MA2321—Analysis in Several Variables
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5. The Riemann Integral in One Dimension

5.1. Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

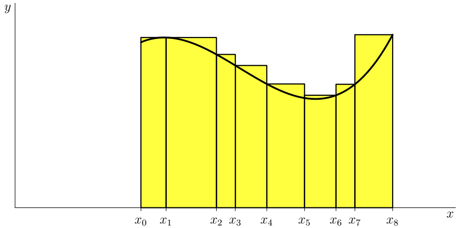
Definition

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

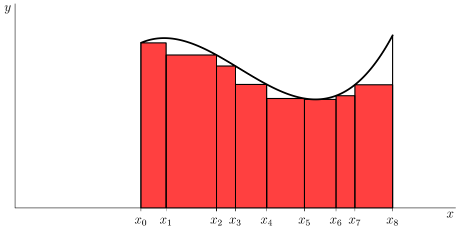
where $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$.



Similarly the *lower sum* (or *lower Darboux sum*) $L(P, f)$ of f for the partition P of $[a, b]$ is defined so that

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

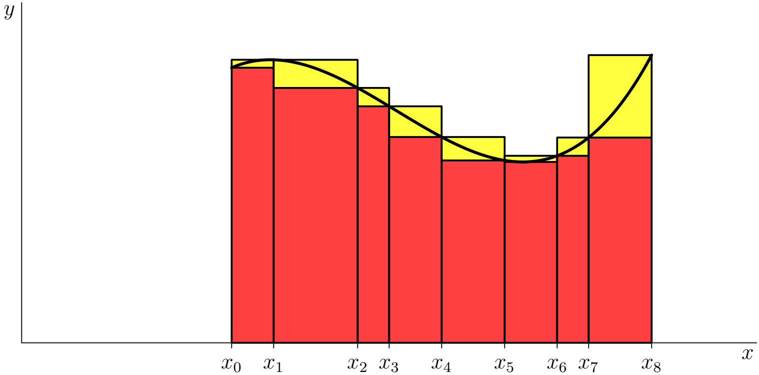
where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$.



Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.



y
x
x_0
x_1
x_2
x_3
x_4
x_5
x_6
x_7
x_8

Definition

Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U} \int_a^b f(x)\, dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L} \int_a^b f(x)\, dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

$$\begin{aligned} \mathcal{U} \int_a^b f(x)\, dx &= \inf \{ U(P, f) : P \text{ is a partition of } [a, b] \}, \\ \mathcal{L} \int_a^b f(x)\, dx &= \sup \{ L(P, f) : P \text{ is a partition of } [a, b] \}. \end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U} \int_a^b f(x)\, dx$ be the infimum of the values of $U(P, f)$ and that $\mathcal{L} \int_a^b f(x)\, dx$ be the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$.

Definition

A bounded function $f\colon [a,b] \to \mathbb{R}$ on a closed bounded interval $[a,b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a,b]$ if

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a,b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x)\,dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

Definition

Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 5.1

Let R be a refinement of some partition P of $[a, b]$. Then

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f\colon [a, b] \to \mathbb{R}$.

Proof

Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 5.2

Let f be a bounded real-valued function on the interval $[a, b]$. Then

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof

Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 5.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L} \int_a^b f(x)\, dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L} \int_a^b f(x)\, dx \leq \mathcal{U} \int_a^b f(x)\, dx$, as required. ∎

Example

Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$\begin{aligned}
L(P_n, f) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ci}{n} + d - \frac{c}{n}\right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ci}{n} + d\right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}$$

It follows that

$$\lim_{n \to +\infty} L(P_n, f) = \frac{c}{2} + d$$

and

$$\lim_{n \to +\infty} U(P_n, f) = \frac{c}{2} + d$$

Now $L(P_n, f) \leq \mathcal{L} \int_a^b f(x)\, dx \leq \mathcal{U} \int_a^b f(x)\, dx \leq U(P_n, f)$ for all positive integers n. It follows that $\mathcal{L} \int_a^b f(x)\, dx = \frac{1}{2}c + d = \mathcal{U} \int_a^b f(x)\, dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\, dx = \frac{1}{2}c + d$.

Example

Let $f\colon [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0,1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0,1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0,1]$.

5.2. Basic Properties of the Riemann Integral

Lemma 5.3

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= -\mathcal{L}\int_a^b f(x)\,dx, \\ \mathcal{L}\int_a^b (-f(x))\,dx &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

Proof

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned} m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \end{aligned}$$

Then the lower and upper sums of f for the partition P are given by the formulae

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Now

$$\begin{aligned} \sup\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &= -\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = -m_i, \\ \inf\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ &= -\sup\{f(x) : x_{i-1} \leq x \leq x_i\} = -M_i \end{aligned}$$

It follows that

$$\begin{aligned} U(P, -f) &= \sum_{i=1}^{n} (-m_i)(x_i - x_{i-1}) = -L(P, f), \\ L(P, -f) &= \sum_{i=1}^{n} (-M_i)(x_i - x_{i-1}) = -U(P, f). \end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval $[a,b]$. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= \inf\{U(P,-f) : P \text{ is a partition of } [a,b]\} \\ &= \inf\{-L(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\sup\{L(P,f) : P \text{ is a partition of } [a,b]\} \\ &= -\mathcal{L}\int_a^b f(x)\,dx \end{aligned}$$

Similarly

$$
\begin{aligned}
\mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P,-f) : P \text{ is a partition of } [a,b]\} \\
&= \sup\{-U(P,f) : P \text{ is a partition of } [a,b]\} \\
&= -\inf\{U(P,f) : P \text{ is a partition of } [a,b]\} \\
&= -\mathcal{U}\int_a^b f(x)\,dx.
\end{aligned}
$$

This completes the proof. ∎

Lemma 5.4

Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, and let P be a partition of the interval $[a,b]$. Then the lower sums of the functions f, g and $f+g$ satisfy

$$L(P, f+g) \geq L(P, f) + L(P, g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P, f) + U(P, g).$$

Proof

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned} L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\ L(P, g) &= \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}), \\ L(P, f + g) &= \sum_{i=1}^{n} m_i(f + g)(x_i - x_{i-1}), \end{aligned}$$

where

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(g) &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(f+g) &= \inf\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\} \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \geq m_i(f) \quad \text{and} \quad g(x) \geq m_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \geq m_i(f) + m_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $m_i(f) + m_i(g)$ is a lower bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The greatest lower bound for this set is $m_i(f + g)$. Therefore

$$m_i(f + g) \geq m_i(f) + m_i(g).$$

It follows that

$$\begin{aligned} L(P, f+g) &= \sum_{i=1}^{n} m_i(f+g)(x_i - x_{i-1}) \\ &\geq \sum_{i=1}^{n} (m_i(f) + m_i(g))(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) \\ &= L(P, f) + L(P, g). \end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\ U(P,g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\ U(P,f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}), \end{aligned}$$

where

$$\begin{aligned} M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i(g) &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i(f+g) &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\} \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \leq M_i(f) \quad \text{and} \quad g(x) \leq M_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \leq M_i(f) + M_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $M_i(f) + M_i(g)$ is an upper bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The least upper bound for this set is $M_i(f+g)$. Therefore

$$M_i(f+g) \leq M_i(f) + M_i(g).$$

It follows that

$$\begin{aligned} U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i(f) + M_i(g))(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}) \\ &= U(P, f) + U(P, g). \end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 5.5

Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof

Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of $[a, b]$ for which

$$L(P, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R,f) \geq L(P,f) \quad \text{and} \quad L(R,g) \geq L(P,g).$$

Applying Lemma 5.4, and the definition of the lower Riemann integral, we see that

$$\begin{aligned} \mathcal{L}\int_a^b (f(x)+g(x))\,dx & \\ &\geq L(R,f+g) \geq L(R,f)+L(R,g) \\ &\geq L(P,f)+L(Q,g) \\ &> \left(\int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon\right) + \left(\int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon\right) \\ &> \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon \end{aligned}$$

We have now shown that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx, \quad \int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

have values that have no dependence whatsoever on the value of ε. It follows that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We can deduce a corresponding inequality involving the upper integral of $f + g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 5.3). We find that

$$\begin{aligned} \mathcal{L}\int_a^b (-f(x) - g(x))\,dx &\geq \int_a^b (-f(x))\,dx + \int_a^b (-g(x))\,dx \\ &= -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{U}\int_a^b (f(x) + g(x))\,dx &= -\mathcal{L}\int_a^b (-f(x) - g(x))\,dx \\ &\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}
&\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
&\qquad \leq \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \leq \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned} \mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\ &= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx. \end{aligned}$$

Thus the function $f+g$ is Riemann-integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Then, replacing g by $-g$, we find that

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

as required. ∎