MA2321—Analysis in Several Variables
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1.7. Absolute Values of Real Numbers

Let x be a real number. The *absolute value* $|x|$ of x is defined so that

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0; \end{cases}$$

Lemma 1.1

Let u and v be real numbers. Then $|u + v| \leq |u| + |v|$ and $|uv| = |u|\,|v|$.

Proof

Let u and v be real numbers. Then

$$-|u| \leq u \leq |u| \quad \text{and} \quad -|v| \leq v \leq |v|.$$

On adding inequalities, we find that

$$-(|u| + |v|) = -|u| - |v| \leq u + v \leq |u| + |v|,$$

and thus

$$u + v \leq |u| + |v| \quad \text{and} \quad -(u + v) \leq |u| + |v|.$$

Now the value of $|u + v|$ is equal to at least one of the numbers $u + v$ and $-(u + v)$. It follows that

$$|u + v| \leq |u| + |v|$$

for all real numbers u and v.

Next we note that $|u|\,|v|$ is the product of one or other of the numbers u and $-u$ with one or other of the numbers v and $-v$, and therefore its value is equal either to uv or to $-uv$. Because both $|u|\,|v|$ and $|uv|$ are non-negative, we conclude that $|uv| = |u|\,|v|$, as required. ∎

Lemma 1.2

Let u and v be real numbers. Then $\bigl||u| - |v|\bigr| \leq |u - v|$.

Proof

It follows from Lemma 1.1 that

$$|u| = |v + (u - v)| \leq |v| + |u - v|.$$

Therefore $|u| - |v| \leq |u - v|$. Interchanging u and v, we find also that

$$|v| - |u| \leq |v - u| = |u - v|.$$

Now $\bigl||u| - |v|\bigr|$ is equal to one or other of the real numbers $|u| - |v|$ and $|v| - |u|$. It follows that $\bigl||u| - |v|\bigr| \leq |u - v|$, as required. ■

1.8. Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* $x_1, x_2, x_3, \ldots$ of real numbers associates to each positive integer j a corresponding real number x_j.

Definition

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number p if and only if the following criterion is satisfied:

> *given any strictly positive real number ε, there exists some positive integer N such that $|x_j - p| < \varepsilon$ for all positive integers j satisfying $j \geq N$.*

If an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p, then p is said to be the *limit* of the sequence, and we can indicate the convergence of the infinite sequence to p by writing '$x_j \to p$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = p$'.

Let x and p be real numbers, and let ε be a strictly positive real number. Then $|x - p| < \varepsilon$ if and only if both $x - p < \varepsilon$ and $p - x < \varepsilon$. It follows that $|x - p| < \varepsilon$ if and only if $p - \varepsilon < x < p + \varepsilon$. The condition $|x - p| < \varepsilon$ essentially requires that the value of the real number x should agree with p to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number p if and only if, given any positive real number ε, there exists some positive integer N such that $p - \varepsilon < x_j < p + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Definition

We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 1.3

Every convergent sequence of real numbers is bounded.

Proof

Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number p. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $p - 1 < x_j < p + 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $p - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $p + 1$. $\blacksquare$

Proposition 1.4

Let $x_1, x_2, x_3, \ldots$ *and* $y_1, y_2, y_3, \ldots$ *be convergent infinite sequences of real numbers. Then the sum and difference of these sequences are convergent, and*

$$\begin{aligned} \lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\ \lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j. \end{aligned}$$

Proof

Throughout this proof let $p = \lim_{j \to +\infty} x_j$ and $q = \lim_{j \to +\infty} y_j$. It follows directly from the definition of limits that $\lim_{j \to +\infty} (-y_j) = -q$.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (p+q)| < \varepsilon$ whenever $j \geq N$. Now $x_j \to p$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - p| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - p| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - q| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (p+q)| &= |(x_j - p) + (y_j - q)| \leq |x_j - p| + |y_j - q| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to p + q$ as $j \to +\infty$.

On replacing y_j by $-y_j$ for all positive integers j, and using the result that $-y_j \to -q$ as $j \to +\infty$, we see that Thus $x_j - y_j \to p - q$ as $j \to +\infty$, as required. ∎

Lemma 1.5

Let $x_1, x_2, x_3, \ldots$ be a convergent infinite sequence of real numbers, and let c be a real number. Then

$$\lim_{j \to +\infty} (c x_j) = c \lim_{j \to +\infty} x_j.$$

Proof

Let some strictly positive real number ε be given. Then a strictly positive real number ε_1 can be chosen so that $|c|\,\varepsilon_1 \leq \varepsilon$. There then exists some positive integer N such that $|x_j - p| < \varepsilon_1$ whenever $j \geq N$, where $p = \lim_{j \to +\infty} x_j$. But then

$$|c x_j - c p| < |c|\,\varepsilon_1 \leq \varepsilon$$

whenever $j \geq N$. We conclude that $\lim_{j \to +\infty} c x_j = c p$, as required. ∎

Proposition 1.6

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the product of these sequences is convergent, and

$$\lim_{j \to +\infty} (x_j y_j) = \left(\lim_{j \to +\infty} x_j \right) \left(\lim_{j \to +\infty} y_j \right).$$

Proof

Let $u_j = x_j - p$ and $v_j = y_j - q$ for all positive integers j where $p = \lim\limits_{j \to +\infty} x_j$ and $q = \lim\limits_{j \to +\infty} y_j$. Then

$$\begin{aligned}
\lim_{j \to +\infty} (u_j v_j) &= \lim_{j \to +\infty} (x_j y_j - x_j q - p y_j + pq) \\
&= \lim_{j \to +\infty} (x_j y_j) - q \lim_{j \to +\infty} x_j - p \lim_{j \to +\infty} y_j + pq \\
&= \lim_{j \to +\infty} (x_j y_j) - pq.
\end{aligned}$$

Let some strictly positive real number ε be given. It follows from the definition of limits that $\lim\limits_{j \to +\infty} u_j = 0$ and $\lim\limits_{j \to +\infty} v_j = 0$. Therefore there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. Thus $\lim\limits_{j \to +\infty} u_j v_j = 0$, and therefore $\lim\limits_{j \to +\infty} (x_j y_j) - pq = 0$. The result follows.

Proposition 1.7

Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers, where $y_j \neq 0$ for all positive integers j and $\lim_{j \to +\infty} y_j \neq 0$. Then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j \to +\infty} \frac{x_j}{y_j} = \frac{\lim\limits_{j \to +\infty} x_j}{\lim\limits_{j \to +\infty} y_j}.$$

Proof

Let $p = \lim_{j \to +\infty} x_j$ and Let $q = \lim_{j \to +\infty} y_j$. Then

$$\frac{x_j}{y_j} - \frac{p}{q} = \frac{qx_j - py_j}{qy_j}$$

for all positive integers j. Now there exists some positive integer N_1 such that $|y_j - q| < \frac{1}{2}|q|$ whenever $j \geq N_1$. Then $|y_j| \geq \frac{1}{2}|q|$ whenever $j \geq N_1$, and therefore

$$\left|\frac{x_j}{y_j} - \frac{p}{q}\right| \leq \frac{2}{|q|^2}\,|qx_j - py_j|$$

whenever $j \geq N_1$.

Let some strictly positive real number ε be given. Applying Lemma 1.5 and Proposition 1.4, we find that

$$\lim_{j\to+\infty}(qx_j - py_j) = q\lim_{j\to+\infty} x_j - p\lim_{j\to+\infty} y_j = qp - pq = 0.$$

Therefore there exists some positive integer N satisfying $N \geq N_1$ with the property that

$$|qx_j - py_j| < \tfrac{1}{2}|q|^2\varepsilon$$

whenever $j \geq N$. But then

$$\left|\frac{x_j}{y_j} - \frac{p}{q}\right| < \varepsilon$$

whenever $j \geq N$. Thus

$$\lim_{j\to+\infty}\frac{x_j}{y_j} = \frac{p}{q},$$

as required. ∎

1.9. Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.8

Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.

Proof

Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound p for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to p.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - p| < \varepsilon$ whenever $j \geq N$. Now $p - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since p is the least upper bound), and therefore there must exist some positive integer N such that $x_N > p - \varepsilon$. But then $p - \varepsilon < x_j \leq p$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by p. Thus $|x_j - p| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to p$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.10. Subsequences of Sequences of Real Numbers

Definition

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots .$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$

$$x_1, x_4, x_9, x_{16}, \ldots$$

Theorem 1.9 (Bolzano-Weierstrass)

Every bounded sequence of real numbers has a convergent subsequence.

Proof

Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.8 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.8. This completes the proof of the Bolzano-Weierstrass Theorem. ∎