Module MA2321: Analysis in Several Real Variables
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Part II (Sections 4 to 7)

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Contents

4 Continuous Functions of Several Real Variables

4.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 4.1 (Schwarz's Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Corollary 4.2 (Triangle Inequality) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$*. Then* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Proof Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 4.2) that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact that the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

4.2 Convergence of Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{p}-\mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 4.3 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*. Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim\limits_{j\to+\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

4.3 Continuity of Functions of Several Real Variables

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f\colon X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f\colon X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 4.4 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f\colon X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.5 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f\colon X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 4.6 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f: X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 4.4. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 4.7 *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $m(x, y) = xy$ are continuous.*

Proof Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x, y) is any point of $\mathbb{R}^2$ whose distance from (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v).

Next we show that $m: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points (x, y) of $\mathbb{R}^2$. Thus if the distance from (x, y) to (u, v) is less than δ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points (x, y) of $\mathbb{R}^2$ whose distance from (u, v) is less than δ. This shows that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u, v). ∎

Proposition 4.8 *Let X be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 4.6, Lemma 4.7 and Lemma 4.4 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f\colon \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 4.8. It then follows from Proposition 4.6 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

Lemma 4.9 *Let X be a subset of $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a continuous function mapping X into $\mathbb{R}^n$, and let $|f|\colon X \to \mathbb{R}$ be defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in X$. Then the real-valued function $|f|$ is continuous on X.*

Proof Let $\mathbf{x}$ and $\mathbf{p}$ be elements of X. Then

$$|f(\mathbf{x})| = |(f(\mathbf{x}) - f(\mathbf{p})) + f(\mathbf{p})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{p})|$$

and

$$|f(\mathbf{p})| = |(f(\mathbf{p}) - f(\mathbf{x})) + f(\mathbf{x})| \leq |f(\mathbf{x}) - f(\mathbf{p})| + |f(\mathbf{x})|,$$

and therefore

$$\Bigl| |f(\mathbf{x})| - |f(\mathbf{p})| \Bigr| \leq |f(\mathbf{x}) - f(\mathbf{p})|.$$

The result now follows from the definition of continuity, using the above inequality. Indeed let $\mathbf{p}$ be a point of X, and let some positive real number ε be given. Then there exists a positive real number δ small enough to ensure that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. But then

$$\Bigl| |f(\mathbf{x})| - |f(\mathbf{p})| \Bigr| \leq |f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$$

for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus the function $|f|$ is continuous, as required. ∎

4.4 Open Sets in Euclidean Spaces

Definition Given a point $\mathbf{p}$ of $\mathbb{R}^n$ and a non-negative real number r, the *open ball* $B(\mathbf{p}, r)$ in $\mathbb{R}^n$ of *radius* r about $\mathbf{p}$ is defined to be the subset of $\mathbb{R}^n$ defined so that

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B(\mathbf{p}, r)$ is the set consisting of all points of $\mathbb{R}^n$ that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

The *open ball* $B(\mathbf{p}, r)$ of radius r about a point $\mathbf{p}$ of $\mathbb{R}^n$ is bounded by the *sphere* of radius r about $\mathbf{p}$. This sphere is the set

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| = r\}.$$

Definition A subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, \delta)$ is the open ball in $\mathbb{R}^n$ of radius δ about the point $\mathbf{p}$, defined so that

$$B(\mathbf{p}, \delta) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}$$

and

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let

$$V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 9\}.$$

Then the subset $\mathbb{R}^3$ of $\mathbb{R}^3$ is the open ball of radius 3 in $\mathbb{R}^3$ about the origin. This open ball is an open set. Indeed let $\mathbf{x}$ be a point of V. Then $|\mathbf{x}| < 3$. Let $\delta = 3 - |\mathbf{x}|$. Then $\delta > 0$. Moreover if $\mathbf{y}$ is a point of $\mathbb{R}^3$ that satisfies $|\mathbf{y} - \mathbf{x}| < \delta$ then

$$|\mathbf{y}| = |\mathbf{x} + (\mathbf{y} - \mathbf{x})| \leq |\mathbf{x}| + |\mathbf{y} - \mathbf{x}| < |\mathbf{x}| + \delta = 3,$$

and therefore $\mathbf{y} \in V$. This proves that V is an open set.

More generally, an open ball of any positive radius about any point of a Euclidean space $\mathbb{R}^n$ of any dimension n is an open set in that Euclidean space. A more general result is proved below (see Lemma 4.10).

4.5 Open Sets in Subsets of Euclidean Spaces

Definition Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X defined so that

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if, given any point $\mathbf{p}$ of V, there exists some strictly positive real number δ such that $B_X(\mathbf{p}, \delta) \subset V$, where $B_X(\mathbf{p}, \delta)$ is the open ball in X of radius δ about on the point $\mathbf{p}$. The empty set $\emptyset$ is also defined to be an open set in X.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 4.10 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 4.11 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 4.12 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both open in X;

(ii) the union of any collection of open sets in X is itself open in X;

(iii) the intersection of any finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Proposition 4.13 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$.

Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 4.10), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 4.12). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \subset V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

4.6 Convergence of Sequences and Open Sets

Lemma 4.14 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 4.10. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

4.7 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 4.10 and Lemma 4.11 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets).

Indeed let $\mathcal{A}$ be some collection of subsets of a set X, and let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcup_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcup_{S \in \mathcal{A}} S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \notin S \\ &\iff \text{for all } S \in \mathcal{A},\ \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcap_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S).$$

Again let $\mathbf{x}$ be a point of X. Then

$$\begin{aligned} \mathbf{x} \in X \setminus \bigcap_{S \in \mathcal{A}} S &\iff \mathbf{x} \notin \bigcap_{S \in \mathcal{A}} S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \notin S \\ &\iff \text{there exists } S \in \mathcal{A} \text{ for which } \mathbf{x} \in X \setminus S \\ &\iff \mathbf{x} \in \bigcup_{S \in \mathcal{A}} (X \setminus S), \end{aligned}$$

and therefore

$$X \setminus \bigcap_{S\in\mathcal{A}} S = \bigcup_{S\in\mathcal{A}} (X \setminus S).$$

The following result therefore follows directly from Proposition 4.12.

Proposition 4.15 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) the empty set $\emptyset$ and the whole set X are both closed in X;

(ii) the intersection of any collection of closed sets in X is itself closed in X;

(iii) the union of any finite *collection of closed sets in X is itself closed in X.*

Lemma 4.16 *Let X be a subset of $\mathbb{R}^n$, and let F be a subset of X which is closed in X. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of F which converges to a point $\mathbf{p}$ of X. Then $\mathbf{p} \in F$.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 4.14 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

4.8 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{x}$ of X satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Thus the function $f\colon X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f\colon X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 4.17 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f\colon X \to Y$ be a function from X to Y. The function f is continuous if and only if $f^{-1}(V)$ is open in X for every open subset V of Y.*

Proof Suppose that $f\colon X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_Y(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 4.10, hence $f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}\left(B_Y(f(\mathbf{p}), \varepsilon)\right)$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

4.9 Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p} \in \mathbb{R}^m$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

It follows easily from the definition of convergence of sequences of points in Euclidean space that if X is a subset of m-dimensional Euclidean space $\mathbb{R}^m$ and if $\mathbf{p}$ is a point of $\mathbb{R}^m$ then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point in $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Lemma 4.18 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of Y, let $f: X \to Y$ be a function satisfying $f(X) \subset Y$, and let $g: Y \to \mathbb{R}^k$ be a function from Y to $\mathbb{R}^k$. Suppose that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and that the function g is continuous at $\mathbf{q}$. Then

$$\lim_{\mathbf{x} \to \mathbf{p}} g(f(\mathbf{x})) = g(\mathbf{q}).$$

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - \mathbf{q}| < \eta$, because the function g is continuous at $\mathbf{q}$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(\mathbf{q})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$, and thus

$$\lim_{\mathbf{x} \to \mathbf{p}} g(\mathbf{x}) = g(\mathbf{q}),$$

as required. ∎

Proposition 4.19 *Let X be a subset of $\mathbb{R}^m$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ be a point of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}^n$ has the property that*

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if

$$\lim_{\mathbf{x} \to \mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$, where $f_1, f_2, \ldots, f_n$ are the components of the function f and $\mathbf{q} = (q_1, q_2, \ldots, q_n)$.

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. It therefore follows from Lemma 4.18 that

$$\lim_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

then

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$.

Conversely suppose that

$$\lim_{\mathbf{x}\to\mathbf{p}} f_i(\mathbf{x}) = q_i$$

for $i = 1, 2, \ldots, n$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $0 < |f_i(\mathbf{x}) - q_i| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - q_i|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$. Thus

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q},$$

as required. ∎

Proposition 4.20 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Proof The result follows directly on comparing the relevant definitions. ∎

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x} - \mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$ satisfying $\mathbf{x} \neq \mathbf{p}$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

Proposition 4.21 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional*

Euclidean space $\mathbb{R}^n$*, let* $\mathbf{p}$ *be a limit point of* X*, and let* $\mathbf{q}$ *and* $\mathbf{r}$ *be points of* $\mathbb{R}^n$*. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r}.$$

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$ and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon$$

whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_2$.

Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - \mathbf{q}| < \tfrac{1}{2}\varepsilon$$

and

$$|g(\mathbf{x}) - \mathbf{r}| < \tfrac{1}{2}\varepsilon,$$

and therefore

$$\begin{aligned} |f(\mathbf{x}) + g(\mathbf{x}) - (\mathbf{q} + \mathbf{r})| &\leq |f(\mathbf{x}) - \mathbf{q}| + |g(\mathbf{x}) - \mathbf{r}| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) = \mathbf{q} + \mathbf{r},$$

as required. ∎

Definition Let $f\colon X \to \mathbb{R}^n$ be a function mapping some subset X of m-dimensional Euclidean space $\mathbb{R}^m$ into $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. We say that $f(\mathbf{x})$ *remains bounded* as $\mathbf{x}$ tends to $\mathbf{p}$ in X if strictly positive constants C and δ can be determined so that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Proposition 4.22 *Let $f: X \to \mathbb{R}^m$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \Bigl(h(\mathbf{x})f(\mathbf{x})\Bigr) = \mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined so that $|h(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|h(\mathbf{x})| \leq C$ and $|f(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.23 *Let $f: X \to \mathbb{R}^m$ be a function mapping some subset X of $\mathbb{R}^m$ into $\mathbb{R}^n$, let $h: X \to \mathbb{R}$ be a real-valued function on X, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = 0$. Suppose also that $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) = \mathbf{0}.$$

Proof Let some strictly positive real number ε be given. Now $f(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|f(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|h(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| \leq C$ and $|h(\mathbf{x})| < \varepsilon_0$, and therefore

$$|h(\mathbf{x})f(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.24 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a limit point of X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{0}$. Suppose also that $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{s}$ in X. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) \cdot g(\mathbf{x}) \Big) = 0.$$

Proof Let some strictly positive real number ε be given. Now $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X, and therefore positive constants C and δ_0 can be determined such that $|g(\mathbf{x})| \leq C$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. A strictly positive real number ε_0 can then be chosen small enough to ensure that $C\varepsilon_0 < \varepsilon$. There then exists a strictly positive real number δ_1 that is small enough to ensure that $|f(\mathbf{x})| < \varepsilon_0$ whenever $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and if $0 < |\mathbf{x} - \mathbf{p}| < \delta$ then $|f(\mathbf{x})| < \varepsilon_0$ and $|g(\mathbf{x})| \leq C$. It then follows from Schwarz's Inequality (Proposition 4.1) that

$$|f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{x})|\,|g(\mathbf{x})| < C\varepsilon_0 < \varepsilon.$$

The result follows. ∎

Proposition 4.25 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $h\colon X \to \mathbb{R}$ be a real-valued function on X, let $\mathbf{p}$ be a limit point of X, let $\mathbf{q}$ be a point of $\mathbb{R}^n$ and let s be a real number. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = s.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) f(\mathbf{x}) = s\mathbf{q}.$$

Proof The functions f and h satisfy the equation

$$h(\mathbf{x}) f(\mathbf{x}) = h(\mathbf{x}) \Big(f(\mathbf{x}) - \mathbf{q} \Big) + (h(\mathbf{x}) - s)\mathbf{q} + s\mathbf{q},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) - \mathbf{q} \Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Big(h(\mathbf{x}) - s \Big) = 0.$$

Moreover there exists a strictly positive constant δ_0 such that $|h(\mathbf{x}) - s| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle

Inequality that $|h(\mathbf{x})| < |s| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $h(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) = \mathbf{0}$$

(see Proposition 4.23). Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x}) - s)\,\mathbf{q} = \mathbf{0}.$$

It follows that

$$\begin{aligned} & \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})f(\mathbf{x})) \\ = \; & \lim_{\mathbf{x}\to\mathbf{p}} (h(\mathbf{x})(f(\mathbf{x}) - \mathbf{q})) + \lim_{\mathbf{x}\to\mathbf{p}} \Big(\Big(h(\mathbf{x}) - s\Big)\mathbf{q}\Big) + s\mathbf{q} \\ = \; & \mathbf{0} + s\mathbf{q}, \end{aligned}$$

as required. ∎

Proposition 4.26 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ and $g\colon X \to \mathbb{R}^n$ be functions mapping X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of X, and let $\mathbf{q}$ and $\mathbf{r}$ be points of $\mathbb{R}^n$. Suppose that*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

and

$$\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) = \mathbf{r}.$$

Then

$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) \cdot g(\mathbf{x})) = \mathbf{q} \cdot \mathbf{r}.$$

Proof The functions f and g satisfy the equation

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = (f(\mathbf{x}) - \mathbf{q}) \cdot g(\mathbf{x}) + \mathbf{q} \cdot (g(\mathbf{x}) - \mathbf{r}) + \mathbf{q} \cdot \mathbf{r},$$

where

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(f(\mathbf{x}) - \mathbf{q}\Big) = \mathbf{0} \quad \text{and} \quad \lim_{\mathbf{x}\to\mathbf{p}} \Big(g(\mathbf{x}) - \mathbf{r}\Big) = \mathbf{0}.$$

Moreover there exists a strictly positive constant δ_0 such that $|g(\mathbf{x}) - \mathbf{r}| < 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. But it then follows from the Triangle Inequality that $|g(\mathbf{x})| < |\mathbf{r}| + 1$ for all $\mathbf{x} \in X$ satisfying $0 < |\mathbf{x} - \mathbf{p}| < \delta_0$. Thus $g(\mathbf{x})$ remains bounded as $\mathbf{x}$ tends to $\mathbf{p}$ in X. It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(\Big(f(\mathbf{x}) - \mathbf{q}\Big) \cdot g(\mathbf{x})\Big) = 0$$

(see Proposition 4.24). Similarly

$$\lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q}\cdot\Big(g(\mathbf{x})-\mathbf{r}\Big)\Big) = 0.$$

It follows that

$$\begin{aligned} & \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})\cdot g(\mathbf{x})) \\ = {} & \lim_{\mathbf{x}\to\mathbf{p}} \Big(\Big(f(\mathbf{x})-\mathbf{q}\Big)\cdot g(\mathbf{x})\Big) + \lim_{\mathbf{x}\to\mathbf{p}} \Big(\mathbf{q}\cdot\Big(g(\mathbf{x})-\mathbf{r}\Big)\Big) + \mathbf{q}\cdot\mathbf{r} \\ = {} & \mathbf{q}\cdot\mathbf{r}, \end{aligned}$$

as required. ∎

Proposition 4.27 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X\to\mathbb{R}$ and $g\colon X\to\mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))$, $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))$ and $\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, and moreover*

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\ \lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\ \lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \end{aligned}$$

If moreover $g(\mathbf{x})\neq 0$ for all $\mathbf{x}\in X$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

First Proof It follows from Proposition 4.21 (applied in the case when the target space is one-dimensional) that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Replacing the function g by $-g$, we deduce that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

It follows from Proposition 4.25 (applied in the case when the target space is one-dimensional), or alternatively from Proposition 4.26, that

$$\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}).$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Let $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be the reciprocal function defined so that $r(t) = 1/t$ for all non-zero real numbers t. Then the reciprocal function r is continuous. Applying the result of Lemma 4.18, we find that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{g(\mathbf{x})} = \lim_{\mathbf{x}\to\mathbf{p}} r(g(\mathbf{x})) = r\left(\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})\right) = \frac{1}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

It follows that

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})},$$

as required. ∎

Second Proof Let $l = \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $m = \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$, and let $h\colon X \to \mathbb{R}^2$ be defined such that

$$h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$$

for all $\mathbf{x} \in X$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x}) = (l, m)$$

(see Proposition 4.19).

Let $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ be the functions from $\mathbb{R}^2$ to $\mathbb{R}$ defined such that $s(u, v) = u + v$ and $m(u, v) = uv$ for all $u, v \in \mathbb{R}$. Then the functions s and m are continuous (see Lemma 4.7). Also $f + g = s \circ h$ and $f \cdot g = m \circ f$. It follows from this that

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) + g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} s(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} s(h(\mathbf{x})) \\ &= s\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = s(l, m) = l + m, \end{aligned}$$

and

$$\begin{aligned} \lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} m(f(\mathbf{x}), g(\mathbf{x})) = \lim_{\mathbf{x}\to\mathbf{p}} m(h(\mathbf{x})) \\ &= m\left(\lim_{\mathbf{x}\to\mathbf{p}} h(\mathbf{x})\right) = m(l, m) = lm \end{aligned}$$

(see Lemma 4.18).

Also

$$\lim_{\mathbf{x}\to\mathbf{p}} (-g(\mathbf{x})) = -m.$$

It follows that
$$\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x}) - g(\mathbf{x})) = l - m.$$

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and that $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$. Representing the function sending $\mathbf{x} \in X$ to $1/g(\mathbf{x})$ as the composition of the function g and the reciprocal function $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, where $r(t) = 1/t$ for all non-zero real numbers t, we find, as in the first proof, that the function sending each point $\mathbf{x}$ of X to

$$\lim_{\mathbf{x}\to\mathbf{p}} \left(\frac{1}{g(\mathbf{x})}\right) = \frac{1}{m}.$$

It then follows that
$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{l}{m},$$
as required. ∎

Proposition 4.28 *Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ and $g: Y \to \mathbb{R}^k$ be functions satisfying $f(X) \subset Y$. Let $\mathbf{p}$ be a limit point of X in $\mathbb{R}^m$, let $\mathbf{q}$ be a limit point of Y in $\mathbb{R}^n$ let $\mathbf{r}$ be a point of $\mathbb{R}^k$. Suppose that the following three conditions are satisfied:*

(i) $\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$;

(ii) $\lim_{\mathbf{y}\to\mathbf{q}} g(\mathbf{y}) = \mathbf{r}$;

(iii) there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_0$.

Then
$$\lim_{\mathbf{x}\to\mathbf{p}} g(f(\mathbf{x})) = \mathbf{r}.$$

Proof Let some positive real number ε be given. Then there exists some positive real number η such that $|g(\mathbf{y}) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{y} \in Y$ satisfies $0 < |\mathbf{y} - \mathbf{q}| < \eta$. There then exists some positive real number δ_1 such that $|f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta_1$. Also there exists some positive real number δ_0 such that $f(\mathbf{x}) \neq \mathbf{q}$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta_0$. Let δ be the minimum of δ_0 and δ_1. Then $\delta > 0$, and $0 < |f(\mathbf{x}) - \mathbf{q}| < \eta$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. But this then ensures that $|g(f(\mathbf{x})) - \mathbf{r}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$. The result follows. ∎

4.10 Limits and Neighbourhoods

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X. A subset N of X is said to be a *neighbourhood* of $\mathbf{p}$ in X if there exists some strictly positive real number δ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta\} \subset N.$$

Lemma 4.29 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of X that is not an isolated point of X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then*

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in X such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$.

Proof This result follows directly from the definitions of limits and neighbourhoods. ∎

Remark Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a limit point of X that does not belong to X. Let $f\colon X \to \mathbb{R}^n$ be a function mapping X into some Euclidean space $\mathbb{R}^n$, and let $\mathbf{q} \in \mathbb{R}^n$. Then

$$\lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) = \mathbf{q}$$

if and only if, given any positive real number ε, there exists a neighbourhood N of $\mathbf{p}$ in $X \cup \{\mathbf{p}\}$ such that

$$|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$$

for all points $\mathbf{x}$ of N that satisfy $\mathbf{x} \neq \mathbf{p}$. Thus the result of Lemma 4.29 can be extended so as to apply to limits of functions taken at limit points of the domain that do not belong to the domain of the function.

5 Compact Subsets of Euclidean Spaces

5.1 The Multidimensional Bolzano-Weierstrass Theorem

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be *bounded* if there exists some constant K such that $|\mathbf{x}_j| \leq K$ for all j.

Example Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\ (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between -1 and 1. Moreover $x_j = 0$ whenever j is the square of a positive integer, $y_j = 1$ whenever j is even and $z_j = 1$ whenever j is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those x_j for which j is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and -1. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those x_j for which j is the square of an even positive integer.

The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when j is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer p.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those x_j for which j is equal to two raised to the power of an even positive integer. Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4),\ \ (x_{16}, y_{16}, z_{16}),\ \ (x_{64}, y_{64}, z_{64}),\ \ (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and it therefore follows from Lemma 4.3 that this sequence is a convergent subsequence of the given sequence of points in $\mathbb{R}^3$.

Example Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$. Then the first components x_j for which the index j is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components y_j for which the index j is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices j for which x_j is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and y_j is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index j for which x_j and y_j both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.3) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded sequence of points in $\mathbb{R}^2$.

Theorem 5.1 (The Multidimensional Bolzano-Weierstrass Theorem)
Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.

Proof We prove the result by induction on the dimension n of the Euclidean space $\mathbb{R}^n$ that contains the infinite sequence in question. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.3) that the theorem is true when $n = 1$. Suppose that $n > 1$, and that every bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of elements of $\mathbb{R}^n$, and let $x_{j,i}$ denote the ith component of $\mathbf{x}_j$ for $i = 1, 2, \ldots, n$ and for all positive integers j. The induction hypothesis requires that all bounded sequences in $\mathbb{R}^{n-1}$ contain convergent subsequences. It follows that there exist real numbers $p_1, p_2, \ldots, p_{n-1}$ and an increasing sequence $m_1, m_2, m_3, \ldots$ of positive integers such that $\lim_{k\to+\infty} x_{m_k,i} = p_i$ for $i = 1, 2, \ldots, n-1$. The nth components

$$x_{m_1,n}, x_{m_2,n}, x_{m_3,n}, \ldots$$

of the members of the subsequence

$$\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$$

then constitute a bounded sequence of real numbers. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 1.3) that there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers for which the sequence

$$x_{m_{k_1},n}, x_{m_{k_2},n}, x_{m_{k_3},n}, \ldots$$

converges.

Let $s_j = m_{k_j}$ for all positive integers j, and let

$$p_n = \lim_{j\to+\infty} x_{m_{k_j},n} = \lim_{j\to+\infty} x_{s_j,n}.$$

Then the sequence $x_{s_1,i}, x_{s_2,i}, x_{s_3,i}, \ldots$ converges for values of i between 1 and $n-1$, because it is a subsequence of the convergent sequence

$$x_{m_1,i}, x_{m_2,i}, x_{m_3,i}, \ldots.$$

Moreover

$$x_{s_1,n}, x_{s_2,n}, x_{s_3,n}, \ldots$$

also converges. Thus the ith components of the infinite sequence

$$\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$$

converge for $i = 1, 2, \ldots, n$. It then follows from Lemma 4.3 that

$$\lim_{j\to+\infty} \mathbf{x}_{s_k} = \mathbf{p},$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The result follows. ∎

5.2 Cauchy Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of n-dimensional Euclidean space $\mathbb{R}^n$ is said to be a *Cauchy sequence* if the following condition is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 5.2 *Every Cauchy sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is bounded.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence of points in $\mathbb{R}^n$. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|\mathbf{x}_j| \leq |\mathbf{x}_N| + 1$ whenever $j \geq N$. Therefore $|\mathbf{x}_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|\mathbf{x}_1|, |\mathbf{x}_2|, \ldots, |\mathbf{x}_{N-1}|$ and $|\mathbf{x}_N| + 1$. Thus the sequence is bounded, as required. ∎

Theorem 5.3 (Cauchy's Criterion for Convergence) *An infinite sequence of points of n-dimensional Euclidean space $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences in $\mathbb{R}^n$ are Cauchy sequences. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a convergent sequence of points in $\mathbb{R}^n$, and let $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ and $|\mathbf{x}_k - \mathbf{p}| < \frac{1}{2}\varepsilon$, and hence

$$|\mathbf{x}_j - \mathbf{x}_k| = |(\mathbf{x}_j - \mathbf{p}) - (\mathbf{x}_k - \mathbf{p})| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{x}_k - \mathbf{p}| < \varepsilon.$$

Thus the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in $\mathbb{R}^n$ is convergent. Now Cauchy sequences are bounded, by Lemma 5.2. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ therefore has a convergent subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$, by the multidimensional Bolzano-Weierstrass Theorem (Theorem 5.1). Let $\mathbf{p} = \lim_{j \to +\infty} \mathbf{x}_{k_j}$. We claim that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ itself converges to $\mathbf{p}$.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|\mathbf{x}_{k_m} - \mathbf{p}| < \frac{1}{2}\varepsilon$. Then

$$|\mathbf{x}_j - \mathbf{p}| \leq |\mathbf{x}_j - \mathbf{x}_{k_m}| + |\mathbf{x}_{k_m} - \mathbf{p}| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$, as required. ∎

5.3 The Multidimensional Extreme Value Theorem

Proposition 5.4 *Let X be a closed bounded set in m-dimensional Euclidean space, and let $f: X \to \mathbb{R}^n$ be a continuous function mapping X into n-dimensional Euclidean space $\mathbb{R}^n$. Then there exists a point $\mathbf{w}$ of X such that $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$.*

Proof Let $g: X \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = \frac{1}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Now the function mapping each $\mathbf{x} \in X$ to $|f(\mathbf{x})|$ is continuous (see Lemma 4.9) and quotients of continuous functions are continuous where they are defined (see Lemma 4.8). It follows that the function $g: X \to \mathbb{R}$ is continuous.

Let

$$m = \inf\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X such that

$$g(\mathbf{x}_j) < m + \frac{1}{j}$$

for all positive integers j. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 5.1) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{w}$ of $\mathbb{R}^n$.

Now the point $\mathbf{w}$ belongs to X because X is closed (see Lemma 4.16). Also

$$m \leq g(\mathbf{x}_{k_j}) < m + \frac{1}{k_j}$$

for all positive integers j. It follows that $g(\mathbf{x}_{k_j}) \to m$ as $j \to +\infty$. It then follows from Lemma 4.5 that

$$g(\mathbf{w}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = m.$$

Then $g(\mathbf{x}) \geq g(\mathbf{w})$ for all $\mathbf{x} \in X$, and therefore $|f(\mathbf{x})| \leq |f(\mathbf{w})|$ for all $\mathbf{x} \in X$, as required. ∎

Theorem 5.5 (The Multidimensional Extreme Value Theorem) *Let X be a closed bounded set in m-dimensional Euclidean space, and let $f: X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof It follows from Proposition 5.4 that the function f is bounded on X. It follows that there exists a real number C large enough to ensure that $f(\mathbf{x}) + C > 0$ for all $\mathbf{x} \in X$. It then follows from Proposition 5.4 that there exists some point $\mathbf{v}$ of X such that

$$f(\mathbf{x}) + C \leq f(\mathbf{v}) + C.$$

for all $\mathbf{x} \in X$. But then $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$. Applying this result with f replaced by $-f$, we deduce that there exists some $\mathbf{u} \in X$ such that $-f(\mathbf{x}) \leq -f(\mathbf{u})$ for all $\mathbf{x} \in X$. The result follows. ∎

5.4 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of $\mathbb{R}^m$. A function $f\colon X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 5.6 *Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f\colon X \to \mathbb{R}^n$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 5.1). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since $\lim\limits_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0$.

But then the sequences

$$f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots \quad \text{and} \quad f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$$

would both converge to $f(\mathbf{p})$, since f is continuous (Lemma 4.5), and thus

$$\lim_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0.$$

But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that

$$|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$$

for all j. We conclude therefore that there must exist some positive real number δ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

5.5 Lebesgue Numbers

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A collection of subsets of $\mathbb{R}^n$ is said to *cover* X if and only if every point of X belongs to at least one of these subsets.

Definition Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An *open cover* of X is a collection of subsets of X that are open in X and cover the set X.

Proposition 5.7 *Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property. Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then

$$B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

of points of X with the property that, for all positive integers j, the open ball

$$B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$$

for all members V of the open cover $\mathcal{V}$. The sequence

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$$

would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 5.1) that there would exist a convergent subsequence

$$\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$$

of

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots.$$

Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 4.16). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 5.7 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\operatorname{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\operatorname{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

A *hypercube* in n-dimensional Euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq u_i + l\},$$

where l is a positive constant that is the length of the edges of the hypercube and $(u_1, u_2, \ldots, u_n)$ is the point in $\mathbb{R}^n$ at which the Cartesian coordinates of points in the hypercube attain their minimum values. The diameter of a hypercube with edges of length l is $l\sqrt{n}$.

Lemma 5.8 *Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the* $\operatorname{diam}(A_i) < \delta$ *for* $i = 1, 2, \ldots, s$ *and*

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof The set X is bounded, and therefore there exists some positive real number M such that that if $(x_1, x_2, \ldots, x_n) \in X$ then $-M \leq x_j \leq M$ for $j = 1, 2, \ldots, n$. Choose k large enough to ensure that $2M/k < \delta_L/\sqrt{n}$. Then the large hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -M \leq x_j \leq M \text{ for } j = 1, 2, \ldots, n\}$$

can be subdivided into k^n hypercubes with edges of length l, where $l = 2M/k$. Each of the smaller hypercubes is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_j \leq x_j \leq u_j + l \text{ for } j = 1, 2, \ldots, n\},$$

where $(u_1, u_2, \ldots, u_n)$ is the corner of the hypercube at which the Cartesian coordinates have their minimum values. If $\mathbf{p}$ is a point belonging to such a small hypercube, then all points of the hypercube lie within a distance $l\sqrt{n}$ of the point $\mathbf{p}$. It follows that the small hypercube is wholly contained within the open ball $B_{\mathbb{R}^n}(\mathbf{p}, \delta_L)$ of radius δ about the point $\mathbf{p}$.

Now the number of small hypercubes resulting from the subdivision is finite. Let $H_1, H_2, \ldots, H_s$ be a listing of the small hypercubes that intersect the set X, and let $A_i = H_i \cap X$. Then $\operatorname{diam}(H_i) \leq \sqrt{n}l < \delta_L$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

as required. ∎

Definition Let $\mathcal{V}$ and $\mathcal{W}$ be open covers of some subset X of a Euclidean space. Then $\mathcal{W}$ is said to be a *subcover* of $\mathcal{V}$ if and only if every open set belonging to $\mathcal{W}$ also belongs to $\mathcal{V}$.

Definition A subset X of a Euclidean space is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Theorem 5.9 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof Let X be a compact subset of $\mathbb{R}^n$ and let

$$V_j = \{\mathbf{x} \in X : |\mathbf{x}| < j\}$$

for all positive integers j. Then the sets $V_1, V_2, V_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset V_{j_1} \cup V_{j_2} \cup \cdots \cup V_{j_k}.$$

Let M be the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x}| \leq M$ for all $\mathbf{x} \in X$. Thus the set X is bounded.

Let $\mathbf{q}$ be a point of $\mathbb{R}^n$ that does not belong to X, and let

$$W_j = \left\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{q}| > \frac{1}{j}\right\}$$

for all positive integers j. Then the sets $W_1, W_2, W_3, \ldots$ constitute an open cover of X. This open cover has a finite subcover, and therefore there exist positive integers $j_1, j_2, \ldots, j_k$ such that

$$X \subset W_{j_1} \cup W_{j_2} \cup \cdots \cup W_{j_k}.$$

Let $\delta = 1/M$, where M is the largest of the positive integers $j_1, j_2, \ldots, j_k$. Then $|\mathbf{x} - \mathbf{q}| \geq \delta$ for all $\mathbf{x} \in X$ and thus the open ball of radius δ about the point $\mathbf{q}$ does not intersect the set X. We conclude that the set X is closed. We have now shown that every compact subset of $\mathbb{R}^n$ is both closed and bounded.

We now prove the converse. Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 5.7 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 5.8 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$.

Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. $\blacksquare$

6 The Multidimensional Riemann Integral

6.1 Rectangles and Partitions

Let X_i be a subset of $\mathbb{R}$ for $i = 1, 2, \ldots, n$, where n is some positive integer. The Cartesian product

$$X_1 \times X_2 \times \cdots \times X_n$$

of the sets $X_1, X_2, \ldots, X_n$ is the subset of $\mathbb{R}^n$ defined such that

$$\begin{aligned} & X_1 \times X_2 \times \cdots \times X_n \\ & \quad = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \in X_i \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

We use the notation

$$\prod_{i=1}^{n} X_i$$

to denote the Cartesian product $X_1 \times X_2 \times \cdots \times X_n$ of sets $X_1, X_2, \ldots, X_n$.

Definition We define a *closed n-dimensional rectangle* in n-dimensional Euclidean space $\mathbb{R}^n$ to be Cartesian product of closed intervals in the real line.

A closed n-dimensional rectangle can thus be represented as a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\},$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers such that $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

An n-dimensional rectangle may be referred to as an *n-rectangle*.

The *interior* of the closed *n-rectangle*

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}$$

is the open set

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i < x_i < b_i \text{ for } i = 1, 2, \ldots, n\},$$

for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$.

In other words, the interior of the closed n-rectangle $\prod_{i=1}^{n} [a_i, b_i]$ is the open set $\prod_{i=1}^{n} (a_i, b_i)$.

Definition Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers, where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let K be the n-rectangle defined so that

$$S = \{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}.$$

The *volume* (or *content*) $v(K)$ of K is defined so that

$$v(S) = \prod_{i=1}^{n} (b_i - a_i) = (a_1 - b_1)(a_1 - b_2) \cdots (a_n - b_n).$$

Let a and b be real numbers, where $a \leq b$. A partition of the closed interval $[a, b]$ is represented as a finite set P which includes the endpoints a and b of the interval and whose elements belong to the interval. The elements of such a partition P can be listed as $x_0, x_1, x_2, \ldots, x_m$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_m = b.$$

Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let P_i be a partition of the closed interval $[a_i, b_i]$ for each i. We can then write

$$P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\},$$

where

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i$$

for $i = 1, 2, \ldots, n$ and $j = 0, 1, \ldots, m(i)$. Let K be the closed n-rectangle defined so that $K = \prod_{i=1}^{n} [a_i, b_i]$. Then the partitions $P_1, P_2, \ldots, P_n$ of the closed intervals

$$[a_1, b_1], \;\; [a_2, b_2], \ldots \;\; [a_n, b_n]$$

determine a partition $P_1 \times P_2 \times \cdots \times P_n$ of the n-rectangle K as a union of smaller closed n-rectangles $K_{j_1, j_2, \ldots, j_n}$, where j_i is an integer between 1 and $m(i)$ for $i = 1, 2, \ldots, n$, and where, for given integers $j_1, j_2, \ldots, j_n$ satisfying $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$, the closed n-rectangle $K_{j_1, j_2, \ldots, j_n}$ is defined so that

$$K_{j_1, j_2, \ldots, j_n} = \prod_{i=1}^{n} [x_{i, j_i - 1}, x_{i, j_i}].$$

Definition Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let K be the closed n-rectangle in $\mathbb{R}^n$ defined such that $K = \prod_{i=1}^{n} [a_i, b_i]$.

A *partition* of K is the decomposition of K as a union of closed n-rectangles $K_{j_1,j_2,\ldots,j_n}$ that is determined by partitions $P_1, P_2, \ldots, P_n$ of

$$[a_1, b_1], [a_2, b_2], \ldots [a_n, b_n]$$

respectively, where, for each integer i between 1 and n, the partition P_i is representable in the form

$$P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\}$$

for real numbers $x_{i,0}, x_{i,1}, x_{i_2}, \ldots, x_{i,m(i)}$ that satisfy

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i,$$

and where $K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$ for all integers $j_1, j_2, \ldots, j_n$ that satisfy $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$.

Proposition 6.1 *Let a_i and b_i be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let $K = \prod_{i=1}^{n} [a_i, b_i]$. Let the partition P_i of $[a_i, b_i]$ be represented in the form $P_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,m(i)}\}$ for $i = 1, 2, \ldots, n$, where*

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i.$$

Then the volume $v(K)$ of the n-rectangle K satisfies

$$v(K) = \sum_{j_1=1}^{m(1)} \sum_{j_2=2}^{m(2)} \cdots \sum_{j_n=1}^{m(n)} v(K_{j_1,j_2,\ldots,j_n}),$$

where $K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$ for all n-tuples $(j_1, j_2, \ldots, j_n)$ of integers satisfying $1 \leq j_i \leq m(i)$ for $i = 1, 2, \ldots, n$.

Proof We must prove that

$$\prod_{i=1}^{n} (b_i - a_i) = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_n=1}^{m(n)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

First we note that

$$b_n - a_n = \sum_{i_n=1}^{m(n)} (x_{n,j_n} - x_{n,j_n-1}).$$

It follows directly that the result holds in the case when $n = 1$.

Suppose that $n > 1$ and that the result is known to hold for all partitions of $(n-1)$-dimensional rectangles in $\mathbb{R}^{n-1}$. Applying the result to the rectangle $\prod_{i=1}^{n-1} [a_i, b_i]$ in $\mathbb{R}^{n-1}$, we find that

$$\begin{aligned} &\prod_{i=1}^{n-1} (b_i - a_i) \\ &\quad = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_{n-1}=1}^{m(n-1)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n-1,j_{n-1}} - x_{n-1,j_{n-1}-1}). \end{aligned}$$

It follows that

$$\prod_{i=1}^{n} (b_i - a_i) = \sum_{j_1=1}^{m(1)} \cdots \sum_{j_n=1}^{m(n)} (x_{1,j_1} - x_{1,j_1-1}) \cdots (x_{n,j_n} - x_{n,j_n-1}).$$

Thus if the result holds all partitions of $(n-1)$-dimensional rectangles in $\mathbb{R}^{n-1}$ then it also holds for all partitions of n-dimensional rectangles in $\mathbb{R}^n$. The result follows. ∎

We now introduce "multi-index" notation in order to reduce the complexity of the notation involved in analysing the properties of n-dimensional rectangles, partitions of such rectangles, and of real-valued functions defined on such rectangles.

Let K be an closed n-dimensional closed rectangle in $\mathbb{R}^n$, let

$$[a_1, b_1], \;\; [a_2, b_2], \ldots, \;\; [a_n, b_n]$$

be closed intervals such that $K = \prod_{i=1}^{n} [a_i, b_i]$, where $a_i \leq b_i$ for $i = 1, 2, \ldots, n$, and let P be a partition of K. Then there exists a partition P_i of $[a_i, b_i]$ for $i = 1, 2, \ldots, n$ such that

$$P = P_1 \times P_2 \times \cdots \times P_k.$$

We let $P_i = \{x_{i,0}, x_{i,1}, x_{i,2}, \ldots, x_{i,m(i)}\}$ for $i = 1, 2, \ldots, n$, where $x_{i,0}, x_{i,1}, x_{i_2}, \ldots, x_{i,m(i)}$ are real numbers that satisfy

$$a_i = x_{i,0} < x_{i,1} < x_{i,2} < \cdots < x_{i,m(i)} = b_i.$$

Now the n-rectangle K is the union of smaller n-rectangles $K_{j_1,j_2,\ldots,j_n}$ determined by the partition P, where

$$K_{j_1,j_2,\ldots,j_n} = \prod_{i=1}^{n} [x_{i,j_i-1}, x_{i,j_i}]$$

for $i = 1, 2, \ldots, n$ and $j_i = 1, 2, \ldots, m(i)$. We refer to these n-rectangles $K_{j_1,j_2,\ldots,j_n}$ as the *cells* determined by the partition P of the rectangle K. Each cell $K_{j_1,j_2,\ldots,j_n}$ is identified by a "multi-index" $(j_1, j_2, \ldots, j_n)$. Such "multi-indices" are typically denoted by Greek letters $\alpha, \beta, \gamma, \ldots$.

Accordingly we let

$$\Omega(P) = \{(j_1, j_2, \ldots, j_n) : i = 1, 2, \ldots, n \text{ and } j_i = 1, 2, \ldots, m(i)\}.$$

Then $\Omega(P)$ is the set consisting of the multi-indices that identify cells of the partition P. Given a multi-index α, where $\alpha = (j_1, j_2, \ldots, j_n)$ for some $(j_1, j_2, \ldots, j_n) \in \Omega(P)$, we can denote by $K_{P,\alpha}$ the cell $K_{j_1,j_2,\ldots,j_n}$ of the partition corresponding to the multi-index α. The result Proposition 6.1 can then be expressed by the identity

$$v(K) = \sum_{\alpha \in \Omega(P)} v(K_{P,\alpha})$$

where $v(K)$ denotes the volume of the rectangle K and $v(K_{P,\alpha})$ denotes the volume of the cell $K_{P,\alpha}$ for all $\alpha \in \Omega(P)$.

Definition Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. We say that the partition R is a *refinement* of P if every cell of the partition R is contained within a cell of the partition P.

Lemma 6.2 *Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. Let the partition P represent K as a union of cells $K_{P,\alpha}$, where the index α ranges over an indexing set $\Omega(P)$, and where the interiors of the cells are disjoint. Similarly let the partition R represent K as a union of cells $K_{R,\beta}$, where the index β ranges over an indexing set $\Omega(R)$, and where the interiors of the cells are disjoint. Suppose that the partition R is a refinement of the partition P. Then there is a well-defined function $\lambda \colon \Omega(R) \to \Omega(P)$ characterized by the requirement that, for every $\beta \in \Omega(R)$, the cell $K_{P,\lambda(\beta)}$ of the partition P is the unique cell of that partition for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$.*

Proof The definition of the cells of the partitions P and R ensures that the interiors of these cells are non-empty. Moreover if a cell $K_{R,\beta}$ of the refinement R is contained in a cell $K_{P,\alpha}$ of the partition P then the interior of $K_{R,\beta}$ is contained in the interior of $K_{P,\alpha}$. But the interiors of the cells of the partition P are disjoint, and therefore the interior of $K_{R,\beta}$ cannot intersect the interiors of two or more cells of the partition P. Therefore $K_{R,\beta}$ can be contained in at most one cell of the partition P. But the definition of refinements ensures that $K_{R,\beta}$ is contained in the interior of at least one cell of the partition P. The result follows. ∎

Lemma 6.3 *Let K be an n-dimensional rectangle in $\mathbb{R}^n$, and let P and Q be partitions of K. Then there exists a partition R of K that is a common refinement of the partitions P and Q.*

Proof Let $K = \prod_{i=1}^n [a_i, b_i]$, where a_i and b_i are real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. It follows from the definition of partitions that there exist partitions P_i and Q_i of the closed bounded interval $[a_i, b_i]$ for $i = 1, 2, \ldots, n$ such that

$$P = P_1 \times P_2 \times \cdots \times P_n$$

and

$$Q = Q_1 \times Q_2 \times \cdots \times Q_n.$$

For each i, P_i and Q_i are finite sets containing the endpoints a_i and b_i of the interval whose other elements all belong to the interval. Let $R_i = P_i \cup Q_i$ for $i = 1, 2, \ldots, n$, and let

$$R = R_1 \times R_2 \times \cdots \times R_n.$$

Then R is a partition of K that is a common refinement of the partitions P and Q of K. The result follows. ∎

6.2 Multidimensional Darboux Sums

Let $f\colon K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$. A partition P of the n-rectangle K represents K as the union of a collection

$$\{K_{P,\alpha} : \alpha \in \Omega(P)\}$$

of n-rectangles contained in K. The interior of each of these n-rectangles is a non-empty open set in $\mathbb{R}^n$, and distinct n-rectangles in this collection intersect, if at all, only along their boundaries. Thus each point of K belongs to the interior of at most one rectangle in the collection

$$\{K_{P,\alpha} : \alpha \in \Omega(P)\}.$$

Also the volume $v(K)$ of the n-dimensional rectangle K is the sum of the volumes of the cells of the partition, and thus

$$v(K) = \sum_{\alpha \in \Omega(P)} K_{P,\alpha}.$$

Let K be an n-dimensional rectangle in $\mathbb{R}^n$ and let P and R be partitions of K. Then the partition R is a refinement of P if every cell of the partition R is contained within a cell of the partition P. We have shown that if the partition R of K is a refinement of a partition P of K, and if the cells of the partitions P and R of K are indexed by indexing sets $\Omega(P)$ and $\Omega(R)$ respectively, then there is a well-defined function $\lambda \colon \Omega(R) \to \Omega(P)$ characterized by the property that, for each $\beta \in \Omega(P)$, the cell $K_{P,\lambda(\beta)}$ is the unique cell of the partition P for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$ (see Lemma 6.2). We have also shown that, given any two partitions P and Q of K, there exists a partition R of K that is a common refinement of P and Q. (see Lemma 6.3.)

Remark The previous discussion contains more details regarding how the partition of K is implemented, and how the cells of the partition are constructed, and how they can be indexed. The results just described will be essential in the following discussion. But the details of how the cells of the partition are indexed is immaterial to the following discussion, and we could at this point choose an ordering of the cells of a given partition, and use this ordering to represent the indexing set $\Omega(P)$ associated with a partition P of an n-dimensional rectangle K as a set of consecutive integers indexing the cells of the partition P in accordance with the chosen ordering of those cells. The cells determined by the partition P of K could then be denoted as

$$K_{P,1}, \;\; K_{P,2}, \;\; \cdots K_{P,r},$$

where r is the number of cells resulting from the partition P of K.

Definition Let $f \colon K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, let P be a partition of K, and let the cells of this partition be indexed by the set $\Omega(P)$. For each element α of the indexing set $\Omega(P)$, let $K_{P,\alpha}$ denotes the cell of the partition indexed by α, let $v(K_P, \alpha)$ denote the volume of $K_{P,\alpha}$, and let

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

Then the Darboux lower sum $L(P, f)$ and the Darboux upper sum $U(P, f)$ are defined by the formulae

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}),$$

Let $f: K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$. Then the definition of the Darboux lower and upper sums ensures that $L(P, f) \leq U(P, f)$ for all partitions P of the n-rectangle K.

Lemma 6.4 *Let $f: K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, and let P and R be partitions of K. Suppose that R is a refinement of P. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f).$$

Proof Let the cells of the partitions P and R be indexed by indexing sets $\Omega(P)$ and $\omega(R)$ respectively. Also, for each $\alpha \in \Omega(P)$, let $K_{P,\alpha}$ be the cell of the partition P determined by α, and, for each $\beta \in \Omega(R)$, let $K_{R,\beta}$ be the cell of the partition R determined by β. Then there is a well-defined function $\lambda: \Omega(R) \to \Omega(P)$ characterized by the requirement that, for every $\beta \in \Omega(R)$, the cell $K_{P,\lambda(\beta)}$ of the partition P is the unique cell of that partition for which $K_{R,\beta} \subset K_{P,\lambda(\beta)}$ (see Lemma 6.2). Now

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha}), \\ U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}), \end{aligned}$$

where

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

for all $\alpha \in \Omega(P)$. Similarly

$$\begin{aligned} L(R, f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, v(K_{R,\beta}), \\ U(R, f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, v(K_{R,\beta}), \end{aligned}$$

where

$$m_{R,\beta} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

and

$$M_{R,\beta} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

for all $\beta \in \Omega(R)$.

Now

$$m_{R,\beta} \geq m_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, because $K_{R,\beta} \subset K_{P,\lambda(\beta)}$. Also the partition R of K determines a partition of each cell $K_{P,\alpha}$ of that partition P, decomposing the cell $K_{P,\alpha}$ as a union of the sets $K_{R,\beta}$ for which $\lambda(\beta) = \alpha$. It follows that

$$K_{P,\alpha} = \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta})$$

where

$$\Omega(R;\alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$ (see Proposition 6.1). Therefore

$$\begin{aligned} L(R,f) &= \sum_{\beta \in \Omega(R)} m_{R,\beta}\, v(K_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} m_{R,\beta}\, v(K_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha}) \\ &= L(P,f). \end{aligned}$$

An analogous argument applies to upper sums. Now

$$M_{R,\beta} \geq M_{P,\lambda(\beta)}$$

for all $\beta \in \Omega(R)$, where

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

for all $\alpha \in \Omega(P)$ and

$$M_{R,\beta} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{R,\beta}\}$$

for all $\beta \in \Omega(R)$, because $K_{R,\beta} \subset K_{P,\lambda(\beta)}$. Also

$$K_{P,\alpha} = \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta})$$

where

$$\Omega(R;\alpha) = \{\beta \in \Omega(R) : \lambda(\beta) = \alpha\}$$

for all $\alpha \in \Omega(P)$, as before. Therefore

$$\begin{aligned} U(R,f) &= \sum_{\beta \in \Omega(R)} M_{R,\beta}\, v(K_{R,\beta}) \\ &= \sum_{\alpha \in \Omega(P)} \sum_{\beta \in \Omega(R;\alpha)} M_{R,\beta}\, v(K_{R,\beta}) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha} \sum_{\beta \in \Omega(R;\alpha)} v(K_{R,\beta}) \\ &\geq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}) \\ &= U(P,f). \end{aligned}$$

This completes the proof. ∎

Lemma 6.5 *Let $f \colon K \to \mathbb{R}$ be a bounded real-valued function defined on an n-dimensional rectangle K in $\mathbb{R}^n$, and let P and Q be partitions of K. Then then the Darboux sums of the function f for the partitions P and Q satisfy $L(P,f) \leq U(Q,f)$.*

Proof There exists a partition R of K that is a common refinement of the partitions P and Q of K. (Lemma 6.3.) Moreover $L(R,f) \geq L(P,f)$ and $U(R,f) \leq U(Q,f)$ (Lemma 6.4). It follows that

$$L(P,f) \leq L(R,f) \leq U(R,f) \leq U(Q,f),$$

as required. ∎

6.3 The Multidimensional Riemann-Darboux Integral

Definition Let K be an n-dimensional rectangle in $\mathbb{R}^n$, and let $f \colon K \to \mathbb{R}$ be a bounded real-valued function on K. The *lower Riemann integral* and the *upper Riemann integral*, denoted by

$$\mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \quad \text{and} \quad \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n$$

respectively, are defined such that

$$\begin{aligned}
&\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\quad = \sup\{L(P,f) : P \text{ is a partition of } K\}, \\
&\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\quad = \inf\{U(P,f) : P \text{ is a partition of } K\}.
\end{aligned}$$

Lemma 6.6 *Let f be a bounded real-valued function on an n-dimensional rectangle K in $\mathbb{R}^n$. Then*

$$\mathcal{L}\int_K f(\mathbf{x})\,dx \leq \mathcal{U}\int_K f(\mathbf{x})\,dx.$$

$$\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \leq \mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.$$

Proof It follows from Lemma 6.5 that $L(P,f) \leq L(Q,f)$ for all partitions P and Q of K. It follows that, for a fixed partition Q, the upper sum $U(Q,f)$ is an upper bound on all the lower sums $L(P,f)$, and therefore

$$\mathcal{L}\int_K f(\mathbf{x})\,dx \leq U(Q,f).$$

The lower Riemann integral is then a lower bound on all the upper sums, and therefore

$$\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \leq \mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.$$

as required. ∎

Definition A bounded function $f\colon K \to \mathbb{R}$ on a closed n-dimensional rectangle K in $\mathbb{R}^n$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on K if

$$\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n = \mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n,$$

in which case the *Riemann integral* $\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$ (or *Darboux integral*) of f on X is defined to be the common value of $\mathcal{U}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$ and $\mathcal{L}\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$.

Lemma 6.7 *Let $f\colon K \to \mathbb{R}$ be a bounded function on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned}\mathcal{U}\int_K (-f(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n &= -\mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n,\\ \mathcal{L}\int_K (-f(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n &= -\mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n.\end{aligned}$$

Proof Let P be a partition of K, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $K_{P,\alpha}$. Then the lower and upper sums of f for the partition P satisfy the equations

$$L(P,f) = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}\, v(K_{P,\alpha}), \quad U(P,f) = \sum_{\alpha\in\Omega(P)} M_{P,\alpha}\, v(K_{P,\alpha}),$$

where

$$\begin{aligned} m_{P,\alpha} &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\},\\ M_{P,\alpha} &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.\end{aligned}$$

Now

$$\begin{aligned}&\sup\{-f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}\\ &\qquad = -\inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} = -m_{P,\alpha},\\ &\inf\{-f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}\\ &\qquad = -\sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\} = -M_{P,\alpha}\end{aligned}$$

It follows that

$$\begin{aligned} U(P,-f) &= \sum_{\alpha\in\Omega(P)} (-m_{P,\alpha}) v(K_{P,\alpha}) = -L(P,f),\\ L(P,-f) &= \sum_{\alpha\in\Omega(P)} (-M_{P,\alpha}) v(K_{P,\alpha}) = -U(P,f).\end{aligned}$$

We have now shown that

$$U(P,-f) = -L(P,f) \quad \text{and} \quad L(P,-f) = -U(P,f)$$

for all partitions P of the interval K. Applying the definition of the upper and lower integrals, we see that

$$\mathcal{U}\int_K (-f(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n$$

$$\begin{aligned}
&= \inf \{U(P,-f) : P \text{ is a partition of } K\} \\
&= \inf \{-L(P,f) : P \text{ is a partition of } K\} \\
&= -\sup \{L(P,f) : P \text{ is a partition of } K\} \\
&= -\mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{L}\int_K (-f(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n & \\
&= \sup \{L(P,-f) : P \text{ is a partition of } K\} \\
&= \sup \{-U(P,f) : P \text{ is a partition of } K\} \\
&= -\inf \{U(P,f) : P \text{ is a partition of } K\} \\
&= -\mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n.
\end{aligned}$$

This completes the proof. ■

Lemma 6.8 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded functions on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P,f) + L(P,g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P,f) + U(P,g).$$

Proof Let P be a partition of K, let $\Omega(P)$ be the indexing set for the cells of the partition P, and let the cell of the partition indexed by $\alpha \in \Omega(P)$ be denoted by $K_{P,\alpha}$. Then

$$\begin{aligned}
L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}), \\
L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g) v(K_{P,\alpha}), \\
L(P,f+g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f+g) v(K_{P,\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
m_{P,\alpha}(f+g) &= \inf\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}
\end{aligned}$$

for $\alpha \in \Omega(P)$.

Now

$$f(\mathbf{x}) \geq m_{P,\alpha}(f) \quad \text{and} \quad g(\mathbf{x}) \geq m_{P,\alpha}(g).$$

for all $\mathbf{x} \in K_{P,\alpha}$. Adding, we see that

$$f(\mathbf{x}) + g(\mathbf{x}) \geq m_{P,\alpha}(f) + m_{P,\alpha}(g)$$

for all $\mathbf{x} \in K_{P,\alpha}$, and therefore $m_{P,\alpha}(f) + m_{P,\alpha}(g)$ is a lower bound for the set

$$\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

The greatest lower bound for this set is $m_{P,\alpha}(f+g)$. Therefore

$$m_{P,\alpha}(f+g) \geq m_{P,\alpha}(f) + m_{P,\alpha}(g).$$

It follows that

$$\begin{aligned}
&L(P, f+g) \\
&\quad = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f+g) v(K_{P,\alpha}) \\
&\quad \geq \sum_{\alpha\in\Omega(P)} (m_{P,\alpha}(f) + m_{P,\alpha}(g)) v(K_{P,\alpha}) \\
&\quad = \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}) + \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g) v(K_{P,\alpha}) \\
&\quad = L(P, f) + L(P, g).
\end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned}
U(P, f) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\
U(P, g) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}), \\
U(P, f+g) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f+g) v(K_{P,\alpha}),
\end{aligned}$$

where

$$\begin{aligned}
M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\
M_{P,\alpha}(f+g) &= \sup\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}
\end{aligned}$$

for $\alpha \in \Omega(P)$.

Now

$$f(\mathbf{x}) \leq M_{P,\alpha}(f) \quad \text{and} \quad g(\mathbf{x}) \leq M_{P,\alpha}(g).$$

for all $\mathbf{x} \in K_{P,\alpha}$. Adding, we see that

$$f(\mathbf{x}) + g(\mathbf{x}) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g)$$

for all $\mathbf{x} \in K_{P,\alpha}$, and therefore $M_{P,\alpha}(f) + M_{P,\alpha}(g)$ is an upper bound for the set

$$\{f(\mathbf{x}) + g(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

The least upper bound for this set is $M_{P,\alpha}(f + g)$. Therefore

$$M_{P,\alpha}(f + g) \leq M_{P,\alpha}(f) + M_{P,\alpha}(g).$$

It follows that

$$\begin{aligned} & U(P, f+g) \\ &\quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f+g) v(K_{P,\alpha}) \\ &\quad \leq \sum_{\alpha \in \Omega(P)} (M_{P,\alpha}(f) + M_{P,\alpha}(g)) v(K_{P,\alpha}) \\ &\quad = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}) + \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}) \\ &\quad = U(P, f) + U(P, g). \end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 6.9 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-rectangle K. Then the functions $f + g$ and $f - g$ are Riemann-integrable on K, and moreover*

$$\begin{aligned} & \int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \cdots dx_n \\ &\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n, \end{aligned}$$

and

$$\int_K (f(\mathbf{x}) - g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n$$
$$= \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of K for which

$$L(P, f) > \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R, f) \geq L(P, f) \quad \text{and} \quad L(R, g) \geq L(P, g).$$

Applying Lemma 6.8, and the definition of the lower Riemann integral, we see that

$$\begin{aligned}
\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n & \\
\geq\ & L(R, f + g) \geq L(R, f) + L(R, g) \\
\geq\ & L(P, f) + L(Q, g) \\
>\ & \left(\int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \tfrac{1}{2}\varepsilon\right) \\
& \quad + \left(\int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \tfrac{1}{2}\varepsilon\right) \\
>\ & \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \varepsilon
\end{aligned}$$

We have now shown that

$$\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n$$
$$> \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n - \varepsilon$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2 \,\cdots\, dx_n, \quad \int_K f(\mathbf{x})\, dx_1\, dx_2 \,\cdots\, dx_n$$

and

$$\int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n$$

have values that have no dependence whatsoever on the value of ε.

It follows that

$$\begin{aligned}
&\mathcal{L}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\geq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

We can deduce a corresponding inequality involving the upper integral of $f+g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 6.7). We find that

$$\begin{aligned}
&\mathcal{L}\int_K (-f(\mathbf{x})-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\quad\geq \int_K (-f(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n + \int_K (-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\quad= -\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n - \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathcal{U}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad= -\mathcal{L}\int_K (-f(\mathbf{x})-g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\leq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}
&\int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\leq \mathcal{L}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\leq \mathcal{U}\int_K (f(\mathbf{x})+g(\mathbf{x}))\,dx_1\,dx_2\,\cdots\,dx_n \\
&\qquad\leq \int_K f(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n + \int_K g(\mathbf{x})\,dx_1\,dx_2\,\cdots\,dx_n.
\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}
&\mathcal{L}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&\qquad = \mathcal{U}\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n.
\end{aligned}$$

Thus the function $f + g$ is Riemann-integrable on K, and

$$\begin{aligned}
&\int_K (f(\mathbf{x}) + g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n + \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n.
\end{aligned}$$

Then, replacing g by $-g$, we find that

$$\begin{aligned}
&\int_K (f(\mathbf{x}) - g(\mathbf{x}))\, dx_1\, dx_2\, \cdots\, dx_n \\
&\qquad = \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \int_K g(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n.
\end{aligned}$$

as required. ∎

Proposition 6.10 *Let $f\colon K \to \mathbb{R}$ be a bounded function on a closed n-dimensional rectangle K in $\mathbb{R}^n$. Then the function f is Riemann-integrable on K if and only if, given any positive real number ε, there exists a partition P of K with the property that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof First suppose that $f\colon K \to \mathbb{R}$ is Riemann-integrable on K. Let some positive real number ε be given. Then

$$\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

is equal to the common value of the lower and upper integrals of the function f on K, and therefore there exist partitions Q and R of K for which

$$L(Q, f) > \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n - \tfrac{1}{2}\varepsilon$$

and

$$U(R, f) < \int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q, f) \leq L(P, f) \leq U(P, f) \leq U(R, f).$$

(see Lemma 6.4). It follows that

$$U(P, f) - L(P, f) \leq U(R, f) - L(Q, f) < \varepsilon.$$

Now suppose that $f: K \to \mathbb{R}$ is a bounded function on K with the property that, given any positive real number ε, there exists a partition P of K for which $U(P, f) - L(P, f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of K for which $U(P, f) - L(P, f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$\begin{aligned} L(P, f) &\leq \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \\ &\leq \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \leq U(P, f), \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n - \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \\ < U(P, f) - L(P, f) < \varepsilon. \end{aligned}$$

Thus the difference between the values of the upper and lower integrals of f on K must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n = \mathcal{L}\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n.$$

This completes the proof. ∎

Lemma 6.11 *Let $f: K \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-dimensional rectangle K in $\mathbb{R}^n$, let $|f|: K \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in K$, and let P be a partition of the n-rectangle K. Then the Darboux sums $U(P, f)$ and $L(P, f)$ of the function f on K and the Darboux sums $U(P, |f|)$ and $L(P, |f|)$ of the function $|f|$ on K satisfy the inequality*

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f).$$

Proof Let P be a partition of K, let $\Omega(P)$ be a set that indexes the cells of the partition P of K, and let

$$\begin{aligned} M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\ M_{P,\alpha}(|f|) &= \sup\{|f(\mathbf{x})| : \mathbf{x} \in K_{P,\alpha}\}, \\ m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}, \\ m_{P,\alpha}(|f|) &= \inf\{|f(\mathbf{x})| : \mathbf{x} \in K_{P,\alpha}\} \end{aligned}$$

for $\alpha \in \Omega(P)$. It follows from Lemma 3.8 that

$$M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|) \leq M_{P,\alpha}(f) - m_{P,\alpha}(f)$$

for $\alpha \in \Omega(P)$. Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned} L(P, f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f) v(K_{P,\alpha}), \\ L(P, |f|) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(|f|) v(K_{P,\alpha}), \\ U(P, f) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\ U(P, |f|) &= \sum_{\alpha\in\Omega(P)} M_{P,\alpha}(|f|) v(K_{P,\alpha}). \end{aligned}$$

It follows that

$$\begin{aligned} U(P, |f|) - L(P, |f|) &= \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(|f|) - m_{P,\alpha}(|f|)) v(K_{P,\alpha}) \\ &\leq \sum_{\alpha\in\Omega(P)} (M_{P,\alpha}(f) - m_{P,\alpha}(f)) v(K_{P,\alpha}) \\ &= U(P, f) - L(P, f), \end{aligned}$$

as required. ∎

Proposition 6.12 *Let $f\colon K \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed n-dimensional rectangle K in $\mathbb{R}^n$, and let $|f|\colon K \to \mathbb{R}$ be the function defined such that $|f|(\mathbf{x}) = |f(\mathbf{x})|$ for all $\mathbf{x} \in K$. Then the function $|f|$ is Riemann-integrable on K, and*

$$\left| \int_a^b f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \right| \leq \int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \cdots dx_n.$$

Proof Let some positive real number ε be given. It follows from Proposition 6.10 that there exists a partition P of K such that

$$U(P, f) - L(P, f) < \varepsilon.$$

It then follows from Lemma 6.11 that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 6.10 then ensures that the function $|f|$ is Riemann-integrable on K.

Now $-|f(\mathbf{x})| \leq f(\mathbf{x}) \leq |f(\mathbf{x})|$ for all $\mathbf{x} \in K$. It follows that

$$\begin{aligned} -\int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \, \cdots \, dx_n &\leq \int_a^b f(\mathbf{x})\, dx_1\, dx_2 \, \cdots \, dx_n \\ &\leq \int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \, \cdots \, dx_n. \end{aligned}$$

It follows that

$$\left| \int_a^b f(\mathbf{x})\, dx_1\, dx_2 \, \cdots \, dx_n \right| \leq \int_a^b |f(\mathbf{x})|\, dx_1\, dx_2 \, \cdots \, dx_n,$$

as required. ∎

Lemma 6.13 *Let $f\colon K \to \mathbb{R}$ and $g\colon K \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed n-dimensional rectangle K in $\mathbb{R}^n$, let B be a positive real number with the property that $|f(\mathbf{x})| \leq B$ and $|g(\mathbf{x})| \leq B$ for all $\mathbf{x} \in K$, and let P be a partition of the n-rectangle K. Then the Darboux sums $U(P, f)$, $U(P, g)$, $U(P, f \cdot g)$, $L(P, f)$, $L(P, g)$ and $L(P, f \cdot g)$ of the functions f, g and $f \cdot g$ on K satisfy the inequality*

$$\begin{aligned} &U(P, f \cdot g) - L(P, f \cdot g) \\ &\quad \leq B\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big). \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned} U(P, f) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f) v(K_{P,\alpha}), \\ U(P, g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(g) v(K_{P,\alpha}), \\ U(P, f \cdot g) &= \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f \cdot g) v(K_{P,\alpha}), \end{aligned}$$

$$\begin{aligned} L(P,f) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f)v(K_{P,\alpha}), \\ L(P,g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(g)v(K_{P,\alpha}), \\ L(P,f\cdot g) &= \sum_{\alpha\in\Omega(P)} m_{P,\alpha}(f\cdot g)v(K_{P,\alpha}), \end{aligned}$$

where

$$\begin{aligned} M_{P,\alpha}(f) &= \sup\{f(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}, \\ M_{P,\alpha}(g) &= \sup\{g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}, \\ M_{P,\alpha}(f\cdot g) &= \sup\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\} \\ m_{P,\alpha}(f) &= \inf\{f(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}, \\ m_{P,\alpha}(g) &= \inf\{g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}, \\ m_{P,\alpha}(f\cdot g) &= \inf\{f(\mathbf{x})g(\mathbf{x}) : \mathbf{x}\in K_{P,\alpha}\}. \end{aligned}$$

for $\alpha\in\Omega(P)$.

Now it follows from Lemma 3.11 that

$$M_{P,\alpha}(f\cdot g) - m_{P,\alpha}(f\cdot g) \leq B\Big(M_{P,\alpha}(f) - m_{P,\alpha}(f) + M_{P,\alpha}(g) - m_{P,\alpha}(g)\Big).$$

for $\alpha\in\Omega(P)$. The required inequality therefore holds on multiplying both sides of the inequality above by $v(K_{P,\alpha})$ and summing over all integers between 1 and n. ∎

Proposition 6.14 *Let $f\colon K\to\mathbb{R}$ and $g\colon K\to\mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded n-dimensional rectangle K in $\mathbb{R}^n$. Then the function $f\cdot g$ is Riemann-integrable on K, where $(f\cdot g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x}\in K$.*

Proof The functions f and g are bounded on K, and therefore there exists some positive real number B with the property that $|f(\mathbf{x})|\leq B$ and $|g(\mathbf{x})|\leq B$ for all $\mathbf{x}\in K$.

Let some positive real number ε be given. It follows from Proposition 6.10 that there exist partitions Q and R of the closed n-rectangle K for which

$$U(Q,f) - L(Q,f) < \frac{\varepsilon}{2B}$$

and

$$U(R,g) - L(R,g) < \frac{\varepsilon}{2B}.$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 6.4 that

$$U(P, f) - L(P, f) \leq U(Q, f) - L(Q, f) < \frac{\varepsilon}{2B}$$

and

$$U(P, g) - L(P, g) \leq U(R, g) - L(R, g) < \frac{\varepsilon}{2B}.$$

It then follows from Proposition 6.13 that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ \leq \;& B\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big) \\ < \;& \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed n-dimensional rectangle K with the property that

$$U(P, f \cdot g) - L(P, f \cdot g) < \varepsilon.$$

It follows from Proposition 6.10 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎

6.4 Integrability of Continuous functions

Theorem 6.15 *Let K be a closed n-dimensional rectangle in $\mathbb{R}^n$. Then any continuous real-valued function on K is Riemann-integrable.*

Proof Let $f\colon K \to \mathbb{R}$ be a continuous real-valued function on K. Then f is bounded above and below on K, and moreover $f\colon K \to \mathbb{R}$ is uniformly continuous on K. (These results follow from Theorem 5.5 and Theorem 5.6.) Therefore there exists some strictly positive real number δ such that $|f(\mathbf{u}) - f(\mathbf{w})| < \varepsilon$ whenever $\mathbf{u}, \mathbf{w} \in K$ satisfy $|\mathbf{u} - \mathbf{w}| < \delta$.

Choose a partition P of the n-rectangle K such that each cell in the partition has diameter less than δ. Let $\Omega(P)$ be an index set which indexes the cells of the partition P and, for each $\alpha \in \Omega(P)$ let $K_{P,\alpha}$ be the corresponding cell of the partition P of K. Also let $\mathbf{p}_\alpha$ be a point of $K_{P,\alpha}$ for all $\alpha \in \Omega(P)$. Then $|\mathbf{x} - \mathbf{p}_\alpha| < \delta$ for all $\mathbf{x} \in K_{P,\alpha}$. Thus if

$$m_{P,\alpha} = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

and

$$M_{P,\alpha} = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}$$

then

$$f(\mathbf{p}_\alpha) - \varepsilon \leq m_{P,\alpha} \leq M_{P,\alpha} \leq f(\mathbf{p}_\alpha) + \varepsilon$$

for all $\alpha \in \Omega(P)$. It follows that

$$\begin{aligned} & \sum_{i=1}^{n} f(\mathbf{p}_\alpha) v(K_{P,\alpha}) - \varepsilon v(K) \\ \leq \; & L(P, f) \leq U(P, f) \\ \leq \; & \sum_{i=1}^{n} f(\mathbf{p}_\alpha) v(K_{P,\alpha}) + \varepsilon v(K), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$\begin{aligned} 0 \;\leq\; & \mathcal{U} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n - \mathcal{L} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n \\ \leq\; & U(P, f) - L(P, f) \leq 2\varepsilon v(K). \end{aligned}$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n = \mathcal{L} \int_K f(x)\, dx_1\, dx_2 \cdots dx_n,$$

and thus the function f is Riemann-integrable on K. ∎

6.5 Repeated Integration

Let K be an n-rectangle in $\mathbb{R}^n$, given by

$$\begin{aligned} K &= \prod_{i=1}^{n} [a_i, b_i] \\ &= \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i \text{ for } i = 1, 2, \ldots, n\}, \end{aligned}$$

where $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers which satisfy $a_i \leq b_i$ for each i. Given any continuous real-valued function f on K, let us denote by $\mathcal{I}_K(f)$ the repeated integral of f over the n-rectangle K whose value is

$$\int_{x_n=a_n}^{b_n} \Big(\cdots \int_{x_2=a_2}^{b_2} \Big(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1 \Big)\, dx_2 \ldots \Big)\, dx_n.$$

(Thus $\mathcal{I}_K(f)$ is obtained by integrating the function f first over the coordinate x_1, then over the coordinate x_2, and so on).

Note that if $m \leq f(\mathbf{x}) \leq M$ on K for some constants m and M then

$$m\, v(K) \leq \mathcal{I}_K(f) \leq M\, v(K).$$

We shall use this fact to show that if f is a continuous function on some n-rectangle K in $\mathbb{R}^n$ then

$$\mathcal{I}_K(f) = \int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

(i.e., $\mathcal{I}_K(f)$ is equal to the Riemann integral of f over K).

Theorem 6.16 *Let f be a continuous real-valued function defined on some n-rectangle K in $\mathbb{R}^n$, where*

$$K = \{\mathbf{x} \in \mathbb{R}^n : a_i \leq x_i \leq b_i\}.$$

Then the Riemann integral

$$\int_K f(\mathbf{x})\, dx_1\, dx_2\, \cdots\, dx_n$$

of f over K is equal to the repeated integral

$$\int_{x_n=a_n}^{b_n} \Bigl(\cdots \int_{x_2=a_2}^{b_2} \Bigl(\int_{x_1=a_1}^{b_1} f(x_1, x_2, \ldots, x_n)\, dx_1\Bigr)\, dx_2 \ldots\Bigr)\, dx_n.$$

Proof Given a partition P of the n-rectangle K, we denote by $L(P, f)$ and $U(P, f)$ the quantities so that

$$L(P, f) = \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, v(K_{P,\alpha})$$

and

$$U(P, f) = \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, v(K_{P,\alpha})$$

where $\Omega(P)$ is an indexing set that indexes the cells of the partition P, and where, for all $\alpha \in \Omega(P)$, $v(K_{P,\alpha})$ is the volume of the cell $K_{P,\alpha}$ of the partition P indexed by α,

$$m_{P,\alpha}(f) = \inf\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\},$$

and

$$M_{P,\alpha}(f) = \sup\{f(\mathbf{x}) : \mathbf{x} \in K_{P,\alpha}\}.$$

Now

$$m_{P,\alpha}(f) \leq f(\mathbf{x}) \leq M_{P,\alpha}(f)$$

for all $\alpha \in \Omega(P)$ and $\mathbf{x} \in K_{P,\alpha}$, and therefore

$$m_{P,\alpha}(f)\, v(K_{P,\alpha}) \leq \mathcal{I}_{K,\alpha}(f) \leq M_{P,\alpha}(f)\, v(K_{P,\alpha})$$

for all $\alpha \in \Omega(P)$. Summing these inequalities as α ranges over the indexing set $\Omega(P)$, we find that

$$\begin{aligned} L(P, f) &= \sum_{\alpha \in \Omega(P)} m_{P,\alpha}(f)\, v(K_{P,\alpha}) \\ &\leq \sum_{\alpha \in \Omega(P)} \mathcal{I}_{K,\alpha}(f) \\ &\leq \sum_{\alpha \in \Omega(P)} M_{P,\alpha}(f)\, v(K_{P,\alpha}) \\ &= U(P, f). \end{aligned}$$

But

$$\sum_{\alpha \in \Omega(P)} \mathcal{I}_{K,\alpha}(f) = \mathcal{I}_K(f).$$

It follows that

$$L(P, f) \leq \mathcal{I}_K(f) \leq U(P, f).$$

The Riemann integral of f is equal to the supremum of the quantities $L(P, f)$ as P ranges over all partitions of the n-rectangle K, hence

$$\int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n \leq \mathcal{I}_K(f).$$

Similarly the Riemann integral of f is equal to the infimum of the quantities $U(P, f)$ as P ranges over all partitions of the n-rectangle K, hence

$$\mathcal{I}_K(f) \leq \int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n.$$

Hence

$$\mathcal{I}_K(f) = \int_K f(\mathbf{x})\, dx_1\, dx_2 \cdots dx_n,$$

as required. ∎

Note that the order in which the integrations are performed in the repeated integral plays no role in the above proof. We may therefore deduce the following important corollary.

Corollary 6.17 *Let f be a continuous real-valued function defined over some closed rectangle K in $\mathbb{R}^2$, where*

$$K = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \quad c \leq y \leq d\}.$$

Then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Proof It follows directly from Theorem 6.16 that the repeated integrals

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \text{ and } \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

are both equal to the Riemann integral of the function f over the rectangle K. Therefore these repeated integrals must be equal. ∎

7 Norms on Finite-Dimensional Vector Spaces

7.1 Norms

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

The Euclidean norm $|.|$ is a norm on $\mathbb{R}^n$ defined so that

$$|(x_1, x_2, \ldots, x_n)| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$$

for all $(x_1, x_2, \ldots, x_n)$. There are other useful norms on $\mathbb{R}^n$. These include the norms $\|.\|_1$ and $\|.\|_{\text{sup}}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\text{sup}} = \text{maximum}(|x_1|, |x_2|, \ldots, |x_n|)$$

for all $(x_1, x_2, \ldots, x_n)$.

Definition Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

Lemma 7.1 *If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.*

Proof let $\|.\|_*$ and $\|.\|_{**}$ be norms on a real vector space X that are both equivalent to a norm $\|.\|$ on X. Then there exist constants c_*, c_{**}, C_* and C_{**}, where $0 < c_* \leq C_*$ and $0 < c_{**} \leq C_{**}$, such that

$$c_*\|x\| \leq \|x\|_* \leq C_*\|x\|$$

and

$$c_{**}\|x\| \leq \|x\|_{**} \leq C_{**}\|x\|$$

for all $x \in X$. But then

$$\frac{c_{**}}{C_*}\|x\|_* \leq \|x\|_{**} \leq \frac{C_{**}}{c_*}\|x\|_*.$$

for all $x \in X$, and thus the norms $\|.\|_*$ and $\|.\|_{**}$ are equivalent to one another. The result follows. ∎

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 7.2 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive real number C with the property that $\|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ be a point of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n).$$

Using Schwarz's Inequality, we see that

$$\begin{aligned} \|\mathbf{x}\| &= \left\| \sum_{j=1}^n x_j \mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x}|, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and

$$|\mathbf{x}| = \left(\sum_{j=1}^n x_j^2 \right)^{\frac{1}{2}}$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The result follows. ∎

Lemma 7.3 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then there exists a positive constant C such that*

$$\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x} - \mathbf{y}\| \leq C|\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

It follows that

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\|$$

and

$$\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\|,$$

and therefore

$$\Big|\|\mathbf{y}\| - \|\mathbf{x}\|\Big| \leq \|\mathbf{x} - \mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. The result therefore follows from Lemma 7.2. ∎

Theorem 7.4 *Any two norms on $\mathbb{R}^n$ are equivalent.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now it follows from Lemma 7.3 that the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous. Also S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. It therefore follows from the Extreme Value Theorem (Theorem 5.5) that there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. ∎

7.2 Linear Transformations

The space $\mathbb{R}^n$ consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers is a vector space over the field $\mathbb{R}$ of real numbers, where addition and multiplication by scalars are defined by

$$\begin{aligned}(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \lambda(x_1, x_2, \ldots, x_n) &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)\end{aligned}$$

for all $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

Definition A map $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is said to be a *linear transformation* if

$$T(\mathbf{x} + \mathbf{y}) = T\mathbf{x} + T\mathbf{y}, \qquad T(\lambda\mathbf{x}) = \lambda T\mathbf{x}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$.

Every linear transformation $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is represented by an $n \times m$ matrix $(T_{i,j})$. Indeed let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ be the standard basis vectors of $\mathbb{R}^m$ defined by

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{e}_m = (0, 0, \ldots, 1).$$

Thus if $\mathbf{x} \in \mathbb{R}^m$ is represented by the m-tuple $(x_1, x_2, \ldots, x_m)$ then

$$\mathbf{x} = \sum_{j=1}^{m} x_j \mathbf{e}_j.$$

Similarly let $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ be the standard basis vectors of $\mathbb{R}^n$ defined by

$$\mathbf{f}_1 = (1, 0, \ldots, 0), \quad \mathbf{f}_2 = (0, 1, \ldots, 0), \ldots, \mathbf{f}_n = (0, 0, \ldots, 1).$$

Thus if $\mathbf{v} \in \mathbb{R}^n$ is represented by the n-tuple $(v_1, v_2, \ldots, v_n)$ then

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{f}_i.$$

Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Define $T_{i,j}$ for all integers i between 1 and n and for all integers j between 1 and m such that

$$T\mathbf{e}_j = \sum_{i=1}^{n} T_{i,j} \mathbf{f}_i.$$

Using the linearity of T, we see that if $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ then

$$T\mathbf{x} = T\left(\sum_{j=1}^{m} x_j \mathbf{e}_j\right) = \sum_{j=1}^{m} (x_j\, T\mathbf{e}_j) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} T_{i,j} x_j\right) \mathbf{f}_i.$$

Thus the ith component of $T\mathbf{x}$ is

$$T_{i,1}x_1 + T_{i,2}x_2 + \cdots + T_{i,m}x_m.$$

Writing out this identity in matrix notation, we see that if $T\mathbf{x} = \mathbf{v}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

then

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} T_{1,1} & T_{1,2} & \ldots & T_{1,m} \\ T_{2,1} & T_{2,2} & \ldots & T_{2,m} \\ \vdots & \vdots & & \vdots \\ T_{n,1} & T_{n,2} & \ldots & T_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}.$$

7.3 The Operator Norm of a Linear Transformation

Definition Given $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The *operator norm* $\|T\|_{\mathrm{op}}$ of T is defined such that

$$\|T\|_{\mathrm{op}} = \sup\{|T\mathbf{x}| : \mathbf{x} \in \mathbb{R}^m \quad \text{and} \quad |\mathbf{x}| = 1\}.$$

Lemma 7.5 *Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ and $U\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$, and let λ be a real number. Then $\|T\|_{\mathrm{op}}$ is the smallest non-negative real number with the property that $|T\mathbf{x}| \leq \|T\|_{\mathrm{op}}|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^m$. Moreover*

$$\|\lambda T\|_{\mathrm{op}} = |\lambda|\, \|T\|_{\mathrm{op}} \quad \text{and} \quad \|T + U\|_{\mathrm{op}} \leq \|T\|_{\mathrm{op}} + \|U\|_{\mathrm{op}}.$$

Proof Let $\mathbf{x}$ be an element of $\mathbb{R}^m$. Then we can express $\mathbf{x}$ in the form $\mathbf{x} = \mu\mathbf{z}$, where $\mu = |\mathbf{x}|$ and $\mathbf{z} \in \mathbb{R}^m$ satisfies $|\mathbf{z}| = 1$. Then

$$|T\mathbf{x}| = |T(\mu\mathbf{z})| = |\mu T\mathbf{z}| = |\mu|\, |T\mathbf{z}| = |\mathbf{x}|\, |T\mathbf{z}| \leq \|T\|_{\mathrm{op}}|\mathbf{x}|.$$

Next let C be a non-negative real number with the property that $|T\mathbf{x}| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^m$. Then C is an upper bound for the set

$$\{|T\mathbf{x}| : \mathbf{x} \in \mathbb{R}^m \quad \text{and} \quad |\mathbf{x}| = 1\},$$

and thus $\|T\|_{\text{op}} \leq C$. Thus $\|T\|_{\text{op}}$ is the smallest non-negative real number C with the property that $|T\mathbf{x}| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^m$.

Next we note that

$$\begin{aligned} \|\lambda T\|_{\text{op}} &= \sup\{|\lambda T\mathbf{x}| : \mathbf{x} \in \mathbb{R}^m \quad \text{and} \quad |\mathbf{x}| = 1\} \\ &= \sup\{|\lambda|\,|T\mathbf{x}| : \mathbf{x} \in \mathbb{R}^m \quad \text{and} \quad |\mathbf{x}| = 1\} \\ &= |\lambda| \sup\{|T\mathbf{x}| : \mathbf{x} \in \mathbb{R}^m \quad \text{and} \quad |\mathbf{x}| = 1\} \\ &= |\lambda|\,\|T\|_{\text{op}}. \end{aligned}$$

Let $\mathbf{x} \in \mathbb{R}^m$. Then

$$\begin{aligned} |(T+U)\mathbf{x}| &\leq |T\mathbf{x}| + |U\mathbf{x}| \leq \|T\|_{\text{op}}|\mathbf{x}| + \|U\|_{\text{op}}|\mathbf{x}| \\ &\leq (\|T\|_{\text{op}} + \|U\|_{\text{op}})|\mathbf{x}| \end{aligned}$$

It follows that

$$\|(T+U)\|_{\text{op}} \leq \|T\|_{\text{op}} + \|U\|_{\text{op}}.$$

This completes the proof. ∎

7.4 The Hilbert-Schmidt Norm of a Linear Transformation

Recall that the *length* (or *norm*) of an element $\mathbf{x} \in \mathbb{R}^n$ is defined such that

$$|\mathbf{x}|^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Definition Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$, and let $(T_{i,j})$ be the $n \times m$ matrix representing this linear transformation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^n$. The *Hilbert-Schmidt norm* $\|T\|_{\text{HS}}$ of the linear transformation is then defined so that

$$\|T\|_{\text{HS}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{m} T_{i,j}^2}.$$

Note that the Hilbert-Schmidt norm is just the Euclidean norm on the real vector space of dimension mn whose elements are $n \times m$ matrices representing linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ with respect to the standard bases of these vector spaces. Therefore it has the standard properties of the Euclidean norm. In particular it follows from the Triangle Inequality (Lemma 4.2) that

$$\|T+U\|_{\text{HS}} \leq \|T\|_{\text{HS}} + \|U\|_{\text{HS}} \quad \text{and} \quad \|\lambda T\|_{\text{HS}} = |\lambda|\,\|T\|_{\text{HS}}$$

for all linear transformations T and U from $\mathbb{R}^m$ to $\mathbb{R}^n$ and for all real numbers λ.

Lemma 7.6 *Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$. Then T is uniformly continuous on $\mathbb{R}^n$. Moreover*

$$|T\mathbf{x} - T\mathbf{y}| \leq \|T\|_{\mathrm{HS}} |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, where $\|T\|_{HS}$ is the Hilbert-Schmidt norm of the linear transformation T.

Proof Let $\mathbf{v} = T\mathbf{x} - T\mathbf{y}$, where $\mathbf{v} \in \mathbb{R}^n$ is represented by the n-tuple $(v_1, v_2, \ldots, v_n)$. Then

$$v_i = T_{i,1}(x_1 - y_1) + T_{i,2}(x_2 - y_2) + \cdots + T_{i,m}(x_m - y_m)$$

for all integers i between 1 and n. It follows from Schwarz's Inequality (Lemma 4.1) that

$$v_i^2 \leq \left(\sum_{j=1}^m T_{i,j}^2\right) \left(\sum_{j=1}^m (x_j - y_j)^2\right) = \left(\sum_{j=1}^m T_{i,j}^2\right) |\mathbf{x} - \mathbf{y}|^2.$$

Hence

$$|\mathbf{v}|^2 = \sum_{i=1}^n v_i^2 \leq \left(\sum_{i=1}^n \sum_{j=1}^m T_{i,j}^2\right) |\mathbf{x} - \mathbf{y}|^2 = \|T\|_{\mathrm{HS}}^2 |\mathbf{x} - \mathbf{y}|^2.$$

Thus $|T\mathbf{x} - T\mathbf{y}| \leq \|T\|_{\mathrm{HS}} |\mathbf{x} - \mathbf{y}|$. It follows from this that T is uniformly continuous. Indeed let some positive real number ε be given. We can then choose δ so that $\|T\|_{\mathrm{HS}}\, \delta < \varepsilon$. If $\mathbf{x}$ and $\mathbf{y}$ are elements of $\mathbb{R}^n$ which satisfy the condition $|\mathbf{x} - \mathbf{y}| < \delta$ then $|T\mathbf{x} - T\mathbf{y}| < \varepsilon$. This shows that $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is uniformly continuous on $\mathbb{R}^m$, as required. ∎

Lemma 7.7 *Let $T\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ and let $S\colon \mathbb{R}^n \to \mathbb{R}^p$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^p$. Then the Hilbert-Schmidt norm of the composition of the linear operators T and S satisfies the inequality $\|ST\|_{\mathrm{HS}} \leq \|S\|_{\mathrm{HS}}\, \|T\|_{\mathrm{HS}}$.*

Proof The composition ST of the linear operators is represented by the product of the corresponding matrices. Thus the component $(ST)_{k,j}$ in the kth row and the jth column of the $p \times m$ matrix representing the linear transformation ST satisfies

$$(ST)_{k,j} = \sum_{i=1}^n S_{k,i} T_{i,j}.$$

It follows from Schwarz's Inequality (Lemma 4.1) that

$$(ST)_{k,j}^2 \leq \left(\sum_{i=1}^{n} S_{k,i}^2 \right) \left(\sum_{i=1}^{n} T_{i,j}^2 \right).$$

Summing over k, we find that

$$\sum_{k=1}^{p} (ST)_{k,j}^2 \leq \left(\sum_{k=1}^{p} \sum_{i=1}^{n} S_{k,i}^2 \right) \left(\sum_{i=1}^{n} T_{i,j}^2 \right) = \|S\|_{\mathrm{HS}}^2 \left(\sum_{i=1}^{n} T_{i,j}^2 \right).$$

Then summing over j, we find that

$$\begin{aligned} \|ST\|_{\mathrm{HS}}^2 &= \sum_{k=1}^{p} \sum_{j=1}^{m} (ST)_{k,j}^2 \leq \|S\|_{\mathrm{HS}}^2 \left(\sum_{i=1}^{n} \sum_{j=1}^{m} T_{i,j}^2 \right) \\ &\leq \|S\|_{\mathrm{HS}}^2 \|T\|_{\mathrm{HS}}^2. \end{aligned}$$

On taking square roots, we find that $\|ST\|_{\mathrm{HS}} \leq \|S\|_{\mathrm{HS}} \|T\|_{\mathrm{HS}}$, as required. ∎