MA2321—Analysis in Several Variables
School of Mathematics, Trinity College
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4.3. Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

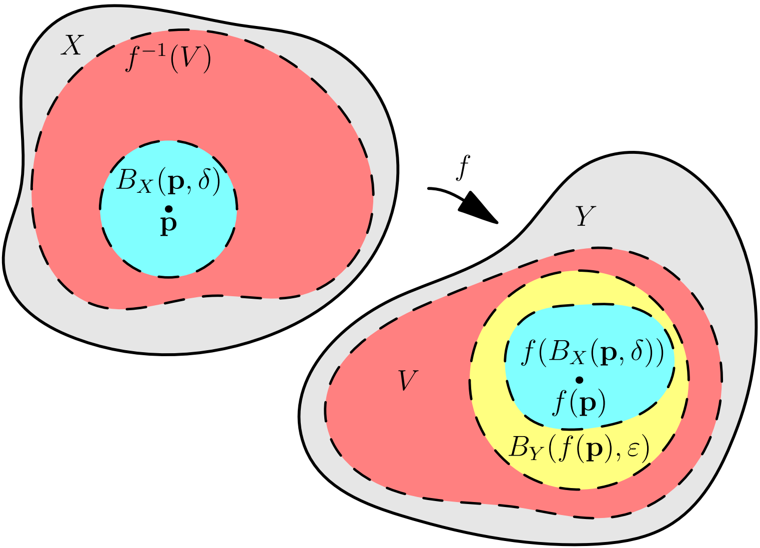
Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 4.18

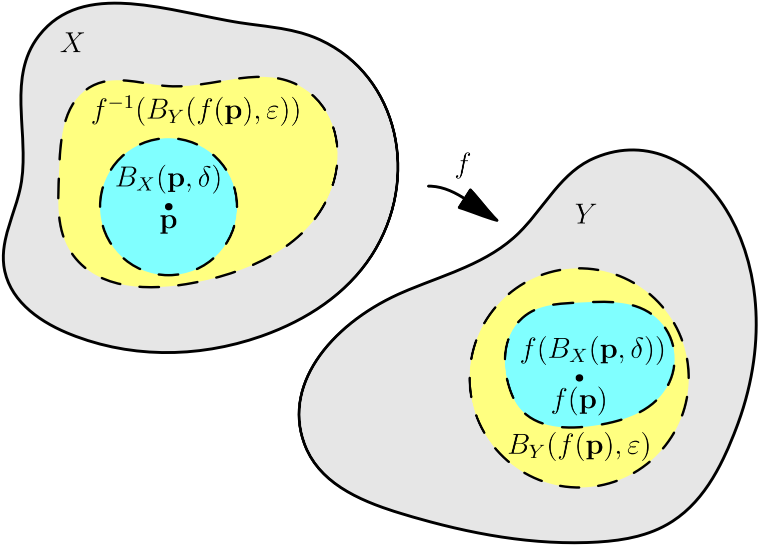
Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.

Proof

Suppose that $f: X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.


X
$f^{-1}(V)$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
V
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 3.1, hence $f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. $\blacksquare$


X
$f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let X be a subset of $\mathbb{R}^n$, let $f \colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

4.4. Limits and Neighbourhoods

Definition

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. A subset N of X is said to be a *neighbourhood* of $\mathbf{p}$ in X if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N.$$

Lemma 4.19

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X that is not an isolated point of X. Let $f \colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in X such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$.

Proof

This result follows directly from the definitions of limits and neighbourhoods. ∎

Remark

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a limit point of X that does not belong to X. Let $f: X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in $X \cup \{\mathbf{p}\}$ such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$. Thus the result of Lemma 4.19 can be extended so as to apply to limits of functions taken at limit points of the domain that do not belong to the domain of the function.

4.5. The Multidimensional Extreme Value Theorem

Proposition 4.20

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$. Then there exists a point $\mathbf{w}$ of X such that $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$.

Proof

Let $g\colon X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the function mapping each $\mathbf{x} \in X$ to $|f(\mathbf{x})|$ is continuous (see Lemma 4.6) and quotients of continuous functions are continuous where they are defined (see Lemma 4.5). It follows that the function $g\colon X \to \mathbb{R}$ is continuous.

Let

$$m = \inf\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$g(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{w}$ of $\mathbb{R}^n$.

Now the point $\mathbf{w}$ belongs to X because X is closed (see Lemma 3.7). Also

$$m \leq g(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $g(\mathbf{x}_{k_j}) \to m$ as $j \to +\infty$. It then follows from Lemma 4.2 that

$$g(\mathbf{w}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = m.$$

Then $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$, and therefore $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$, as required. ∎

Theorem 4.21 (The Multidimensional Extreme Value Theorem)

Let X be a closed bounded set in $\mathbb{R}^m$, and let $f \colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

Proof

It follows from Proposition 4.20 that the function f is bounded on X. It follows that there exists a real number C large enough to ensure that $f(\mathbf{x}) + C > 0$ for all $\mathbf{x} \in X$. It then follows from Proposition 4.20 that there exists some point $\mathbf{v}$ of X such that

$$f(\mathbf{x}) + C \leq f(\mathbf{v}) + C.$$

for all $\mathbf{x} \in X$. But then $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$. Applying this result with f replaced by $-f$, we deduce that there exists some $\mathbf{u} \in X$ such that $-f(\mathbf{x}) \leq -f(\mathbf{u})$ for all $\mathbf{x} \in X$. The result follows. ∎

4.6. Uniform Continuity for Functions of Several Real Variables

Definition

Let X be a subset of $\mathbb{R}^m$. A function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 4.22

Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f\colon X \to \mathbb{R}^n$ is uniformly continuous.

Proof

Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 2.6). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since

$$\lim_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0.$$

But then the sequences

$$f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$$

and

$$f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$$

would both converge to $f(\mathbf{p})$, since f is continuous (Lemma 4.2), and thus

$$\lim_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that

$$|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$$

for all j. We conclude therefore that there must exist some positive real number δ such that such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

4.7. Norms on Vector Spaces

Definition

A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

The Euclidean norm $|.|$ is a norm on $\mathbb{R}^n$ defined so that

$$|(x_1, x_2, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

for all $(x_1, x_2, \ldots, x_n)$. There are other useful norms on $\mathbb{R}^n$. These include the norms $\|.\|_1$ and $\|.\|_{\mathrm{sup}}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\mathrm{sup}} = \mathrm{maximum}(|x_1|, |x_2|, \ldots, |x_n|)$$

for all $(x_1, x_2, \ldots, x_n)$.

Definition

Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 4.23

If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.

Proof

let $\|.\|_*$ and $\|.\|_{**}$ be norms on a real vector space X that are both equivalent to a norm $\|.\|$ on X. Then there exist constants c_*, c_{**}, C_* and C_{**}, where $0 < c_* \leq C_*$ and $0 < c_{**} \leq C_{**}$, such that

$$c_*\|x\| \leq \|x\|_* \leq C_*\|x\|$$

and

$$c_{**}\|x\| \leq \|x\|_{**} \leq C_{**}\|x\|$$

for all $x \in X$. But then

$$\frac{c_{**}}{C_*}\|x\|_* \leq \|x\|_{**} \leq \frac{C_{**}}{c_*}\|x\|_*.$$

for all $x \in X$, and thus the norms $\|.\|_*$ and $\|.\|_{**}$ are equivalent to one another. The result follows. ∎

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 4.24

Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive real number C with the property that $\|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots,$$

$$\mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ be a point of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

Using Schwarz's Inequality, we see that

$$
\begin{aligned}
\|\mathbf{x}\| &= \left\| \sum_{j=1}^{n} x_j \mathbf{e}_j \right\| \leq \sum_{j=1}^{n} |x_j| \, \|\mathbf{e}_j\| \\
&\leq \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x}|,
\end{aligned}
$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and

$$|\mathbf{x}| = \left(\sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The result follows. ∎

Lemma 4.25

Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive constant C such that

$$\bigl|\|\mathbf{x}\| - \|\mathbf{y}\|\bigr| \leq \|\mathbf{x} - \mathbf{y}\| \leq C|\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

It follows that

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

and

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|,$$

and therefore

$$\Big| \|\mathbf{y}\| - \|\mathbf{x}\| \Big| \leq \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The result therefore follows from Lemma 4.24. ∎

Theorem 4.26

Any two norms on $\mathbb{R}^n$ are equivalent.

Proof

Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now it follows from Lemma 4.25 that the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous. Also S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. It therefore follows from the Extreme Value Theorem (Theorem 4.21) that there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. $\blacksquare$