MA2321—Analysis in Several Variables
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6.4. Integrability of Continuous Functions

Theorem 6.17

Let C be a closed n-cell in $\mathbb{R}^n$. Then any continuous real-valued function on C is Riemann-integrable.

Proof

Let $f: C \to \mathbb{R}$ be a continuous real-valued function on C. Then f is bounded above and below on C, and moreover $f: C \to \mathbb{R}$ is uniformly continuous on C. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(\mathbf{u}) - f(\mathbf{w})| < \varepsilon$ whenever $\mathbf{u}, \mathbf{w} \in C$ satisfy $|\mathbf{u} - \mathbf{w}| < \delta$.

Choose a partition P of the n-cell C such that each cell in the partition has diameter less than δ. Let $\Omega(P)$ be an index set which indexes the cells of the partition P and, for each $\alpha \in \Omega(P)$ let $C_{P,\alpha}$ be the corresponding cell of the partition P of C. Also let $\mathbf{p}_\alpha$ be a point of $C_{P,\alpha}$ for all $\alpha \in \Omega(P)$. Then $|\mathbf{x} - \mathbf{p}_\alpha| < \delta$ for all $\mathbf{x} \in C_{P,\alpha}$. Thus if

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}$$

then

$$f(\mathbf{p}_\alpha) - \varepsilon \leq m_{P,\alpha} \leq M_{P,\alpha} \leq f(\mathbf{p}_\alpha) + \varepsilon$$

for all $\alpha \in \Omega(P)$. It follows that

$$\begin{aligned}
&\sum_{i=1}^{n} f(\mathbf{p}_\alpha)\mu(C_{P,\alpha}) - \varepsilon\mu(C) \\
\leq\ & L(P,f) \leq U(P,f) \\
\leq\ & \sum_{i=1}^{n} f(\mathbf{p}_\alpha)\mu(C_{P,\alpha}) + \varepsilon\mu(C),
\end{aligned}$$

where $L(P,f)$ and $U(P,f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$\begin{aligned} 0 &\leq \mathcal{U}\int_C f(x)\,d\mu - \mathcal{L}\int_C f(x)\,d\mu \\ &\leq U(P,f) - L(P,f) \leq 2\varepsilon\mu(C). \end{aligned}$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_C f(x)\,d\mu = \mathcal{L}\int_C f(x)\,d\mu,$$

and thus the function f is Riemann-integrable on C. ∎

6.5. Repeated Integration

Let C be an n-cell in $\mathbb{R}^n$, given by

$$\begin{aligned} C &= \prod_{i=1}^{n} [a_i, b_i] \\ &= \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}, \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers which satisfy $a_i \leq b_i$ for each i. Given any continuous real-valued function f on C, let us denote by $\mathcal{I}_C(f)$ the repeated integral of f over the n-cell C whose value is

$$\int_{x_n=a_n}^{b_n} \left(\cdots \int_{x_2=a_2}^{b_2} \left(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1 \right) dx_2 \ldots \right) dx_n.$$

(Thus $\mathcal{I}_C(f)$ is obtained by integrating the function f first over the coordinate x_1, then over the coordinate x_2, and so on).

Note that if $m \leq f(\mathbf{x}) \leq M$ on C for some constants m and M then

$$m\,\mu(C) \leq \mathcal{I}_C(f) \leq M\,\mu(C).$$

We shall use this fact to show that if f is a continuous function on some n-cell C in $\mathbb{R}^n$ then

$$\mathcal{I}_C(f) = \int_C f(\mathbf{x})\,d\mu$$

(i.e., $\mathcal{I}_C(f)$ is equal to the Riemann integral of f over C).

Theorem 6.18

Let f be a continuous real-valued function defined on some n-cell C in $\mathbb{R}^n$, where

$$C = \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

Then the Riemann integral

$$\int_C f(\mathbf{x})\, d\mu$$

of f over C is equal to the repeated integral

$$\int_{x_n=a_n}^{b_n} \left(\cdots \int_{x_2=a_2}^{b_2} \left(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1 \right) dx_2 \ldots \right) dx_n.$$

Proof

Given a partition P of the n-cell C, we denote by $L(P, f)$ and $U(P, f)$ the quantities so that

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, \mu(C_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, \mu(C_{P,\alpha})$$

where $\Omega(P)$ is an indexing set that indexes the cells of the partition P, and where, for all $\alpha \in \Omega(P)$, $\mu(C_{P,\alpha})$ is the content of the cell $C_{P,\alpha}$ of the partition P indexed by α,

$$m_{P,\alpha}(f) = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\},$$

and

$$M_{P,\alpha}(f) = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.$$

Now

$$m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f)$$

for all $\alpha \in \Omega(P)$ and $\mathbf{x} \in C_{P,\alpha}$, and therefore

$$m_{P,\alpha}(f)\,\mu(C_{P,\alpha}) \leq \mathcal{I}_{C,\alpha}(f) \leq M_{P,\alpha}(f)\,\mu(C_{P,\alpha})$$

for all $\alpha \in \Omega(P)$.

Summing these inequalities as α ranges over the indexing set $\Omega(P)$, we find that

$$\begin{aligned} L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)\,\mu(C_{P,\alpha}) \\ &\leq \sum_{\alpha\in\Omega(P)} \mathcal{I}_{C,\alpha}(f) \\ &\leq \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f)\,\mu(C_{P,\alpha}) \\ &= U(P,f). \end{aligned}$$

But

$$\sum_{\alpha\in\Omega(P)} \mathcal{I}_{C,\alpha}(f) = \mathcal{I}_C(f).$$

It follows that

$$L(P,f) \leq \mathcal{I}_C(f) \leq U(P,f).$$

The Riemann integral of f is equal to the supremum of the quantities $L(P, f)$ as P ranges over all partitions of the n-cell C, hence

$$\int_C f(\mathbf{x})\, d\mu \leq \mathcal{I}_C(f).$$

Similarly the Riemann integral of f is equal to the infimum of the quantities $U(P, f)$ as P ranges over all partitions of the n-cell C, hence

$$\mathcal{I}_C(f) \leq \int_C f(\mathbf{x})\, d\mu.$$

Hence

$$\mathcal{I}_C(f) = \int_C f(\mathbf{x})\, d\mu,$$

as required. ■

Note that the order in which the integrations are performed in the repeated integral plays no role in the above proof. We may therefore deduce the following important corollary.

Corollary 6.19

Let f be a continuous real-valued function defined over some closed rectangle C in $\mathbb{R}^2$, where

$$C = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\}.$$

Then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Proof

It follows directly from Theorem 6.18 that the repeated integrals

$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx \text{ and } \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

are both equal to the Riemann integral of the function f over the rectangle C. Therefore these repeated integrals must be equal. ∎

Example

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that

$$f(x,y) = \begin{cases} \dfrac{4xy(x^2-y^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Set $u = x^2 + y^2$. Then

$$f(x,y) = \frac{2x(2x^2-u)}{u^3}\,\frac{\partial u}{\partial y},$$

and therefore, when $x \neq 0$,

$$
\begin{aligned}
\int_{y=0}^{1} f(x,y)\,dy &= \int_{u=x^2}^{x^2+1} \left(\frac{4x^3}{u^3} - \frac{2x}{u^2} \right) du \\
&= \left[-\frac{2x^3}{u^2} + \frac{2x}{u} \right]_{u=x^2}^{x^2+1} \\
&= -\frac{2x^3}{(x^2+1)^2} + \frac{2x}{x^2+1} \\
&= \frac{2x}{(x^2+1)^2}
\end{aligned}
$$

It follows that

$$\begin{aligned}\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx &= \int_{x=0}^{1}\frac{2x}{(x^2+1)^2}\,dx \\ &= \left[-\frac{1}{x^2+1}\right]_0^1 = \frac{1}{2}.\end{aligned}$$

Now $f(y,x) = -f(x,y)$ for all x and y. Interchanging x and y in the above evaluation, we find that

$$\begin{aligned}\int_{y=0}^{1}\left(\int_{x=0}^{1} f(x,y)\,dx\right)\,dy &= \int_{x=0}^{1}\left(\int_{y=0}^{1} f(y,x)\,dy\right)\,dx \\ &= -\int_{x=0}^{1}\left(\int_{y=0}^{1} f(x,y)\,dy\right)\,dx \\ &= -\frac{1}{2}.\end{aligned}$$

Thus

$$\int_{x=0}^{1} \left(\int_{y=0}^{1} f(x,y)\,dy \right) dx \neq \int_{y=0}^{1} \left(\int_{x=0}^{1} f(x,y)\,dx \right) dy.$$

when

$$f(x,y) = \frac{4xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

for all $(x, y) \in \mathbb{R}^2$ distinct from $(0, 0)$. Note that, in this case $f(2t, t) \to +\infty$ as $t \to 0^+$, and $f(t, 2t) \to -\infty$ as $t \to 0^-$. Thus the function f is not continuous at $(0, 0)$ and does not remain bounded as $(x, y) \to (0, 0)$.