MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2017 Lecture 13 (October 26, 2017)

David R. Wilkins

6.3. The Multidimensional Riemann-Darboux Integral

Definition

Let C be an n-cell in $\mathbb{R}^n$, and let $f\colon C \to \mathbb{R}$ be a bounded real-valued function on C. The *lower Riemann integral* and the *upper Riemann integral*, denoted by

$$\mathcal{L}\int_C f(\mathbf{x})\,d\mu \quad \text{and} \quad \mathcal{U}\int_C f(\mathbf{x})\,d\mu$$

respectively, are defined such that

$$\begin{aligned}\mathcal{L}\int_C f(\mathbf{x})\,d\mu &= \sup\{L(P,f) : P \text{ is a partition of } C\},\\ \mathcal{U}\int_C f(\mathbf{x})\,d\mu &= \inf\{U(P,f) : P \text{ is a partition of } C\}.\end{aligned}$$

Lemma 6.7

Let f be a bounded real-valued function on an n-cell C in $\mathbb{R}^n$. Then

$$\mathcal{L}\int_C f(\mathbf{x})\,dx \leq \mathcal{U}\int_C f(\mathbf{x})\,dx.$$

$$\mathcal{L}\int_C f(\mathbf{x})\,d\mu \leq \mathcal{U}\int_C f(\mathbf{x})\,d\mu.$$

Proof

The inequality $L(P, f) \leq L(Q, f)$ holds for all partitions P and Q of the closed n-cell C (Lemma 6.6). It follows that, for a fixed partition Q, the upper sum $U(Q, f)$ is an upper bound on all the lower sums $L(P, f)$, and therefore

$$\mathcal{L} \int_C f(\mathbf{x}) \, dx \leq U(Q, f).$$

The lower Riemann integral is then a lower bound on all the upper sums, and therefore

$$\mathcal{L} \int_C f(\mathbf{x}) \, d\mu \leq \mathcal{U} \int_C f(\mathbf{x}) \, d\mu.$$

as required. ∎

Definition

A bounded function $f\colon C \to \mathbb{R}$ on a closed n-cell C in $\mathbb{R}^n$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on C if

$$\mathcal{U}\int_C f(\mathbf{x})\,d\mu = \mathcal{L}\int_C f(\mathbf{x})\,d\mu,$$

in which case the *Riemann integral* $\int_C f(\mathbf{x})\,d\mu$ (or *Darboux integral*) of f on X is defined to be the common value of $\mathcal{U}\int_C f(\mathbf{x})\,d\mu$ and $\mathcal{L}\int_C f(\mathbf{x})\,d\mu$.

Lemma 6.8

Let $f\colon C \to \mathbb{R}$ be a bounded function on a closed n-cell C in $\mathbb{R}^n$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities

$$\begin{aligned} \mathcal{U}\int_C (-f(\mathbf{x}))\,d\mu &= -\mathcal{L}\int_C f(\mathbf{x})\,d\mu, \\ \mathcal{L}\int_C (-f(\mathbf{x}))\,d\mu &= -\mathcal{U}\int_C f(\mathbf{x})\,d\mu. \end{aligned}$$

Proof

Let P be a partition of C, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $C_{P,\alpha}$. Then the lower and upper sums of f for the partition P satisfy the equations

$$L(P,f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha}), \quad U(P,f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, \mu(C_{P,\alpha}),$$

where

$$\begin{aligned} m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}. \end{aligned}$$

Now

$$\begin{aligned}\sup\{-f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} &= -\inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} = -m_{P,\alpha},\\ \inf\{-f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} &= -\sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} = -M_{P,\alpha}\end{aligned}$$

It follows that

$$\begin{aligned}U(P,-f) &= \sum_{\alpha\in\Omega(P)} (-m_{P,\alpha})\mu(C_{P,\alpha}) = -L(P,f),\\ L(P,-f) &= \sum_{\alpha\in\Omega(P)} (-M_{P,\alpha})\mu(C_{P,\alpha}) = -U(P,f).\end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval C. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned} & \mathcal{U}\int_C (-f(\mathbf{x}))\,d\mu \\ = \; & \inf\{U(P,-f) : P \text{ is a partition of } C\} \\ = \; & \inf\{-L(P,f) : P \text{ is a partition of } C\} \\ = \; & -\sup\{L(P,f) : P \text{ is a partition of } C\} \\ = \; & -\mathcal{L}\int_C f(\mathbf{x})\,d\mu \end{aligned}$$

Similarly

$$\begin{aligned}
&\mathcal{L}\int_C (-f(\mathbf{x}))\,d\mu \\
= &\ \sup\{L(P,-f) : P \text{ is a partition of } C\} \\
= &\ \sup\{-U(P,f) : P \text{ is a partition of } C\} \\
= &\ -\inf\{U(P,f) : P \text{ is a partition of } C\} \\
= &\ -\mathcal{U}\int_C f(\mathbf{x})\,d\mu.
\end{aligned}$$

This completes the proof. ∎

Lemma 6.9

Let $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$ be bounded functions on a closed n-cell C in $\mathbb{R}^n$. Then the lower sums of the functions f, g and $f + g$ satisfy

$$L(P, f + g) \geq L(P, f) + L(P, g),$$

and the upper sums of these functions satisfy

$$U(P, f + g) \leq U(P, f) + U(P, g).$$

Proof

Let P be a partition of C, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $C_{P,\alpha}$. Then

$$\begin{aligned} L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}), \\ L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g)\mu(C_{P,\alpha}), \\ L(P,f+g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f+g)\mu(C_{P,\alpha}), \end{aligned}$$

$$
\begin{aligned}
U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}), \\
U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g)\mu(C_{P,\alpha}), \\
U(P, f + g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f + g)\mu(C_{P,\alpha}),
\end{aligned}
$$

where

$$
\begin{aligned}
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
m_{P,\alpha}(f+g) &= \inf\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \\
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\
M_{P,\alpha}(f+g) &= \sup\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}
\end{aligned}
$$

for $\alpha \in \Omega(P)$.

Now

$$m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f) \quad \text{and} \quad m_{P,\alpha}(g) \leq g(\mathbf{x}) \leq M_{P,\alpha}(g).$$

for all $\mathbf{x} \in C_{P,\alpha}$. Adding, we see that

$$m_{P,\alpha}(f) + m_{P,\alpha}(g) \leq f(\mathbf{x}) + g(\mathbf{x}) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g)$$

for all $\mathbf{x} \in C_{P,\alpha}$, and therefore $M_{P,\alpha}(f) + M_{P,\alpha}(g)$ is an upper bound for the set

$$\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}.$$

and $m_{P,\alpha}(f) + m_{P,\alpha}(g)$ is a lower bound for the same set. The least upper bound and greatest lower bound for this set are $M_{P,\alpha}(f + g)$ and $m_{P,\alpha}(f + g)$ respectively.

Therefore

$$\begin{aligned} m_{P,\alpha}(f) + m_{P,\alpha}(g) &\leq m_{P,\alpha}(f+g) \\ &\leq M_{P,\alpha}(f+g) \\ &\leq M_{P,\alpha}(f) + M_{P,\alpha}(g). \end{aligned}$$

It follows that

$$\begin{aligned} &U(P, f+g) \\ &\quad = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f+g)\mu(C_{P,\alpha}) \\ &\quad \leq \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(f) + M_{P,\alpha}(g))\mu(C_{P,\alpha}) \\ &\quad = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f)\mu(C_{P,\alpha}) + \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(g)\mu(C_{P,\alpha}) \\ &\quad = U(P, f) + U(P, g). \end{aligned}$$

Similarly

$$\begin{aligned}
&L(P, f+g) \\
&\quad = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f+g)\mu(C_{P,\alpha}) \\
&\quad \geq \sum_{\alpha\in\Omega(P)} (m_{P,\alpha}(f) + m_{P,\alpha}(g))\mu(C_{P,\alpha}) \\
&\quad = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)\mu(C_{P,\alpha}) + \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g)\mu(C_{P,\alpha}) \\
&\quad = L(P, f) + L(P, g).
\end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \qquad \blacksquare$$

Proposition 6.10

Let $f\colon C \to \mathbb{R}$ and $g\colon C \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-cell C. Then the functions $f + g$ and $f - g$ are Riemann-integrable on C, and moreover

$$\begin{aligned} &\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu \\ = &\int_C f(\mathbf{x})\, d\mu + \int_C g(\mathbf{x})\, d\mu, \end{aligned}$$

and

$$\begin{aligned} &\int_C (f(\mathbf{x}) - g(\mathbf{x}))\, d\mu \\ = &\int_C f(\mathbf{x})\, d\mu - \int_C g(\mathbf{x})\, d\mu. \end{aligned}$$

Proof

Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of C for which

$$L(P, f) > \int_C f(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_C g(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R,f) \geq L(P,f) \quad \text{and} \quad L(R,g) \geq L(P,g).$$

Applying Lemma 6.9, and the definition of the lower Riemann integral, we see that

$$\begin{aligned} & \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\, d\mu \\ \geq \; & L(R, f+g) \geq L(R,f) + L(R,g) \\ \geq \; & L(P,f) + L(Q,g) \\ > \; & \left(\int_C f(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon\right) \\ & \qquad + \left(\int_C g(\mathbf{x})\, d\mu - \tfrac{1}{2}\varepsilon\right) \\ > \; & \int_C f(\mathbf{x})\, d\mu + \int_C g(\mathbf{x})\, d\mu - \varepsilon \end{aligned}$$

We have now shown that

$$\begin{aligned} & \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ > \quad & \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu - \varepsilon \end{aligned}$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu, \quad \int_C f(\mathbf{x})\,d\mu$$

and

$$\int_C g(\mathbf{x})\,d\mu$$

have values that have no dependence whatsoever on the value of ε.

It follows that

$$\begin{aligned} &\mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ \geq\ & \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

We can deduce a corresponding inequality involving the upper integral of $f + g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 6.8). We find that

$$\begin{aligned} & \mathcal{L}\int_C (-f(\mathbf{x}) - g(\mathbf{x}))\,d\mu \\ \geq \;& \int_C (-f(\mathbf{x}))\,d\mu + \int_C (-g(\mathbf{x}))\,d\mu \\ = \;& -\int_C f(\mathbf{x})\,d\mu - \int_C g(\mathbf{x})\,d\mu \end{aligned}$$

and therefore

$$\begin{aligned} & \mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ = \;& -\mathcal{L}\int_C (-f(\mathbf{x}) - g(\mathbf{x}))\,d\mu \\ \leq \;& \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu. \end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}\int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu &\leq \mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &\leq \mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &\leq \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu.\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}\mathcal{L}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu &= \mathcal{U}\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &= \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu.\end{aligned}$$

Thus the function $f + g$ is Riemann-integrable on C, and

$$\begin{aligned}&\int_C (f(\mathbf{x}) + g(\mathbf{x}))\,d\mu \\ &= \int_C f(\mathbf{x})\,d\mu + \int_C g(\mathbf{x})\,d\mu.\end{aligned}$$

Then, replacing g by $-g$, we find that

$$\begin{aligned}\int_C (f(\mathbf{x}) - g(\mathbf{x}))\, d\mu \\ = \int_C f(\mathbf{x})\, d\mu - \int_C g(\mathbf{x})\, d\mu.\end{aligned}$$

as required. $\blacksquare$

Proposition 6.11

Let $f : C \to \mathbb{R}$ be a bounded function on a closed n-cell C in $\mathbb{R}^n$. Then the function f is Riemann-integrable on C if and only if, given any positive real number ε, there exists a partition P of C with the property that

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof

First suppose that $f\colon C \to \mathbb{R}$ is Riemann-integrable on C. Let some positive real number ε be given. Then

$$\int_C f(\mathbf{x})\,d\mu$$

is equal to the common value of the lower and upper integrals of the function f on C, and therefore there exist partitions Q and R of C for which

$$L(Q, f) > \int_C f(\mathbf{x})\,d\mu - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_C f(\mathbf{x})\,d\mu + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 6.5). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f: C \to \mathbb{R}$ is a bounded function on C with the property that, given any positive real number ε, there exists a partition P of C for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of C for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$\begin{aligned} L(P, f) &\leq \mathcal{L} \int_C f(\mathbf{x}) \, d\mu \\ &\leq \mathcal{U} \int_C f(\mathbf{x}) \, d\mu \leq U(P, f), \end{aligned}$$

and therefore

$$\begin{aligned} &\mathcal{U} \int_C f(\mathbf{x}) \, d\mu - \mathcal{L} \int_C f(\mathbf{x}) \, d\mu \\ &\qquad < \quad U(P, f) - L(P, f) < \varepsilon. \end{aligned}$$

Thus the difference between the values of the upper and lower integrals of f on C must be less than every strictly positive real number ε, and therefore

$$\mathcal{U} \int_C f(\mathbf{x})\, d\mu = \mathcal{L} \int_C f(\mathbf{x})\, d\mu.$$

This completes the proof. ∎