MA2321—Analysis in Several Variables
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2. Convergence in Euclidean Spaces

2.1. Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\
\mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\
\lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\
|\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.
\end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 2.1

(Schwarz's Inequality) Let $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$*. Then* $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$*.*

Proof

We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x}\cdot\mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x}\cdot\mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x}\cdot\mathbf{y})^2 + (\mathbf{x}\cdot\mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x}\cdot\mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x}\cdot\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Proposition 2.2

(Triangle Inequality) Let $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$*. Then* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$*.*

Proof

Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Proposition 2.2) that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact that the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

2.2. Convergence of Sequences in Euclidean Spaces

Definition

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> *given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$.*

We refer to $\mathbf{p}$ as the *limit* $\displaystyle\lim_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 2.3

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. Then a sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to $\mathbf{p}$ if and only if the ith components of the elements of this sequence converge to p_i for $i = 1, 2, \ldots, n$.

Proof

Let $(\mathbf{x}_j)_i$ denote the ith components of $\mathbf{x}_j$. Then $|(\mathbf{x}_j)_i - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for $i = 1, 2, \ldots, n$ and for all positive integers j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each integer i between 1 and n, $(\mathbf{x}_j)_i \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|(\mathbf{x}_j)_i - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then $j \geq N_i$ for $i = 1, 2, \ldots, n$, and therefore

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} ((\mathbf{x}_j)_i - p_i)^2 < n \left(\frac{\varepsilon}{\sqrt{n}} \right)^2 = \varepsilon^2.$$

Thus $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

2.3. Limit Points of Subsets of Euclidean Spaces

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in \mathbb{R}^n$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Lemma 2.4

Let X be a subset of n-dimensional Euclidean space $\mathbf{R}^n$. A point $\mathbf{p}$ is a limit point of the set X if and only if, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set.

Proof

Suppose that, given any positive real number δ, the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\}$$

is an infinite set. Then, for each positive real number δ, the set thus determined by δ must consist of more than just the single point $\mathbf{p}$, and therefore there exists $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. Thus $\mathbf{p}$ is a limit point of the set X.

Now let $\mathbf{p}$ be an arbitrary point of $\mathbb{R}^n$. Suppose that there exists some positive real number δ_0 for which the set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

is finite. If this set does not contain any points of X distinct from the point $\mathbf{p}$ then $\mathbf{p}$ is not a limit point of the set X. Otherwise let δ be the minimum value of $|\mathbf{x} - \mathbf{p}|$ as $\mathbf{x}$ ranges over all points of the finite set

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_0\}$$

that are distinct from $\mathbf{p}$. Then $\delta > 0$, and $|\mathbf{x} - \mathbf{p}| \geq \delta$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. Thus the point $\mathbf{p}$ is not a limit point of the set X. The result follows. ∎

Lemma 2.5

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$ and let $\mathbf{p} \in \mathbb{R}^n$. Then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

Proof

Suppose that $\mathbf{p}$ is a limit point of X. Then, for each positive integer j, there exists a point $\mathbf{x}_j$ of X for which $0 < |\mathbf{x}_j - \mathbf{p}| < 1/j$. The points $\mathbf{x}_j$ satisfying this condition then constitute an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, that converge to the point $\mathbf{p}$.

Conversely suppose that $\mathbf{p}$ is some point of $\mathbb{R}^n$ that is the limit of some infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X that are all distinct from the point $\mathbf{p}$. Let some positive number δ be given. The definition of convergence ensures that there exists a positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$. Moreover $|\mathbf{x}_j - \mathbf{p}| > 0$ for all positive integers j. Thus $0 < |\mathbf{x}_j - \mathbf{p}| < \delta$ when the positive integer j is sufficiently large. Thus the point $\mathbf{p}$ is a limit point of the set X, as required. $\blacksquare$

Definition

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A point $\mathbf{p}$ of X is said to be an *isolated point* of X if it is not a limit point of X.

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p} \in X$. It follows immediately from the definition of isolated points that the point $\mathbf{p}$ is an isolated point of the set X if and only if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} = \{\mathbf{p}\}.$$