MA2321—Analysis in Several Variables
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9.3. Properties of Differentiable Functions of Several Real Variables

Lemma 9.4

Let $\varphi\colon X \to \mathbb{R}^n$ be a function which maps an open subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$ which is differentiable at some point $\mathbf{p}$ of X. Then φ is continuous at $\mathbf{p}$.

Proof

Let $\Omega\colon X \to \mathbb{R}^n$ be defined so that $\Omega(\mathbf{p}) = 0$ and

$$\Omega(\mathbf{x}) = \frac{1}{|\mathbf{x} - \mathbf{p}|} \left(\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}} (\mathbf{x} - \mathbf{p})\right)$$

for all points $\mathbf{x}$ of X satisfying $\mathbf{x} \neq \mathbf{p}$. If $\varphi\colon X \to \mathbb{R}^n$ is differentiable at $\mathbf{p}$ then $\Omega\colon X \to \mathbb{R}^n$ is continuous at $\mathbf{p}$ (see Lemma 9.3). Moreover

$$\varphi(\mathbf{x}) = \varphi(\mathbf{p}) + (D\varphi)_{\mathbf{p}} (\mathbf{x} - \mathbf{p}) + |\mathbf{x} - \mathbf{p}|\, \Omega(\mathbf{x})$$

for all $\mathbf{x} \in X$. It follows that $\varphi\colon X \to \mathbb{R}^n$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 9.5

Let $\varphi\colon X \to \mathbb{R}^n$ be a function which maps an open subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$ which is differentiable at some point $\mathbf{p}$ of X. Let $(D\varphi)_{\mathbf{p}}\colon \mathbb{R}^m \to \mathbb{R}^n$ be the derivative of φ at $\mathbf{p}$. Let $\mathbf{u}$ be an element of $\mathbb{R}^m$. Then

$$(D\varphi)_{\mathbf{p}}\mathbf{u} = \lim_{t\to 0} \frac{1}{t} \left(\varphi(\mathbf{p} + t\mathbf{u}) - \varphi(\mathbf{p})\right).$$

Thus the derivative $(D\varphi)_{\mathbf{p}}$ of φ at $\mathbf{p}$ is uniquely determined by the map φ.

Proof

It follows from the differentiability of φ at $\mathbf{p}$ that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|} \left(\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,(\mathbf{x}-\mathbf{p})\right) = \mathbf{0}.$$

In particular, if we set $(\mathbf{x}-\mathbf{p}) = t\mathbf{u}$, and $(\mathbf{x}-\mathbf{p}) = -t\mathbf{u}$, where t is a real variable, we can conclude that

$$\lim_{t\to 0^+} \frac{1}{t} \left(\varphi(\mathbf{p}+t\mathbf{u}) - \varphi(\mathbf{p}) - t(D\varphi)_{\mathbf{p}}\mathbf{u}\right) = \mathbf{0},$$

$$\lim_{t\to 0^-} \frac{1}{t} \left(\varphi(\mathbf{p}+t\mathbf{u}) - \varphi(\mathbf{p}) - t(D\varphi)_{\mathbf{p}}\mathbf{u}\right) = \mathbf{0},$$

It follows that

$$\lim_{t\to 0} \frac{1}{t} \left(\varphi(\mathbf{p}+t\mathbf{u}) - \varphi(\mathbf{p})\right) = (D\varphi)_{\mathbf{p}}\mathbf{u},$$

as required. ∎

We now show that given two differentiable functions mapping X into $\mathbb{R}$, where X is an open set in $\mathbb{R}^m$, the sum, difference and product of these functions are also differentiable.

Proposition 9.6

Let X be an open set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be functions mapping X into $\mathbb{R}$. Let $\mathbf{p}$ be a point of X. Suppose that f and g are differentiable at $\mathbf{p}$. Then the functions $f + g$ and $f - g$ are differentiable at $\mathbf{p}$, and

$$D(f+g)_{\mathbf{p}} = (Df)_{\mathbf{p}} + (Dg)_{\mathbf{p}}$$

and

$$D(f-g)_{\mathbf{p}} = (Df)_{\mathbf{p}} - (Dg)_{\mathbf{p}}.$$

Proof

The limit of a sum of functions is the sum of the limits of those functions, provided that these limits exist. Applying the definition of differentiability, it therefore follows that

$$\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|}\Big(f(\mathbf{x}) + g(\mathbf{x}) - (f(\mathbf{p}) + g(\mathbf{p})) - ((Df)_{\mathbf{p}} + (Dg)_{\mathbf{p}})(\mathbf{x}-\mathbf{p})\Big) \\
&\quad = \lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|}\Big(f(\mathbf{x}) - f(\mathbf{p}) - (Df)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})\Big) \\
&\qquad\quad + \lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x}-\mathbf{p}|}\Big(g(\mathbf{x}) - g(\mathbf{p}) - (Dg)_{\mathbf{p}}(\mathbf{x}-\mathbf{p})\Big) \\
&\quad = 0.
\end{aligned}$$

Therefore

$$D(f+g)_{\mathbf{p}} = (Df)_{\mathbf{p}} + (Dg)_{\mathbf{p}}.$$

Also the function $-g$ is differentiable, with derivative $-(Dg)_{\mathbf{p}}$. It follows that $f - g$ is differentiable, with derivative $(Df)_{\mathbf{p}} - (Dg)_{\mathbf{p}}$. This completes the proof. ∎