Module MA2321: Analysis in Several Real Variables
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Section II: The Mean Value Theorem

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2 The Mean Value Theorem

2.1 Interior Points and Open Sets in the Real Line

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers. We say that D is an *open set* in $\mathbb{R}$ if every point of D is an interior point of D.

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

2.2 Differentiable Functions of a Single Real Variable

We recall basic results of the theory of differentiable functions.

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h\to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Let s be some real number, and let f and g be real-valued functions defined around s that are differentiable at s. The basic rules of differential calculus then ensure that the functions $f+g$, $f-g$ and $f\cdot g$ are differentiable at s (where

$$(f{+}g)(x) = f(x){+}g(x), \quad (f{-}g)(x) = f(x){-}g(x) \quad \text{and} \quad (f.g)(x) = f(x)g(x)$$

for all real numbers x at which both $f(x)$ and $g(x)$ are defined), and

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s) \quad (\textit{Product Rule}).$$

If moreover $g(s) \neq 0$ then the function f/g is differentiable at s (where $(f/g)(x) = f(x)/g(x)$ where both $f(x)$ and $g(x)$ are defined), and

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2} \quad (\textit{Quotient Rule}).$$

Moreover if h is a real-valued function defined around $f(s)$ which is differentiable at $f(s)$ then the composition function $h \circ f$ is differentiable at $f(s)$ and

$$(h \circ f)'(s) = h'(f(s))f'(s) \quad (\textit{Chain Rule}).$$

Derivatives of some standard functions are as follows:—

$$\frac{d}{dx}(x^m) = mx^{m-1}, \quad \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\exp x) = \exp x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \ (x > 0).$$

2.3 Rolle's Theorem

Theorem 2.1 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a,b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a,b]$ with the property that $f(u) \leq$

$f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$ (see Theorem 1.7). If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

2.4 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 2.2 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 2.1) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

2.5 Concavity and the Second Derivative

Proposition 2.3 *Let s and h be real numbers, and let f be a twice differentiable real-valued function defined on some open interval containing s and $s + h$. Then there exists a real number θ satisfying $0 < \theta < 1$ for which*

$$f(s + h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s + \theta h).$$

Proof Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s + th)$ is defined for all $t \in I$, and let $q: I \to \mathbb{R}$ be defined so that

$$q(t) = f(s + th) - f(s) - thf'(s) - t^2(f(s + h) - f(s) - hf'(s)).$$

for all $t \in I$. Differentiating, we find that

$$q'(t) = hf'(s + th) - hf'(s) - 2t(f(s + h) - f(s) - hf'(s))$$

and

$$q''(t) = h^2 f''(s + th) - 2(f(s + h) - f(s) - hf'(s)).$$

Now $q(0) = q(1) = 0$. It follows from Rolle's Theorem, applied to the function q on the interval $[0, 1]$, that there exists some real number φ satisfying $0 < \varphi < 1$ for which $q'(\varphi) = 0$.

Then $q'(0) = q'(\varphi) = 0$, and therefore Rolle's Theorem can be applied to the function q' on the interval $[0, \varphi]$ to prove the existence of some real number θ satisfying $0 < \theta < \varphi$ for which $q''(\theta) = 0$. Then

$$0 = q''(\theta) = h^2 f''(s + \theta h) - 2(f(s + h) - f(s) - hf'(s)).$$

Rearranging, we find that

$$f(s + h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s + \theta h),$$

as required. ∎

Corollary 2.4 *Let $f: (s-\delta_0, s+\delta_0)$ be a twice-differentiable function throughout some open interval $(s - \delta_0, s + \delta_0)$ centred on a real number s. Suppose that $f''(s + h) > 0$ for all real numbers h satisfying $|h| < \delta_0$. Then*

$$f(s + h) \geq f(s) + hf'(s)$$

for all real numbers h satisfying $|h| < \delta_0$.

It follows from Corollary 2.4 that if a twice-differentiable function has positive second derivative throughout some open interval, then it is concave upwards throughout that interval. In particular the function has a local minimum at any point of that open interval where the first derivative is zero and the second derivative is positive.

Corollary 2.5 *Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined over a subset D of $\mathbb{R}$, and let s be a point in the interior of D. Suppose that $f'(s) = 0$ and that $f''(x) > 0$ for all real numbers x belonging to some sufficiently small neighbourhood of x. Then s is a local minimum for the function f.*