MA2321—Analysis in Several Variables School of Mathematics, Trinity College Michaelmas Term 2017 Lecture 4 (October 2, 2017)

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2.4. The Multidimensional Bolzano-Weierstrass Theorem

We introduce some terminology and notation for discussing convergence along subsequences of bounded sequences of points in Euclidean spaces. This will be useful in proving the multi-dimensional version of the Bolzano-Weierstrass Theorem.

Definition

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in $\mathbb{R}^n$, let J be an infinite subset of the set $\mathbb{N}$ of positive integers, and let $\mathbf{p}$ be a point of $\mathbb{R}^n$. We say that $\mathbf{p}$ is the *limit* of $\mathbf{x}_j$ as j *tends to infinity in the set* J, and write "$\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ in J" if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \in J$ and $j \geq N$.*

The one-dimensional version of the Bolzano-Weierstrass Theorem (Theorem 1.9) is equivalent to the following statement:

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, there exists an infinite subset J of the set $\mathbb{N}$ of positive integers and a real number p such that $x_j \to p$ as $j \to +\infty$ in J.*

Given an infinite subset J of $\mathbb{N}$, the elements of J can be labelled as $k_1, k_2, k_3, \ldots$, where $k_1 < k_2 < k_3 < \cdots$, so that k_1 is the smallest positive integer belonging of J, k_2 is the next smallest, etc. Therefore any standard result concerning convergence of sequences of points can be applied in the context of the convergence of subsequences of a given sequence of points.

The following result is therefore a direct consequence of the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9):

> *Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers, and given an infinite subset J of the set $\mathbb{N}$ of positive integers, there exists an infinite subset K of J and a real number p such that $x_j \to p$ as $j \to +\infty$ in K.*

The above statement in fact corresponds to the following assertion:—

Given any bounded infinite sequence $x_1, x_2, x_3, \ldots$ *of real numbers, and given any subsequence*

$$x_{k_1}, x_{k_2}, x_{k_3}, \cdots$$

of the given infinite sequence, there exists a convergent subsequence

$$x_{k_{m_1}}, x_{k_{m_2}}, x_{k_{m_3}}, \ldots$$

of the given subsequence. Moreover this convergent subsequence of the given subsequence is itself a convergent subsequence of the given infinite sequence, and it contains only members of the given subsequence of the given sequence.

The basic principle can be presented purely in words as follows:

> *Given a bounded sequence of real numbers, and given a subsequence of that original given sequence, there exists a convergent subsequence of the given subsequence. Moreover this subsequence of the subsequence is a convergent subsequence of the original given sequence.*

We employ this principle in the following proof of the Multidimensional Bolzano-Weierstrass Theorem.

Theorem 2.6 (Multidimensional Bolzano-Weierstrass Theorem)

Every bounded sequence of points in a Euclidean space has a convergent subsequence.

Proof

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of points in $\mathbb{R}^n$, and, for each positive integer j, and for each integer i between 1 and n, let $(\mathbf{x}_j)_i$ denote the ith component of $\mathbf{x}_j$. Then

$$\mathbf{x}_j = \Big((\mathbf{x}_j)_1, (\mathbf{x}_j)_2, \ldots, (\mathbf{x}_j)_n\Big).$$

for all positive integers j. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.9) that there exists an infinite subset J_1 of the set $\mathbb{N}$ of positive integers and a real number p_1 such that $(\mathbf{x}_j)_1 \to p_1$ as $j \to +\infty$ in J_1.

Let k be an integer between 1 and $n-1$. Suppose that there exists an infinite subset J_k of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_k$ such that, for each integer i between 1 and k, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_k. It then follows from the one-dimensional Bolzano-Weierstrass Theorem that there exists an infinite subset J_{k+1} of J_k and a real number p_{k+1}, such that $(\mathbf{x}_j)_{k+1} \to p_{k+1}$ as $j \to +\infty$ in J_{k+1}. Moreover the requirement that $J_{k+1} \subset J_k$ then ensures that, for each integer i between 1 and $k+1$, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_{k+1}. Repeated application of this result then ensures the existence of an infinite subset J_n of $\mathbb{N}$ and real numbers $p_1, p_2, \ldots, p_n$ such that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$ in J_n. Let

$$J_n = \{k_1, k_2, k_3, \ldots\},$$

where $k_1 < k_2 < k_3 < \cdots$. Then $\displaystyle\lim_{j\to+\infty} (\mathbf{x}_{k_j})_i = p_i$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 2.3 that $\displaystyle\lim_{j\to+\infty} \mathbf{x}_{k_j} = \mathbf{p}$. The result follows. ∎

2.5. Cauchy Sequences in Euclidean Spaces

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if the following condition is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.*

Lemma 2.7

Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.

Proof

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 2.8

(Cauchy's Criterion for Convergence) An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ *is convergent if and only if it is a Cauchy sequence.*

Proof

First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 2.7. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 2.6). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎