(b^2 + c^2 t^2) - 2(k + 1)c^2 t p_k(t).$$

Thus the function $u_{b,c} \colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders.

Moreover the first derivative $u'_{b,c}(0)$ of $u_{b,c}(t)$ at $t = 0$ is given by the formula

$$u'_{b,c}(0) = \begin{cases} \dfrac{c^2}{b} & \text{if } b \neq 0; \\ 0 & \text{if } b = 0. \end{cases}$$

We have shown that the restriction of the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ to any line passing through the origin determines a function that may be differentiated any number of times with respect to distance along the line. Analogous arguments show that the restriction of the function g to any other line in the plane also determines a function that may be differentiated any number of times with respect to distance along the line.

Now $g(x, y) = \frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x > 0$ and $y = \pm\sqrt{x}$, and similarly $g(x, y) = -\frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x < 0$ and $y = \pm\sqrt{-x}$. It follows that every open disk about the origin $(0, 0)$ contains some points at which the function g takes the value $\frac{1}{2}$, and other points at which the function takes the value $-\frac{1}{2}$, and indeed the function g will take on all real values between $-\frac{1}{2}$ and $\frac{1}{2}$ on any open disk about the origin, no matter how small the disk. Therefore the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ is not continuous at zero, even though the partial derivatives of the function g with respect to x and y exist at each point of $\mathbb{R}^2$.

Remark
These last two examples exhibits an important point. They show that *even if all the partial derivatives of a function exist at some point, this does not necessarily imply that the function is differentiable at that point.* However we shall show that if the first order partial derivatives of the components of a function exist *and are continuous* throughout some neighbourhood of a given point then the function is differentiable at that point.