MA2321—Analysis in Several Variables
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Definition

Let $f\colon X \to \mathbb{R}^n$ be a function mapping some subset X of m-dimensional Euclidean space $\mathbb{R}^m$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. We say that $f(\mathbf{x})$ *remains bounded* as $\mathbf{x}$ tends to $\mathbf{p}$ in X if strictly positive constants C and δ can be determined so that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Proposition 4.9

Let $f\colon X \to \mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(h(\mathbf{x}) f(\mathbf{x}) \Big) = \mathbf{0}.$$

Proof

Let some strictly positive real number ε be given. Now $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined so that $|h(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|h(\mathbf{x})| \leq C$ and $|f(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.10

Let $f: X \to \mathbb{R}^n$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x}) = 0$. Suppose also that $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} (h(\mathbf{x}) f(\mathbf{x})) = \mathbf{0}.$$

Proof

Let some strictly positive real number ε be given. Now $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|h(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| \leq C$ and $|h(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.11

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) \cdot g(\mathbf{x}) \Big) = 0.$$

Proof

Let some strictly positive real number ε be given. Now $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|g(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| < \varepsilon_0$ and $|g(\mathbf{x})| \leq C$. It then follows from Schwarz's Inequality (Proposition 2.1) that

$$|f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{x})|\,|g(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.12

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of $\mathbb{R}^n$ and let s be a real number. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = s.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})f(\mathbf{x}) = s\mathbf{q}.$$

Proof

The functions f and h satisfy the equation

$$h(\mathbf{x})f(\mathbf{x}) = h(\mathbf{x})\Big(f(\mathbf{x}) - \mathbf{q}\Big) + (h(\mathbf{x}) - s)\mathbf{q} + s\mathbf{q},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}}\Big(f(\mathbf{x}) - \mathbf{q}\Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}}\Big(h(\mathbf{x}) - s\Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that $|h(\mathbf{x}) - s| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|h(\mathbf{x})| < |s| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}}\left(h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})\right) = \mathbf{0}$$

(see Proposition 4.10).

Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x}) - s)\,\mathbf{q} = \mathbf{0}.$$

It follows that

$$\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) \\
&= \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) + \lim_{\mathbf{x}\to\mathbf{p}} \left(\Big(h(\mathbf{x}) - s\Big)\mathbf{q}\right) + s\mathbf{q} \\
&= \mathbf{0} + s\mathbf{q},
\end{aligned}$$

as required. ∎

Lemma 4.13

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f\colon X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g\colon Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof

Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}),$$

as required. ∎

Proposition 4.14

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) = \mathbf{q} \cdot \mathbf{r}.$$

Proof

The functions f and g satisfy the equation

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = (f(\mathbf{x}) - \mathbf{q}) \cdot g(\mathbf{x}) + \mathbf{q} \cdot (g(\mathbf{x}) - \mathbf{r}) + \mathbf{q} \cdot \mathbf{r},$$

where

$$\lim_{\mathbf{x} \to \mathbf{p}} \Bigl(f(\mathbf{x}) - \mathbf{q} \Bigr) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x} \to \mathbf{p}} \Bigl(g(\mathbf{x}) - \mathbf{r} \Bigr) = \mathbf{0}.$$

Moreover there exists a strictly positive constant δ_0 such that $|g(\mathbf{x}) - \mathbf{r}| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|g(\mathbf{x})| < |\mathbf{r}| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} \left(\Bigl(f(\mathbf{x}) - \mathbf{q} \Bigr) \cdot g(\mathbf{x}) \right) = 0$$

(see Proposition 4.11).

Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(\mathbf{q} \cdot \bigl(g(\mathbf{x}) - \mathbf{r}\bigr)\Bigr) = 0.$$

It follows that

$$\begin{aligned}
&\lim_{\mathbf{x}\to\mathbf{p}} \left(f(\mathbf{x}) \cdot g(\mathbf{x})\right) \\
= \; &\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(\bigl(f(\mathbf{x}) - \mathbf{q}\bigr) \cdot g(\mathbf{x})\Bigr) + \lim_{\mathbf{x}\to\mathbf{p}} \Bigl(\mathbf{q} \cdot \bigl(g(\mathbf{x}) - \mathbf{r}\bigr)\Bigr) + \mathbf{q} \cdot \mathbf{r} \\
= \; &\mathbf{q} \cdot \mathbf{r},
\end{aligned}$$

as required. ∎

Proposition 4.15

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))$ and $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, and moreover

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

First Proof

It follows from Proposition 4.8 (applied in the case when the target space is one-dimensional) that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

It follows from Proposition 4.12 (applied in the case when the target space is one-dimensional), or alternatively from Proposition 4.14, that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Let $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the reciprocal function defined so that $e(t) = 1/t$ for all non-zero real numbers t. Then the reciprocal function e is continuous. Applying the result of Lemma 4.13, we find that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{g(\mathbf{x})} = \lim_{\mathbf{x}\to\mathbf{p}} e(g(\mathbf{x})) = e\left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\right) = \frac{1}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})},$$

as required. $\blacksquare$

Second Proof

Let $q = \lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x})$ and $r = \lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x})$, and let $h \colon X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} h(\mathbf{x}) = (q, r)$$

(see Proposition 4.7).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u,v) = u + v$ and $m(u,v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 4.4). Also $f + g = s \circ h$ and $f \cdot g = m \circ f$. It follows from this that

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(h(\mathbf{x})) \\
&= s\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = s(q, m) = q + r,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(h(\mathbf{x})) \\
&= m\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = m(q, r) = qr
\end{aligned}$$

(see Lemma 4.13).

Also

$$\lim_{\mathbf{x}\to\mathbf{p}} (-g(\mathbf{x})) = -r.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = q - r.$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $e\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $e(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{r}.$$

It then follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{q}{r},$$

as required. $\blacksquare$

Proposition 4.16

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to \mathbb{R}^k$ be functions satisfying $f(X) \subset Y$. Let $\mathbf{p}$ be a limit point of X in $\mathbb{R}^m$, let $\mathbf{q}$ be a limit point of Y in $\mathbb{R}^n$ let $\mathbf{r}$ be a point of $\mathbb{R}^k$. Suppose that the following three conditions are satisfied:

- **(i)** $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$;
- **(ii)** $\lim\limits_{\mathbf{y}\to\mathbf{q}} g(\mathbf{y}) = \mathbf{r}$;
- **(iii)** *there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$.*

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = \mathbf{r}.$$

Proof

Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{y} \in Y$ satisfies $0 < |\mathbf{y} - \mathbf{q}| < \eta$. There then exists some positive real number δ_1 such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Also there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and $0 < |f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. But this then ensures that $|g(f(\mathbf{x})) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. The result follows. $\blacksquare$

Proposition 4.17

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.

Proof

The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x} - \mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.