MA2321—Analysis in Several Variables
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4. Limits and Continuity for Functions of Several Variables

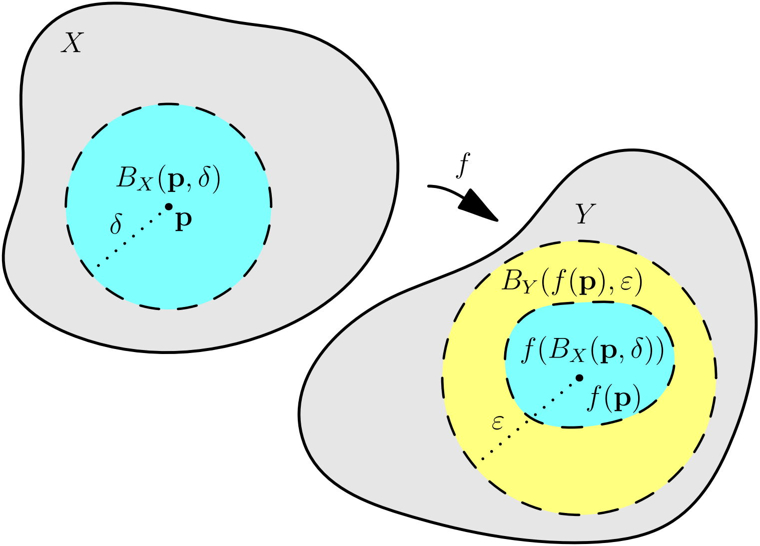
4.1. Continuity of Functions of Several Real Variables

Definition

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.*

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.


X
$B_X(\mathbf{p}, \delta)$
δ
$\mathbf{p}$
f
Y
$B_Y(f(\mathbf{p}), \varepsilon)$
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{p})$
ε

Lemma 4.1

Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.

Proof

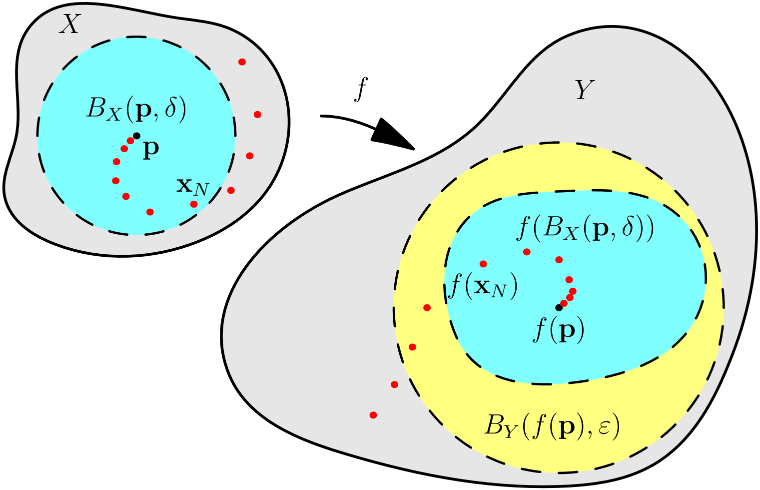
Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.2

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.

Proof

Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎


X
$B_X(\mathbf{p}, \delta)$
$\mathbf{p}$
$\mathbf{x}_N$
f
Y
$f(B_X(\mathbf{p}, \delta))$
$f(\mathbf{x}_N)$
$f(\mathbf{p})$
$B_Y(f(\mathbf{p}), \varepsilon)$

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 4.3

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f \colon X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.

Proof

Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 4.1. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.4

The functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x,y) = x + y$ and $m(x,y) = xy$ are continuous.

Proof

Let $(u,v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x,y) is any point of $\mathbb{R}^2$ whose distance from (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence

$$|s(x,y) - s(u,v)| = |x+y-u-v| \leq |x-u| + |y-v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ be given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 4.5

Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.

Proof

Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 4.3, Lemma 4.4 and Lemma 4.1 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example

Consider the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 4.5. It then follows from Proposition 4.3 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Lemma 4.6

Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.

Proof

Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Big| |f(\mathbf{x})| - |f(\mathbf{p})| \Big| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

4.2. Limits of Functions of Several Real Variables

Definition

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> *given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.*

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Proposition 4.7

Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f\colon X \to \mathbb{R}^n$ has the property that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof

Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$. Let i be an integer between 1 and n, and let some positive real number ε be given. Then there exists some positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It then follows from the definition of the Euclidean norm that

$$|f_i(\mathbf{x}) - q_i| \leq |f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$ then $\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$ for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < |f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 4.8

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof

Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (\mathbf{q} + \mathbf{r})| &\leq |f(\mathbf{x}) - \mathbf{q}| + |g(\mathbf{x}) - \mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x} \to \mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r},$$

as required. ∎