Module MA2321: Analysis in Several Real Variables
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Section I: The Real Number System

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Contents

1 The Real Number System

1.1 A Concise Characterization of the Real Number System

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field.

We describe below what a *field* is, what an *ordered field* is, and what is meant by saying that an ordered field is *Dedekind-complete.*

1.2 Fields

Definition A *field* is a set $\mathbb{F}$ on which are defined operations of addition and multiplication, associating elements $x + y$ and xy of $\mathbb{F}$ to each pair x, y of elements of $\mathbb{F}$, for which the following axioms are satisfied:

(i) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *commutative*);

(ii) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *associative*);

(iii) there exists an element 0 of $\mathbb{F}$ with the property that $0 + x = x$ for all $x \in \mathbb{F}$ (i.e., there exists a *zero element* for the operation of addition on $\mathbb{F}$);

(iv) given any $x \in \mathbb{F}$, there exists an element $-x$ of $\mathbb{F}$ satisfying $x+(-x) = 0$ (i.e., *negatives* of elements of $\mathbb{F}$ always exist);

(v) $xy = yx$ for all $x, y \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *commutative*);

(vi) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *associative*);

(vii) there exists an element 1 of $\mathbb{F}$ with the property that $1x = x$ for all $x \in \mathbb{F}$ (i.e., there exists an *identity element* for the operation of multiplication on $\mathbb{F}$);

(viii) given any $x \in \mathbb{F}$ satisfying $x \neq 0$, there exists an element x^{-1} of $\mathbb{F}$ satisfying $xx^{-1} = 1$;

(ix) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$ (i.e., multiplication is *distributive* over addition).

The operations of subtraction and division are defined on a field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $x-y = x+(-y)$ for all elements x and y of $\mathbb{F}$, and moreover $x/y = xy^{-1}$ provided that $y \neq 0$.

1.3 Ordered Fields

Definition An *ordered field* consists of a field $\mathbb{F}$ together with an ordering $<$ on that field that satisfies the following axioms:—

(x) if x and y are elements of $\mathbb{F}$ then one and only one of the three statements $x < y$, $x = y$ and $y < x$ is true (i.e., the ordering satisfies the *Trichotomy Law*);

(xi) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ and $y < z$ then $x < z$ (i.e., the ordering is *transitive*);

(xii) if x, y and z are elements of $\mathbb{F}$ and if $x < y$ then $x + z < y + z$;

(xiii) if x and y are elements of $\mathbb{F}$ which satisfy $0 < x$ and $0 < y$ then $0 < xy$.

We can write $x > y$ in cases where $y < x$. we can write $x \leq y$ in cases where either $x = y$ or $x < y$. We can write $x \geq y$ in cases where either $x = y$ or $y < x$.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

1.4 Least Upper Bounds

Let S be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let S be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The following property is satisfied in some ordered fields but not in others.

> **Least Upper Bound Property**: given any non-empty subset S of $\mathbb{F}$ that is bounded above, there exists an element $\sup S$ of $\mathbb{F}$ that is the least upper bound for the set S.

Definition A *Dedekind-complete* ordered field $\mathbb{F}$ is an ordered field which has the Least Upper Bound Property.

1.5 Greatest Lower Bounds

Let S be a subset of an ordered field $\mathbb{F}$. A *lower bound* of S is an element l of $\mathbb{F}$ with the property that $l \leq x$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists. A *greatest lower bound* (or *infimum*) for a set S is a lower bound for that set that is greater than every other lower bound for that set. The greatest lower bound of the set S (if it exists) is denoted by $\inf S$.

Let $\mathbb{F}$ be a Dedekind-complete ordered field. Then, given any non-empty subset S of $\mathbb{F}$ that is bounded below, there exists a greatest lower bound (or *infimum*) $\inf S$ for the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

Remark It can be proved that any two Dedekind-complete ordered fields are isomorphic via an isomorphism that respects the ordering and the algebraic operations on the fields. The theory of *Dedekind cuts* provides a construction that yields a Dedekind-complete ordered field that can represent the system of real numbers. For an account of this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.6 Bounded Sets of Real Numbers

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field. Thus every non-empty subset S of $\mathbb{R}$ that is bounded above has a *least upper bound* (or *supremum*) $\sup S$, and every non-empty subset S of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or *infimum*) $\inf S$.

Let S be a non-empty subset of the real numbers that is bounded (both above and below). Then the closed interval $[\inf S, \sup S]$ is the smallest closed interval in the set $\mathbb{R}$ of real numbers that contains the set S. Indeed if $S \subset [a, b]$, where a and b are real numbers satisfying $a \leq b$, then $a \leq \inf S \leq \sup S \leq b$, and therefore

$$S \subset [\inf S, \sup S] \subset [a, b].$$

1.7 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l-\varepsilon < x_j < l+\varepsilon$ for all positive integers j satisfying $j \geq N$.

Lemma 1.1 *Let S be a subset of the set $\mathbb{R}$ of real numbers which is non-empty and bounded above, and let $\sup S$ denote the least upper bound of the set S. Then there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_j \in S$ for all positive integers j and $\lim_{j\to+\infty} x_j = \sup S$.*

Proof Let $s = \sup S$. For each positive integer j, the real number $s - 1/j$ is not an upper bound for the set S (because s is the least upper bound of S), and therefore there exists some element x_j of S satisfying $x_j > s - 1/j$. Moreover $x_j \leq s$ for all positive integers j, because s is an upper bound for the set S. It follows that $s - 1/j < x_j \leq s$ for all positive integers j. Given any positive real number ε, let N be a positive integer chosen so that $N > 1/\varepsilon$. Then $|x_j - s| < \varepsilon$ whenever $j \geq N$. It follows that $\lim_{j\to+\infty} x_j = s$, as required. ∎

1.8 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.2 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.9 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

1.10 The Bolzano-Weierstrass Theorem

Theorem 1.3 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 <$

$j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.2 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.2. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

1.11 The Definition of Continuity for Functions of a Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $f\colon D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be *continuous* at s if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Lemma 1.4 *Let $f\colon D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$, and let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers belonging to D. Suppose that $x_j \to s$ as $j \to +\infty$, where $s \in D$, and that f is continuous at s. Then $f(x_j) \to f(s)$ as $j \to +\infty$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then there exists some positive integer N such that $|x_j - s| < \delta$ for all j satisfying $j \geq N$. Thus $|f(x_j) - f(s)| < \varepsilon$ whenever $j \geq N$. Hence $f(x_j) \to f(s)$ as $j \to +\infty$. ∎

1.12 The Intermediate Value Theorem

Theorem 1.5 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous function* *defined on the interval* $[a, b]$*. Let* c *be a real number which lies between* $f(a)$ *and* $f(b)$ *(so that either* $f(a) \leq c \leq f(b)$ *or else* $f(a) \geq c \geq f(b)$*.) Then there exists some* $s \in [a, b]$ *for which* $f(s) = c$*.*

Proof If $f(a) = c$ then we may take $s = a$, and if $f(b) = c$ then we may take $s = b$.

It remains to consider cases where $f(a) < c < f(b)$ or $f(a) > c > f(b)$. In the case where $f(a) < c < f(b)$ let $g: [a, b] \to \mathbb{R}$ be defined such that $g(x) = f(x) - c$. In the case where $f(a) > c > f(b)$ let $g: [a, b] \to \mathbb{R}$ be defined such that $g(x) = c - f(x)$. In both cases the function g is a continuous function on $[a, b]$ defined so that $g(a) < 0$ and $g(b) > 0$, and in both cases we must prove the existence of a real number s belonging to the interval $[a, b]$ for which $g(s) = 0$.

Let

$$S = \{x \in [a, b] : g(x) \leq 0\}.$$

Then $a \in S$, and $x \leq b$ for all $x \in S$. The set S is thus non-empty and bounded above, and therefore there exists a least upper bound $\sup S$ for the set S. Let $s = \sup S$.

Now it follows from Lemma 1.1 that there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_j \in S$ for all positive integers j and $\lim_{j\to+\infty} x_j = s$. Now $g(x_j) \leq 0$ for all positive integers j (because $x_j \in S$). Moreover the continuity of the function g ensures that $g(s) = \lim_{j\to+\infty} g(x_j)$. It follows that $g(s) \leq 0$. Moreover $s < b$ (because $g(b) > 0$), and therefore there exists an infinite sequence $y_1, y_2, y_3, \ldots$ such that $s < y_j \leq b$ for all positive integers j and $\lim_{j\to+\infty} y_j = s$. (Indeed we could take $y_j = s + (b - s)/j$ for all positive integers j.) Now $g(y_j) > 0$ for all positive integers j (because $y_j \notin S$), and $g(s) = \lim_{j\to+\infty} g(y_j)$, and therefore $g(s) \geq 0$. We have now shown that both $g(s) \leq 0$ and $g(s) \geq 0$. It follows that $g(s) = 0$, and thus $f(s) = c$, as required. ∎

1.13 The Extreme Value Theorem

Proposition 1.6 *Let* a *and* b *be real numbers satisfying* $a < b$*, and let* $f: [a, b] \to \mathbb{R}$ *be a continuous real-valued function defined on the closed interval* $[a, b]$*. Then there exists a positive constant* M *with the property that* $|f(x)| \leq M$ *for all* $x \in [a, b]$*.*

Proof Let S be the set consisting of those real numbers t satisfying $a \leq t \leq b$ for which the function f is bounded on $[a, t]$. A real number t therefore belongs to the set S if and only if $a \leq t \leq b$ and also there exists some positive real number K_t with the property that $|f(x)| \leq K_t$ for all $x \in [a, t]$. Now $a \in S$ and $t \leq b$ for all $t \in S$. Thus set S is non-empty and bounded above. It follows from the Least Upper Bound Principle that the set S has a least upper bound $\sup S$. Let $s = \sup S$. Then $s \in [a, b]$.

Now the function f is continuous at s. Therefore there exists some positive real number δ such that $|f(x)| \leq |f(s)| + 1$ for all whenever $x \in [a, b]$ and $s - \delta < x < s + \delta$. Also $s - \delta$ is not an upper bound for the set S and therefore there exists some element t of S satisfying $s - \delta < t \leq s$. There then exists some positive real number K_t with the property that $|f(x)| \leq K_t$ for all $x \in [a, t]$.

Let $M = \max(K_t, |f(s)|+1)$. Then $|f(x)| \leq M$ for all $x \in [a, b]$ satisfying $x < s + \delta$, and therefore $x \in S$ for all $x \in [a, b]$ satisfying $x < s + \delta$. If it were the case that $s < b$ then s would not be an upper bound for the set S, contradicting the definition of s as the least upper bound of S. Therefore $s = b$. It follows that $|f(x)| \leq M$ for all $x \in [a, b]$. Thus the function f is bounded on $[a, b]$, as required. ∎

Theorem 1.7 (The Extreme Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then there exist real numbers u and v belonging to the interval $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.*

Proof It follows from Proposition 1.6 that the set

$$\{f(x) : x \in [a, b]\}$$

is bounded above and below. This set is also non-empty. It follows that there exist real numbers M and m such that

$$M = \sup\{f(x) : x \in [a, b]\} \quad \text{and} \quad m = \inf\{f(x) : x \in [a, b]\}.$$

If it were the case that $f(x) < M$ for all $x \in [a, b]$ then there would exist a well-defined function $g\colon [a, b] \to \mathbb{R}$ satisfying

$$g(x) = \frac{1}{M - f(x)}$$

for all $x \in [a, b]$. This function would not be bounded, because, given any positive constant K, there would exist $x \in [a, b]$ for which $f(x) > M - 1/K$ and $g(x) > K$. The existence of such a function g would contradict the result of Proposition 1.6. Therefore there must exist $v \in [a, b]$ with the property that $f(x) \leq f(v)$ for all $x \in [a, b]$.

Similarly there cannot exist any continuous function $h\colon [a,b] \to \mathbb{R}$ with the property that

$$h(x) = \frac{1}{f(x) - m}$$

for all $x \in [a,b]$, and therefore there must exist $u \in [a,b]$ with the property that $f(u) \leq f(x)$ for all $x \in [a,b]$. This completes the proof. ∎

1.14 Uniform Continuity

Definition A function $f\colon D \to \mathbb{R}$ is said to be *uniformly continuous* over a subset D of $\mathbb{R}$ if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in [a,b]$ satisfying $|u - v| < \delta$. (where δ does not depend on u or v).

A continuous function defined over a subset D of $\mathbb{R}$ is not necessarily uniformly continuous on D. (One can verify for example that the function sending a non-zero real number x to $1/x$ is not uniformly continuous on the set of all non-zero real numbers.) However we show that continuity does imply uniform continuity when $D = [a,b]$ for some real numbers a and b satisfying $a < b$.

Theorem 1.8 *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on a closed bounded interval $[a,b]$. Then the function f is uniformly continuous on $[a,b]$.*

Proof Let some strictly positive real number ε be given. Suppose that there did not exist any strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$. Then, for each positive integer j, there would exist values u_j and v_j in the interval $[a,b]$ such that $|u_j - v_j| < 1/j$ and $|f(u_j) - f(v_j)| \geq \varepsilon$. But the sequence $u_1, u_2, u_3, \ldots$ would be bounded (since $a \leq u_j \leq b$ for all j) and thus would possess a convergent subsequence $u_{k_1}, u_{k_2}, u_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 1.3).

Let $l = \lim\limits_{j \to +\infty} u_{k_j}$. Then $l = \lim\limits_{j \to +\infty} v_{k_j}$ also, since $\lim\limits_{j \to +\infty} (v_{k_j} - u_{k_j}) = 0$. Moreover $a \leq l \leq b$. It follows from the continuity of f that $\lim\limits_{j \to +\infty} f(u_{k_j}) = \lim\limits_{j \to +\infty} f(v_{k_j}) = f(l)$ (see Lemma 1.4). Thus $\lim\limits_{j \to +\infty} \bigl(f(u_{k_j}) - f(v_{k_j})\bigr) = 0$. But this is impossible, since u_j and v_j have been chosen so that $|f(u_j) - f(v_j)| \geq \varepsilon$ for all positive integers j. We conclude therefore that there must exist some strictly positive real number δ with the required property. ∎

1.15 Historical Note on the Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techniques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second. Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convergence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.