MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2017 Lecture 12 (October 23, 2017)

David R. Wilkins

6. The Multidimensional Riemann Integral

6.1. Partitions of Closed Cells

Definition

We define a *closed n-cell* in $\mathbb{R}^n$ to be a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, n\},$$

where $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are real numbers satisfying $u_i < v_i$ for $i = 1, 2, \ldots, n$.

Definition

Let C be a closed n-cell in $\mathbb{R}^n$. Then there are uniquely-determined real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ satisfying $u_i < v_i$ for $i = 1, 2, \ldots, n$ for which

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, n\}.$$

We define the *interior* of the n-cell C to be the open set $\mathrm{int}(C)$ defined such that

$$\mathrm{int}(C) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i < x_i < v_i \text{ for } i = 1, 2, \ldots, n\}.$$

Definition

Let C be a closed n-cell in $\mathbb{R}^n$. Then there are uniquely-determined real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ satisfying $u_i < v_i$ for $i = 1, 2, \ldots, n$ for which

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, n\}.$$

We define the *content* of the n-cell C to be the positive real number $\mu(C)$ defined by the formula

$$\mu(C) = \prod_{i=1}^{n} (v_i - u_i),$$

where $\prod\limits_{i=1}^{n} (v_i - u_i)$ denotes the product of the quantities $v_i - u_i$ for $i = 1, 2, \ldots, n$.

We now develop some notation and terminology for use in discussing partitions of closed n-cells in $\mathbb{R}^n$.

Given sets $X_1, X_2, \ldots, X_n$, the *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ of those sets is the set consisting of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ with the property that $x_i \in X_i$ for $i = 1, 2, \ldots, n$.

Thus for example let $[a, b]$ and $[c, d]$ be closed intervals, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. The Cartesian product of these two closed intervals is a closed rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$, where

$$[a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b_1 \text{ and } c \leq y \leq d\}.$$

This closed rectangle is a closed 2-cell in $\mathbb{R}^2$, and moreover any closed 2-cell in $\mathbb{R}^2$ is the Cartesian product of 2 closed intervals in $\mathbb{R}^2$.

More generally, any n-cell in $\mathbb{R}^n$ is the Cartesian product of n closed intervals of positive length. The content of the n-cell is then the product of the lengths of those closed intervals.

Indeed let C be a closed n-cell in $\mathbb{R}^n$. This closed cell is determined by real numbers u_i and v_i for $i = 1, 2, \ldots, n$, where $u_i < v_i$ for all i and

$$C = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq v_i \text{ for } i = 1, 2, \ldots, n\}.$$

The n-cell C is thus the Cartesian product

$$[u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n]$$

of the closed intervals $[u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n]$.

6. The Multidimensional Riemann Integral (continued)

Let P_i be a partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ induce a partition P of the closed n-cell C, where

$$C = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n],$$

partitions this n-cell as a collection of closed subcells that meet one another only along parts of their boundaries. Specifically let

$$P_i = \{w_{i,0}, w_{i,1}, \ldots, w_{i,k_i}\}$$

for $i = 1, 2, \ldots, n$, where

$$u_i = w_{i,0} < w_{i,1} < \cdots < w_{i,k_i} = v_i.$$

The partition P_i then decomposes the closed interval $[u_i, v_i]$ as a collection of subintervals $[w_{i,j_i-1}, w_{i,j_i}]$ where the index j_i ranges over the integers from 1 to k_i.

Let

$$\Omega(P) = \{(j_1, j_2, \ldots, j_n) \in \mathbb{Z}^n : 1 \leq j_i \leq k_i \text{ for } i = 1, 2, \ldots, n\}.$$

Given $\alpha \in \Omega(P)$, there exist integers $j_1, j_2, \ldots, j_n$ for which $1 \leq j_i \leq k_i$ for $i = 1, 2, \ldots, n$ and $\alpha = (j_1, j_2, \ldots, j_n)$. Let $C_{P,\alpha}$, or $C_{(j_1, j_2, \ldots, j_n)}$, denote the closed n-cell in $\mathbb{R}^n$ defined so that

$$\begin{aligned} C_{P,\alpha} &= C_{(j_1, j_2, \ldots, j_n)} \\ &= [w_{1,j_1-1}, w_{1,j_1}] \times [w_{2,j_2-1}, w_{2,j_2}] \times \cdots \times [w_{n,j_n-1}, w_{n,j_n}] \\ &= \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \\ &\qquad\qquad w_{i,j_i-1} \leq x_i \leq w_{i,j_i} \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Then the closed n-cell C is the union of the closed subcells $C_{P,\alpha}$ as α ranges over the set $\Omega(P)$. Moreover, if two of these subcells intersect one another, then they intersect only along parts of their boundaries, and thus the interiors of these subcells are disjoint.

Proposition 6.1

Let C be a closed n-cell in $\mathbb{R}^n$, let

$$[u_1, v_1] \times [u_2, v_2], \ldots, [u_n, v_n]$$

be the closed intervals of positive length whose Cartesian product is the n-cell C, and let P_i be a partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ induce a partition P of the closed n-cell C as the union of closed subcells $C_{P,\alpha}$, where the index α ranges over a finite set $\Omega(P)$. Each element α of this indexing set $\Omega(P)$ is an n-tuple of integers $(j_1, j_2, \ldots, j_n)$, where j_i numbers the corresponding subinterval in the partition P_i of the interval $[u_i, v_i]$, and the corresponding subcell $C_{P,\alpha}$ of C is the Cartesian product of those subintervals. Moreover the subcells $C_{P,\alpha}$ for $\alpha \in \Omega(P)$ meet, if at all, only along parts of their boundaries, and thus the interiors of these subcells are disjoint.

Let C be a closed n-cell in $\mathbb{R}^n$. This n-cell is a product of n closed intervals

$$[u_1, v_1], [u_2, v_2], \ldots, [u_n, v_n].$$

Let P_i be a partition of the interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Then the partitions $P_1, P_2, \ldots, P_n$ determine a partition P of the closed n-cell with indexing set $\Omega(P)$ in the manner described in Proposition 6.1. The elements of this indexing set $\Omega(P)$ are n-tuples of integers. These n-tuples label the closed subcells of C determined by the partition P. We refer to these elements of $\Omega(P)$ as *multi-indices*.

Let α be a multi-index in the indexing set $\Omega(P)$ for the partition P of the closed n-cell induced by partitions of the closed intervals $[u_i, v_i]$ whose Cartesian product is the n-cell C. Let k_i denote the number of subintervals in the partition of the ith interval $[u_i, v_i]$ occurring as a factor in the Cartesian product. Then $\alpha = (j_1, j_2, \ldots, j_n)$, where j_i is an integer between 1 and k_i for $i = 1, 2, \ldots, n$. The closed subcell $C_{P,\alpha}$ that corresponds to the multi-index α is then determined as follows:

$$C_{P,\alpha} = [w_{1,j_1-1}, w_{1,j_1}] \times [w_{2,j_2-1}, w_{2,j_2}] \times \cdots \times [w_{n,j_n-1}, w_{n,j_n}],$$

where $[w_{i,j_i-1}, w_{i,j_i}]$ is the j_ith subinterval occuring in the partition of the closed interval $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. The content $\mu(C_{P,\alpha})$ of the closed n-cell $C_{P,\alpha}$ is then given by the formula

$$\mu(C_{P,\alpha}) = \prod_{i=1}^{n} (w_{i,j_i} - w_{i,j_i-1}).$$

Proposition 6.2

Let C be a closed n-cell in $\mathbb{R}^n$ with content $\mu(C)$, and let P be a partition of C induced by partitions of the closed intervals whose Cartesian product is the closed n-cell C. Let $\Omega(P)$ be the indexing set for the partition P, and for all multi-indices $\alpha \in \Omega(P)$, let $C_{P,\alpha}$ be the corresponding closed subcell in the partition of the closed n-cell C, and let $\mu(C_{P,\alpha})$ denote the content of $C_{P,\alpha}$. Then

$$\mu(C) = \sum_{\alpha \in \Omega(P)} \mu(C_{P,\alpha}).$$

Proof

Let

$$C = [u_1, v_1] \times [u_2, v_2], \ldots, [u_n, v_n],$$

where, for each i between 1 and n, u_i and v_i are real numbers satisfying $u_i < v_i$. Then

$$\mu(C) = \prod_{i=1}^{n} (v_i - u_i).$$

Let the partition P of C be induced by partitions P_i of $[u_i, v_i]$ for $i = 1, 2, \ldots, n$. Moreover let

$$P_i = \{w_{i,0}, w_{i,1}, \ldots, w_{i,k_i}\},$$

where $w_{i,0}, w_{i,1}, w_{i,2}, \ldots, w_{i,k_i}$ are real numbers for $j = 1, 2, \ldots, k_i$ and

$$u_i = w_{i,0} < w_{i,1} < \cdots < w_{i,k_i} = v_i.$$

The content $\mu(C_{(j_1,j_2,\ldots,j_n)})$ of the closed subcell $C_{(j_1,j_2,\ldots,j_n)}$ in the partition of C corresponding to the multi-index $(j_1, j_2, \ldots, j_n)$ is then given by the formula

$$\mu(C_{(j_1,j_2,\ldots,j_n)}) = \prod_{i=1}^{n}(w_{i,j_i} - w_{i,j_i-1}).$$

It follows that

$$\begin{aligned}
&\sum_{j_n=1}^{k_n} \mu(C_{(j_1,j_2,\ldots,j_n)}) \\
&\qquad = \left(\prod_{i=1}^{n-1}(w_{i,j_i} - w_{i,j_i-1})\right) \times \left(\sum_{j_n=1}^{k_n}(w_{i,n_i} - w_{i,j_n-1})\right) \\
&\qquad = \left(\prod_{i=1}^{n-1}(w_{i,j_i} - w_{i,j_i-1})\right) \times (v_n - u_n).
\end{aligned}$$

The proposition therefore follows from a straightforward application of the Principle of Mathematical Induction, using induction on the dimension n of the n-cell, and making use of the above identity in establishing the inductive step. ∎

Definition

Let C be an n-cell in $\mathbb{R}^n$ and let P and R be partitions of C, where P is induced by partitions $P_1, P_2, \ldots, P_n$ of the closed intervals whose Cartesian product is the n-cell C and the partition R is induced by partitions $R_1, R_2, \ldots, R_n$ of those same closed intervals. We say that the partition R is a *refinement* of the partition P if $P_i \subset R_i$ for $i = 1, 2, \ldots, n$.

The following result follows directly from the definition of refinements of partitions of closed n-cells in $\mathbb{R}^n$.

Lemma 6.3

Let C be an n-cell in $\mathbb{R}^n$ and let P and R be partitions of C. Then, for each multi-index β belonging to the indexing set $\Omega(R)$ for the partition R of C, there exists a unique multi-index α belonging to the indexing set $\Omega(P)$ for the partition P of C for which the subcells $C_{R,\beta}$ and $C_{P,\alpha}$ of C for the partitions P and R determined by the multi-indices β and α respectively satisfy the inclusion $C_{R,\beta} \subset C_{P,\alpha}$.

Lemma 6.4

Let C be a closed n-cell in $\mathbb{R}^n$, and let P and Q be partitions of C. Then there exists a partition R of C that is a common refinement of the partitions P and Q.

Proof

Let

$$C = [u_1, v_1] \times [u_2, v_2], \dots, [u_n, v_n],$$

where, for each i between 1 and n, u_i and v_i are real numbers satisfying $u_i < v_i$. Then there are partitions P_i and Q_i of the closed interval $[u_i, v_i]$ for $i = 1, 2, \dots, n$ so that the partitions $P_1, P_2, \dots, P_n$ of the respective closed intervals induce the partition P of C and the partitions $Q_1, Q_2, \dots, Q_n$ of those same closed intervals induce the partition Q of C. Let $R_i = P_i \cup Q_i$ for $i = 1, 2, \dots, n$. Then R_i is a partition of the interval $[u_i, v_i]$ for $i = 1, 2, \dots, n$ that is a common refinement of the partitions P_i and Q_i of the interval $[u_i, v_i]$. Let R be the partition of the closed n-cell C induced by the partitions $R_1, R_2, \dots, R_n$ of the respective closed intervals. Then the partition R of C is the required common refinement of the partitions P and Q of C. ∎

6.2. Multidimensional Darboux Sums

Let $f\colon C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$. A partition P of the n-cell C represents C as the union of a collection of closed n-cells $C_{P,\alpha}$ contained in C indexed by a finite set $\Omega(P)$. Distinct n-cells in this collection intersect, if at all, only along parts of their boundaries, and therefore the interiors of the subcells of C determined by the partition P are disjoint. Thus each point of C belongs to the interior of at most one cell in the collection of closed subcells into which the n-cell C is partitioned. Also the content $\mu(C)$ of the n-cell C is the sum of the contents of the subcells determined by the partition, and thus

$$\mu(C) = \sum_{\alpha \in \Omega(P)} \mu(C_{P,\alpha})$$

(see Proposition 6.2).

Definition

Let $f\colon C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, let P be a partition of C, and let $\Omega(P)$ denote the indexing set for the partition P, and, for each $\alpha \in \Omega(P)$, let

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\} \quad \text{and} \quad M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\},$$

where $\mu(C_{P,\alpha})$ denotes the content of the closed subcell $C_{P,\alpha}$ of C indexed by α. Then the Darboux lower sum $L(P, f)$ and the Darboux upper sum $U(P, f)$ are defined by the formulae

$$L(P, f) = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\, \mu(C_{P,\alpha}).$$

Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$. Then the definition of the Darboux lower and upper sums ensures that $L(P, f) \leq U(P, f)$ for all partitions P of the n-cell C.

Let C be a closed n-cell in $\mathbb{R}^n$, and let P and R be partitions of C, where P is determined by partitions $P_1, P_2, \ldots, P_n$ of the closed intervals whose Cartesian product is the closed n-cell C and R is determined by partitions $R_1, R_2, \ldots, R_n$ of those same closed intervals. We recall that the partition R is a *refinement* of P if and only if $P_i \subset R_i$ for $i = 1, 2, \ldots, n$.

Lemma 6.5

Let $f : C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, and let P and R be partitions of C. Suppose that R is a refinement of P. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f).$$

Proof

Let the cells of the partitions P and R be indexed by indexing sets $\Omega(P)$ and $\omega(R)$ respectively. Also, for each $\alpha \in \Omega(P)$, let $C_{P,\alpha}$ be the cell of the partition P determined by α, and, for each $\beta \in \Omega(R)$, let $C_{R,\beta}$ be the cell of the partition R determined by β. Then, given a subcell $C_{R,\beta}$ of C, indexed by some element β of the indexing set $\Omega(R)$ for the partition R, there exists a uniquely-determined subcell $C_{P,\alpha}$ of C, indexed by some element α of the indexing set $\Omega(P)$ for the partition P, for which $C_{R,\beta} \subset C_{P,\alpha}$. (see Lemma 6.3). It follows that there is a unique well-defined function $\lambda\colon \Omega(R) \to \Omega(P)$ characterized by the requirement that, for each multi-index β belonging to the indexing set $\Omega(R)$ for the partition R, the element $\lambda(\beta)$ of the indexing set $\Omega(P)$ for the partition P is the unique multi-index in $\Omega(P)$ for which $C_{R,\beta} \subset C_{P,\lambda(\beta)}$.

Now

$$\begin{aligned}
U(P,f) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\,\mu(C_{P,\alpha}),\\
L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\,\mu(C_{P,\alpha}),\\
U(R,f) &= \sum_{\beta\in\Omega(R)} M_{R,\beta}\,\mu(C_{R,\beta}),\\
L(R,f) &= \sum_{\beta\in\Omega(R)} m_{R,\beta}\,\mu(C_{R,\beta}),
\end{aligned}$$

where

$$\begin{aligned} M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{P,\alpha}\}, \\ M_{R,\beta} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in C_{R,\beta}\}, \\ m_{R,\beta} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in C_{R,\beta}\} \end{aligned}$$

for all $\alpha \in \Omega(P)$ and $\beta \in \Omega(R)$. Also

$$M_{R,\beta} \leq M_{P,\lambda(\beta)} \quad \text{and} \quad m_{R,\beta} \geq m_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, because $C_{R,\beta} \subset C_{P,\lambda(\beta)}$.

Now the partition R of C determines a partition of each cell $C_{P,\alpha}$ of the partition P, decomposing the cell $C_{P,\alpha}$ as a union of the sets $C_{R,\beta}$ for which $\lambda(\beta) = \alpha$. It follows from Proposition 6.2 that

$$C_{P,\alpha} = \sum_{\beta \in \Omega(R;\alpha)} \mu(C_{R,\beta})$$

where

$$\Omega(R; \alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$.

Therefore

$$\begin{aligned} L(R, f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, \mu(C_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} m_{R,\beta}\, \mu(C_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} \mu(C_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, \mu(C_{P,\alpha}) \\ &= L(P, f). \end{aligned}$$

Similarly

$$\begin{aligned}
U(R,f) &= \sum_{\beta\in\Omega(R)} M_{R,\beta}\,\mu(C_{R,\beta}) \\
&= \sum_{\alpha\in\Omega(P)} \sum_{\beta\in\Omega(R;\alpha)} M_{R,\beta}\,\mu(C_{R,\beta}) \\
&\leq \sum_{\alpha\in\Omega(P)} M_{P,\alpha} \sum_{\beta\in\Omega(R;\alpha)} \mu(C_{R,\beta}) \\
&\geq \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\,\mu(C_{P,\alpha}) \\
&= U(P,f).
\end{aligned}$$

This completes the proof. ■

Lemma 6.6

Let $f: C \to \mathbb{R}$ be a bounded real-valued function defined on an n-cell C in $\mathbb{R}^n$, and let P and Q be partitions of C. Then then the Darboux sums of the function f for the partitions P and Q satisfy $L(P, f) \leq U(Q, f)$.

Proof

There exists a partition R of C that is a common refinement of the partitions P and Q of C. (Lemma 6.4.) Moreover $L(R, f) \geq L(P, f)$ and $U(R, f) \leq U(Q, f)$ (Lemma 6.5). It follows that

$$L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f),$$

as required. ∎