MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2017 Lecture 11 (October 19, 2017)

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Proposition 5.6

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof

First suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a,b]$, and therefore there exist partitions Q and R of $[a,b]$ for which

$$L(Q,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q,f) \leq L(P,f) \leq U(P,f) \leq U(R,f).$$

(see Lemma 5.1). It follows that

$$U(P,f) - L(P,f) \leq U(R,f) - L(Q,f) < \varepsilon.$$

Now suppose that $f\colon [a,b] \to \mathbb{R}$ is a bounded function on $[a,b]$ with the property that, given any positive real number ε, there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P,f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P,f) - L(P,f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a, b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Proposition 5.7

Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof

Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\, dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{u_0, u_1, \ldots, u_m\}$, where $u_0, u_1, \ldots, u_m$ are real numbers satisfying

$$a = u_0 < u_1 < u_2 < \cdots u_{m-1} < u_m = b,$$

and $R = \{v_0, v_1, \ldots, v_n\}$, where $v_0, v_1, \ldots, v_n$ are real numbers satisfying

$$b = v_0 < v_1 < v_2 < \cdots v_{n-1} < v_n = c.$$

Then

$$P = \{a, u_1, u_2, \ldots, u_{m-1}, b, v_1, v_2, \ldots, v_{n-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\displaystyle\int_a^c f(x)\,dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx \geq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma 5.3). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

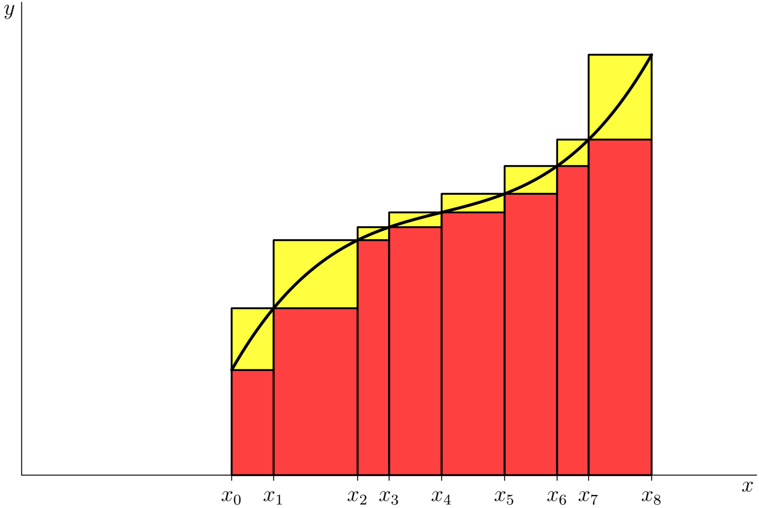
But
$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$
It follows that
$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$
The result follows. ∎

5.3. Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a,b] \to \mathbb{R}$ defined on the closed bounded interval $[a,b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a,b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a,b] \to \mathbb{R}$ is said to be *monotonic* on $[a,b]$ if either it is non-decreasing on $[a,b]$ or else it is non-increasing on $[a,b]$.

Proposition 5.8

Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a,b]$ is Riemann-integrable on $[a,b]$.



y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8

Proof

Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$.

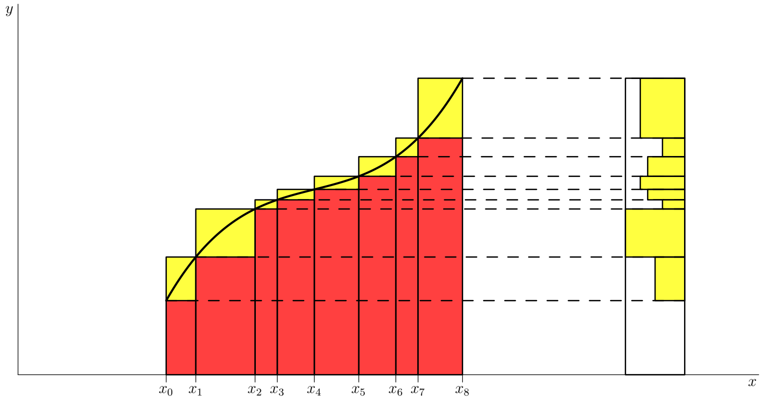
The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that


y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8

$$\begin{aligned} & U(P,f) - L(P,f) \\ = & \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ < & \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a,b]$.

Now let $f\colon [a,b] \to \mathbb{R}$ be a non-increasing function on $[a,b]$. Then $-f$ is a non-decreasing function on $[a,b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a,b]$. It follows that the function f itself must be Riemann-integrable on $[a,b]$, as required. ∎

Corollary 5.9

Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function on the interval $[a,b]$, where a and b are real numbers satisfying $a < b$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a,b]$.

Proof

The result follows immediately on applying the results of Proposition 5.7 and Proposition 5.8. ∎

Remark

The result and proof-strategy of Proposition 5.8 are to be found in their essentials in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

5.4. Integrability of Continuous functions

Theorem 5.10

Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.

Proof

Let f be a continuous real-valued function on $[a, b]$. Then f is bounded above and below on the interval $[a, b]$, and moreover $f: [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$. (These results follow from Theorem 4.21 and Theorem 4.22.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) & \\ \leq \quad & L(P, f) \leq U(P, f) \\ \leq \quad & \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P,f) - L(P,f) \leq 2\varepsilon(b-a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a,b]$. $\blacksquare$

5.5. The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 5.10). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 5.11 (The Fundamental Theorem of Calculus)

Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof

Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$.

Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a,b)$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a,b]$. Also

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{f(x)}{h}\int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x + h$. It follows that

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt$$

whenever $x + h \in [a,b]$.

But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x + h$, and therefore

$$-\varepsilon_0 |h| \leq \int_x^{x+h} (f(t) - f(x))\, dt \leq \varepsilon_0 |h|.$$

It follows that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \varepsilon_0 < \varepsilon$$

whenever $x + h \in [a, b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx} \left(\int_a^x f(t)\, dt \right) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎