MA2321—Analysis in Several Variables
School of Mathematics, Trinity College
Michaelmas Term 2017
Lecture 17 (November 16, 2017)

David R. Wilkins

7.6. Taylor's Theorem

The result obtained in Proposition 7.3 is a special case of a more general result. That more general result is a version of Taylor's Theorem with remainder. The proof of this theorem will make use of the following lemma.

Lemma 7.6

Let s and h be real numbers, let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$, let $c_0, c_1, \ldots, c_{k-1}$ be real numbers, and let

$$p(t) = f(s + th) - \sum_{n=0}^{k-1} c_n t^n.$$

for all real numbers t belonging to some open interval D for which $0 \in D$ and $1 \in D$. Then $p^{(n)}(0) = 0$ for all integers n satisfying $0 \leq n < k$ if and only if

$$c_n = \frac{h^n f^{(n)}(s)}{n!}$$

for all integers n satisfying $0 \leq n < k$.

Proof

On setting $t = 0$, we find that $p(0) = f(s) - c_0$, and thus $p(0) = 0$ if and only if $c_0 = f(s)$.

Let the integer n satisfy $0 < n < k$. On differentiating $p(t)$ n times with respect to t, we find that

$$p^{(n)}(t) = h^n f^{(n)}(s + th) - \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} c_j t^{j-n}.$$

Then, on setting $t = 0$, we find that only the term with $j = n$ contributes to the value of the sum on the right hand side of the above identity, and therefore

$$p^{(n)}(0) = h^n f^{(n)}(s) - n! c_n.$$

The result follows. ∎

Theorem 7.7

(Taylor's Theorem) Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s+h$. Then

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof

Let D be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s+th)$ is defined for all $t \in D$, and let $p\colon D \to \mathbb{R}$ be defined so that

$$p(t) = f(s+th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

for all $t \in D$. A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$ (see Lemma 7.6). Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0, 1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$, and $q(1) = 0$. We can therefore apply Rolle's Theorem (Theorem 7.1) to the function q on the interval $[0, 1]$ to deduce the existence of some real number t_1 satisfying $0 < t_1 < 1$ for which $q'(t_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, t_1]$ to deduce the existence of some real number t_2 satisfying $0 < t_2 < t_1$ for which $q''(t_2) = 0$.

By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_k < t_{k-1} < \cdots < t_1 < 1$ with the property that $q^{(n)}(t_n) = 0$ for $n = 1, 2, \ldots, k$. Let $\theta = t_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1),$$

hence

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) \\ &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h), \end{aligned}$$

as required. ∎

Corollary 7.8

Let $f\colon D \to \mathbb{R}$ be a k-times continuously differentiable function defined over an open subset D of $\mathbb{R}$ and let $s \in \mathbb{R}$. Then given any strictly positive real number ε, there exists some strictly positive real number δ such that

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| < \varepsilon |h|^k$$

whenever $|h| < \delta$.

Proof

The function f is k-times continuously differentiable, and therefore its kth derivative $f^{(k)}$ is continuous. Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ that is small enough to ensure that $s + h \in D$ and $|f^{(k)}(s + h) - f^{(k)}(s)| < k!\varepsilon$ whenever $|h| < \delta$. If h is an real number satisfying $|h| < \delta$, and if θ is a real number satisfying $0 < \theta < 1$, then $s + \theta h \in D$ and $|f^{(k)}(s + \theta h) - f^{(k)}(s)| < k!\varepsilon$. Now it follows from Taylor's Theorem (Theorem 7.7) that, given any real number h satisfying $|h| < \delta$ there exists some real number θ satisfying $0 < \theta < 1$ for which

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h).$$

Then

$$\begin{aligned}\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| \\ &= \frac{|h|^k}{k!} |f^{(k)}(s+\theta h) - f^{(k)}(s)| \\ &< \varepsilon |h|^k,\end{aligned}$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \\
f'(b) &= \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}
\end{aligned}
$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h \to 0^+} \frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h \to 0^-} \frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Proposition 7.9

Let f be a continuously differentiable real-valued function on the interval $[a, b]$. Then

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof

Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\, dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\, dt\right) = 0$$

for all x satisfying $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 7.2) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b - a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 7.10 (Integration by Parts)

Let f and g be continuously differentiable real-valued functions on the interval $[a, b]$. Then

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof

This result follows from Proposition 7.9 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \quad \blacksquare$$

Corollary 7.11 (Integration by Substitution)

Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof

Let F and G be the functions on $[a, b]$ defined by

$$F(x) = \int_c^{u(x)} f(y) dy, \qquad G(x) = \int_a^x f(u(t)) \frac{du(t)}{dt} dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\, dy,$$

and $H'(s) = f(s)$ for all $s \in [a, b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a, b)$. On applying the Mean Value Theorem (Theorem 7.2) to the function $F - G$ on the interval $[a, b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

Proposition 7.12

Let s and h be real numbers, and let f be a function whose first k derivatives are continuous on an interval containing s and $s+h$. Then

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) \\ &\qquad + \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\, dt. \end{aligned}$$

Proof

Let

$$r_m(s,h) = \frac{h^m}{(m-1)!} \int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt$$

for $m = 1, 2, \ldots, k-1$. Then

$$r_1(s,h) = h \int_0^1 f'(s+th)\,dt = \int_0^1 \frac{d}{dt} f(s+th)\,dt = f(s+h) - f(s).$$

Let m be an integer between 1 and $k-2$. It follows from the rule for Integration by Parts (Corollary 7.10) that

$$\begin{aligned}
r_{m+1}(s,h) &= \frac{h^{m+1}}{m!}\int_0^1 (1-t)^m f^{(m+1)}(s+th)\,dt \\
&= \frac{h^m}{m!}\int_0^1 (1-t)^m \frac{d}{dt}\left(f^{(m)}(s+th)\right)\,dt \\
&= \frac{h^m}{m!}\left[(1-t)^m f^{(m)}(s+th)\right]_0^1 \\
&\qquad - \frac{h^m}{m!}\int_0^1 \frac{d}{dt}\left((1-t)^m\right) f^{(m)}(s+th)\,dt \\
&= -\frac{h^m}{m!} f^{(m)}(s) \\
&\qquad + \frac{h^m}{(m-1)!}\int_0^1 (1-t)^{m-1} f^{(m)}(s+th)\,dt \\
&= r_m(s,h) - \frac{h^m}{m!} f^{(m)}(s).
\end{aligned}$$

Thus

$$r_m(s,h) = \frac{h^m}{m!} f^{(m)}(s) + r_{m+1}(s,h)$$

for $m = 1, 2, \ldots, k-1$. It follows by induction on k that

$$\begin{aligned} f(s+h) &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + r_k(s,h) \\ &= f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) \\ &\qquad + \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(s+th)\, dt, \end{aligned}$$

as required. ■