MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2017 Lecture 14 (November 2, 2017)

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Lemma 6.12

Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}. \end{aligned}$$

Then

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

for all $u, v \in X$.

Proof

Let $u, v \in X$. Then either $f(v) \geq f(u)$ or $f(u) \geq f(v)$. In the case where $f(v) \geq f(u)$ the inequalities $m_X(f) \leq f(u) \leq f(v) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. In the case where $f(u) \geq f(v)$ the inequalities $m_X(f) \leq f(v) \leq f(u) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. The result follows. ∎

Lemma 6.13

Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(|f|) &= \sup\{|f(x)| : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(|f|) &= \inf\{|f(x)| : x \in X\}. \end{aligned}$$

Then

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f).$$

Proof

Let δ be a positive real number. Then there exist $u, v \in X$ such that

$$m_X(|f|) \leq |f(u)| < m_X(|f|) + \delta$$

and

$$M_X(|f|) - \delta < |f(v)| \leq M_X(|f|).$$

Then

$$|f(v)| - |f(u)| > M_X(|f|) - m_X(|f|) - 2\delta.$$

But

$$|f(v)| - |f(u)| \leq |f(v) - f(u)|,$$

(because $|f(v)| \leq |f(u)| + |f(v) - f(u)|$) and

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

(see Lemma 6.12). It follows that

$$\begin{aligned} M_X(|f|) - m_X(|f|) - 2\delta &< |f(v)| - |f(u)| \leq |f(v) - f(u)| \\ &\leq M_X(f) - m_X(f). \end{aligned}$$

But the values of $M_X(|f|) - m_X(|f|)$ and $M_X(f) - m_X(f)$ are independent of δ, where $\delta > 0$. It follows that

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f),$$

as required. ∎

Let X be a non-empty set, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X. We denote by $f \cdot g\colon X\mathbb{R}$ the product function defined such that We denote by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.

Lemma 6.14

Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be bounded real-valued functions defined on a non-empty set X, let C be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in X$, and let

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(g) &= \sup\{g(x) : x \in X\}, \\ M_X(f \cdot g) &= \sup\{f(x)g(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(g) &= \inf\{g(x) : x \in X\}, \\ m_X(f \cdot g) &= \inf\{f(x)g(x) : x \in X\}. \end{aligned}$$

Then

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Proof

Let u and v be elements of the set X. Then

$$f(v)g(v) - f(u)g(u) = (f(v) - f(u))g(v) + f(u)(g(v) - g(u)),$$

and therefore

$$\begin{aligned} &|f(v)g(v) - f(u)g(u)| \\ &\qquad \leq \quad |f(v) - f(u)|\,|g(v)| + |f(u)|\,|g(v) - g(u)|, \\ &\qquad \leq \quad K\Big(|f(v) - f(u)| + |g(v) - g(u)|\Big). \end{aligned}$$

Now $|f(v) - f(u)| \leq M_X(f) - m_X(f)$ and $|g(v) - g(u)| \leq M_X(g) - m_X(g)$ and (see Lemma 6.12). Therefore

$$|f(v)g(v) - f(u)g(u)| \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Now, given any positive real number δ, elements u and v of X can be chosen so that

$$m_X(f \cdot g) \leq f(u)g(u) < m_X(f \cdot g) + \delta$$

and

$$M_X(f \cdot g) - \delta < f(v)g(v) \leq M_X(f \cdot g).$$

Then

$$f(v)g(v) - f(u)g(u) > M_X(f \cdot g) - m_X(f \cdot g) - 2\delta.$$

It follows that

$$M_X(f \cdot g) - m_X(f \cdot g) - 2\delta < K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big)$$

for all positive real numbers δ, and therefore

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big),$$

as required. ∎

Proposition 6.15

Let $f\colon C \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-cell C in $\mathbb{R}^n$, and let $|f|\colon C \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in C$. Then the function $|f|$ is Riemann-integrable on C, and

$$\left| \int_C f(\mathbf{x})\, d\mu \right| \leq \int_C |f(\mathbf{x})|\, d\mu.$$

Proof

Let P be a partition of the n-cell C. We first show that the Darboux sums $U(P, f)$ and $L(P, f)$ of the function f on C and the Darboux sums $U(P, |f|)$ and $L(P, |f|)$ of the function $|f|$ on C satisfy the inequality

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f).$$

Let $\Omega(P)$ be the indexing set for the partition P of C, and let

$$\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(|f|) &= \sup\{|f(\mathbf{x})| : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(|f|) &= \inf\{|f(\mathbf{x})| : \mathbf{x} \in C_{P,\alpha}\}
\end{aligned}$$

for $\alpha \in \Omega(P)$. It follows from Lemma 6.13 that

$$M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|) \leq M_{P,\alpha}(f) - m_{P,\alpha}(f)$$

for $\alpha \in \Omega(P)$.

Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned}
L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}), \\
L(P, |f|) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(|f|)\mu(C_{P,\alpha}), \\
U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}), \\
U(P, |f|) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(|f|)\mu(C_{P,\alpha}).
\end{aligned}$$

It follows that

$$\begin{aligned} U(P,|f|) - L(P,|f|) &= \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|))\mu(C_{P,\alpha}) \\ &\leq \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(f) - m_{P,\alpha}(f))\mu(C_{P,\alpha}) \\ &= U(P,f) - L(P,f). \end{aligned}$$

Let some positive real number ε be given. It follows from Proposition 6.11 that there exists a partition P of C such that

$$U(P, f) - L(P, f) < \varepsilon.$$

Then

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 6.11 then ensures that the function $|f|$ is Riemann-integrable on C.

Now $-|f(\mathbf{x})| \leq f(\mathbf{x}) \leq |f(\mathbf{x})|$ for all $\mathbf{x} \in C$. It follows that

$$\begin{aligned} -\int_C |f(\mathbf{x})|\,d\mu &\leq \int_C f(\mathbf{x})\,d\mu \\ &\leq \int_C |f(\mathbf{x})|\,d\mu. \end{aligned}$$

It follows that

$$\left|\int_C f(\mathbf{x})\,d\mu\right| \leq \int_C |f(\mathbf{x})|\,d\mu,$$

as required. ∎

Proposition 6.16

Let $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded n-cell C in $\mathbb{R}^n$. Then the function $f \cdot g$ is Riemann-integrable on C, where $(f \cdot g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in C$.

Proof

The functions f and g are bounded on C, and therefore there exists some positive real number K with the property that $|f(\mathbf{x})| \leq K$ and $|g(\mathbf{x})| \leq K$ for all $\mathbf{x} \in C$.

Let P be a partition of the n-cell C. We first show that the Darboux sums $U(P, f)$, $U(P, g)$, $U(P, f \cdot g)$, $L(P, f)$, $L(P, g)$ and $L(P, f \cdot g)$ of the functions f, g and $f \cdot g$ on C satisfy the inequality

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \quad K\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big). \end{aligned}$$

Let $\Omega(P)$ be the indexing set of the partition P of the n-cell C, and for all $\alpha \in \Omega(P)$, let $\mu(C_{P,\alpha})$ denote the content of the closed subcell of C for the partition P that corresponds to the multi-index α, and let

$$\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(f \cdot g) &= \sup\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \\
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(f \cdot g) &= \inf\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.
\end{aligned}$$

Now it follows from Lemma 6.14 that

$$\begin{aligned} & M_{P,\alpha}(f \cdot g) - m_{P,\alpha}(f \cdot g) \\ & \qquad \leq \quad K\Bigl(M_{P,\alpha}(f) - m_{P,\alpha}(f) + M_{P,\alpha}(g) - m_{P,\alpha}(g)\Bigr). \end{aligned}$$

for $\alpha \in \Omega(P)$. On multiplying both sides of this inequality by the content $\mu(C_{P,\alpha})$ of the subcell $C_{P,\alpha}$ of the partition indexed by α and summing over all integers between 1 and n, we find that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \qquad \leq \quad K\Bigl(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Bigr), \end{aligned}$$

where

$$\begin{aligned}
U(P,f) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}),\\
U(P,g) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(g)\mu(C_{P,\alpha}),\\
U(P,f\cdot g) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f\cdot g)\mu(C_{P,\alpha}),\\
L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}),\\
L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g)\mu(C_{P,\alpha}),\\
L(P,f\cdot g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f\cdot g)\mu(C_{P,\alpha}),
\end{aligned}$$

Let some positive real number ε be given. It follows from Proposition 6.11 that there exist partitions Q and R of the closed n-cell C for which

$$U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 6.5 that

$$U(P, f) - L(P, f) \leq U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(P, g) - L(P, g) \leq U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Combining the various inequalities obtained in the course of the proof, we find that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \; K\Big(U(P,f) - L(P,f) + U(P,g) - L(P,g) \Big) \\ & \quad < \; \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed n-cell C with the property that

$$U(P, f \cdot g) - L(P, f \cdot g) < \varepsilon.$$

It follows from Proposition 6.11 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎