MA2321—Analysis in Several Variables
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1.1. A Concise Characterization of the Real Number System

The set $\mathbb{R}$ of *real numbers*, with its usual ordering and algebraic operations of addition and multiplication, is a Dedekind-complete ordered field.

We describe below what a *field* is, what an *ordered field* is, and what is meant by saying that an ordered field is *Dedekind-complete*.

1.2. Fields

Definition

A *field* is a set $\mathbb{F}$ on which are defined operations of addition and multiplication, associating elements $x + y$ and xy of $\mathbb{F}$ to each pair x, y of elements of $\mathbb{F}$, for which the following axioms are satisfied:

(i) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *commutative*);

(ii) $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *associative*);

(iii) there exists an element 0 of $\mathbb{F}$ with the property that $0 + x = x$ for all $x \in \mathbb{F}$ (i.e., there exists a *zero element* for the operation of addition on $\mathbb{F}$);

(iv) given any $x \in \mathbb{F}$, there exists an element $-x$ of $\mathbb{F}$ satisfying $x + (-x) = 0$ (i.e., *negatives* of elements of $\mathbb{F}$ always exist);

(v) $xy = yx$ for all $x, y \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *commutative*);

(vi) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *associative*);

(vii) there exists an element 1 of $\mathbb{F}$ with the property that $1x = x$ for all $x \in \mathbb{F}$ (i.e., there exists an *identity element* for the operation of multiplication on $\mathbb{F}$);

(viii) given any $x \in \mathbb{F}$ satisfying $x \neq 0$, there exists an element x^{-1} of $\mathbb{F}$ satisfying $xx^{-1} = 1$;

(ix) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$ (i.e., multiplication is *distributive* over addition).

The operations of subtraction and division are defined on a field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $x - y = x + (-y)$ for all elements x and y of $\mathbb{F}$, and moreover $x/y = xy^{-1}$ provided that $y \neq 0$.

1.3. Ordered Fields

Definition

An *ordered field* consists of a field $\mathbb{F}$ together with an ordering $<$ on that field that satisfies the following axioms:—

(x) if x and y are elements of $\mathbb{F}$ then one and only one of the three statements $x < y$, $x = y$ and $y < x$ is true (i.e., the ordering satisfies the *Trichotomy Law*);

(xi) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ and $y < z$ then $x < z$ (i.e., the ordering is *transitive*);

(xii) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ then $x + z < y + z$;

(xiii) if x and y are elements of $\mathbb{F}$ which satisfy $0 < x$ and $0 < y$ then $0 < xy$.

We can write $x > y$ in cases where $y < x$. we can write $x \leq y$ in cases where either $x = y$ or $x < y$. We can write $x \geq y$ in cases where either $x = y$ or $y < x$.

Example
The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example

Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b + c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

1.4. Least Upper Bounds

Let S be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition

Let $\mathbb{F}$ be an ordered field, and let S be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example

The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The following property is satisfied in some ordered fields but not in others.

> **Least Upper Bound Property**: *given any non-empty subset S of $\mathbb{F}$ that is bounded above, there exists an element $\sup S$ of $\mathbb{F}$ that is the least upper bound for the set S.*

Definition

A *Dedekind-complete* ordered field $\mathbb{F}$ is an ordered field which has the Least Upper Bound Property.

1.5. Greatest Lower Bounds

Let S be a subset of an ordered field $\mathbb{F}$. A *lower bound* of S is an element l of $\mathbb{F}$ with the property that $l \leq x$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists. A *greatest lower bound* (or *infimum*) for a set S is a lower bound for that set that is greater than every other lower bound for that set. The greatest lower bound of the set S (if it exists) is denoted by $\inf S$.

Let $\mathbb{F}$ be a Dedekind-complete ordered field. Then, given any non-empty subset S of $\mathbb{F}$ that is bounded below, there exists a greatest lower bound (or *infimum*) $\inf S$ for the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

Remark

It can be proved that any two Dedekind-complete ordered fields are isomorphic via an isomorphism that respects the ordering and the algebraic operations on the fields. The theory of *Dedekind cuts* provides a construction that yields a Dedekind-complete ordered field that can represent the system of real numbers. For an account of this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.6. Bounded Sets of Real Numbers

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field. Thus every non-empty subset S of $\mathbb{R}$ that is bounded above has a *least upper bound* (or *supremum*) $\sup S$, and every non-empty subset S of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or *infimum*) $\inf S$.

Let S be a non-empty subset of the real numbers that is bounded (both above and below). Then the closed interval $[\inf S, \sup S]$ is the smallest closed interval in the set $\mathbb{R}$ of real numbers that contains the set S. Indeed if $S \subset [a, b]$, where a and b are real numbers satisfying $a \leq b$, then $a \leq \inf S \leq \sup S \leq b$, and therefore

$$S \subset [\inf S, \sup S] \subset [a, b].$$