Module MA2321: Analysis in Several Real Variables
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Part I (Sections 1 to 3)

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Contents

1 The Real Number System

1.1 A Concise Characterization of the Real Number System

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field.

We describe below what a *field* is, what an *ordered field* is, and what is meant by saying that an ordered field is *Dedekind-complete.*

1.2 Fields

Definition A *field* is a set $\mathbb{F}$ on which are defined operations of addition and multiplication, associating elements $x + y$ and xy of $\mathbb{F}$ to each pair x, y of elements of $\mathbb{F}$, for which the following axioms are satisfied:

(i) $x + y = y + x$ for all $x, y \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *commutative*);

(ii) $(x+y)+z = x+(y+z)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of addition on $\mathbb{F}$ is *associative*);

(iii) there exists an element 0 of $\mathbb{F}$ with the property that $0 + x = x$ for all $x \in \mathbb{F}$ (i.e., there exists a *zero element* for the operation of addition on $\mathbb{F}$);

(iv) given any $x \in \mathbb{F}$, there exists an element $-x$ of $\mathbb{F}$ satisfying $x+(-x) = 0$ (i.e., *negatives* of elements of $\mathbb{F}$ always exist);

(v) $xy = yx$ for all $x, y \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *commutative*);

(vi) $(xy)z = x(yz)$ for all $x, y, z \in \mathbb{F}$ (i.e., the operation of multiplication on $\mathbb{F}$ is *associative*);

(vii) there exists an element 1 of $\mathbb{F}$ with the property that $1x = x$ for all $x \in \mathbb{F}$ (i.e., there exists an *identity element* for the operation of multiplication on $\mathbb{F}$);

(viii) given any $x \in \mathbb{F}$ satisfying $x \neq 0$, there exists an element x^{-1} of $\mathbb{F}$ satisfying $xx^{-1} = 1$;

(ix) $x(y + z) = xy + xz$ for all $x, y, z \in \mathbb{F}$ (i.e., multiplication is *distributive* over addition).

The operations of subtraction and division are defined on a field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $x-y=x+(-y)$ for all elements x and y of $\mathbb{F}$, and moreover $x/y=xy^{-1}$ provided that $y\neq 0$.

1.3 Ordered Fields

Definition An *ordered field* consists of a field $\mathbb{F}$ together with an ordering $<$ on that field that satisfies the following axioms:—

(x) if x and y are elements of $\mathbb{F}$ then one and only one of the three statements $x<y$, $x=y$ and $y<x$ is true (i.e., the ordering satisfies the *Trichotomy Law*);

(xi) if x, y and z are elements of $\mathbb{F}$ and if $x<y$ and $y<z$ then $x<z$ (i.e., the ordering is *transitive*);

(xii) if x, y and z are elements of $\mathbb{F}$ and if $x<y$ then $x+z<y+z$;

(xiii) if x and y are elements of $\mathbb{F}$ which satisfy $0<x$ and $0<y$ then $0<xy$.

We can write $x>y$ in cases where $y<x$. we can write $x\leq y$ in cases where either $x=y$ or $x<y$. We can write $x\geq y$ in cases where either $x=y$ or $y<x$.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x|=\begin{cases} x & \text{if } x\geq 0;\\ -x & \text{if } x<0.\end{cases}$$

Note that $|x|\geq 0$ for all x and that $|x|=0$ if and only if $x=0$. Also $|x+y|\leq|x|+|y|$ and $|xy|=|x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b+c\sqrt{2})(e+f\sqrt{2})=(be+2cf)+(bf+ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

1.4 Least Upper Bounds

Let S be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set S if $x \leq u$ for all $x \in S$. The set S is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let S be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of S (denoted by $\sup S$) if s is an upper bound of S and $s \leq u$ for all upper bounds u of S.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The following property is satisfied in some ordered fields but not in others.

> **Least Upper Bound Property**: given any non-empty subset S of $\mathbb{F}$ that is bounded above, there exists an element $\sup S$ of $\mathbb{F}$ that is the least upper bound for the set S.

Definition A *Dedekind-complete* ordered field $\mathbb{F}$ is an ordered field which has the Least Upper Bound Property.

1.5 Greatest Lower Bounds

Let S be a subset of an ordered field $\mathbb{F}$. A *lower bound* of S is an element l of $\mathbb{F}$ with the property that $l \leq x$ for all $x \in S$. The set S is said to be *bounded below* if such a lower bound exists. A *greatest lower bound* (or *infimum*) for a set S is a lower bound for that set that is greater than every other lower bound for that set. The greatest lower bound of the set S (if it exists) is denoted by $\inf S$.

Let $\mathbb{F}$ be a Dedekind-complete ordered field. Then, given any non-empty subset S of $\mathbb{F}$ that is bounded below, there exists a greatest lower bound (or *infimum*) $\inf S$ for the set S. Indeed $\inf S = -\sup\{x \in \mathbb{R} : -x \in S\}$.

Remark It can be proved that any two Dedekind-complete ordered fields are isomorphic via an isomorphism that respects the ordering and the algebraic operations on the fields. The theory of *Dedekind cuts* provides a construction that yields a Dedekind-complete ordered field that can represent the system of real numbers. For an account of this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.6 Bounded Sets of Real Numbers

The set $\mathbb{R}$ of *real numbers*, with its usual ordering algebraic operations, constitutes a Dedekind-complete ordered field. Thus every non-empty subset S of $\mathbb{R}$ that is bounded above has a *least upper bound* (or *supremum*) $\sup S$, and every non-empty subset S of $\mathbb{R}$ that is bounded below has a *greatest lower bound* (or *infimum*) $\inf S$.

Let S be a non-empty subset of the real numbers that is bounded (both above and below). Then the closed interval $[\inf S, \sup S]$ is the smallest closed interval in the set $\mathbb{R}$ of real numbers that contains the set S. Indeed if $S \subset [a, b]$, where a and b are real numbers satisfying $a \leq b$, then $a \leq \inf S \leq \sup S \leq b$, and therefore

$$S \subset [\inf S, \sup S] \subset [a, b].$$

1.7 Convergence of Infinite Sequences of Real Numbers

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l-\varepsilon < x_j < l+\varepsilon$ for all positive integers j satisfying $j \geq N$.

Lemma 1.1 *Let S be a subset of the set $\mathbb{R}$ of real numbers which is non-empty and bounded above, and let $\sup S$ denote the least upper bound of the set S. Then there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_j \in S$ for all positive integers j and $\lim_{j \to +\infty} x_j = \sup S$.*

Proof Let $s = \sup S$. For each positive integer j, the real number $s - 1/j$ is not an upper bound for the set S (because s is the least upper bound of S), and therefore there exists some element x_j of S satisfying $x_j > s - 1/j$. Moreover $x_j \leq s$ for all positive integers j, because s is an upper bound for the set S. It follows that $s - 1/j < x_j \leq s$ for all positive integers j. Given any positive real number ε, let N be a positive integer chosen so that $N > 1/\varepsilon$. Then $|x_j - s| < \varepsilon$ whenever $j \geq N$. It follows that $\lim_{j \to +\infty} x_j = s$, as required. ∎

1.8 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 1.2 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

1.9 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

1.10 The Bolzano-Weierstrass Theorem

Theorem 1.3 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers. We define a *peak index* to be a positive integer j with the property that $a_j \geq a_k$ for all positive integers k satisfying $k \geq j$. Thus a positive integer j is a peak index if and only if the jth member of the infinite sequence $a_1, a_2, a_3, \ldots$ is greater than or equal to all succeeding members of the sequence. Let S be the set of all peak indices. Then

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}.$$

First let us suppose that the set S of peak indices is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 <$

$j_2 < j_3 < j_4 < \cdots$. It follows from the definition of peak indices that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 1.2 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S of peak indices is finite. Choose a positive integer j_1 which is greater than every peak index. Then j_1 is not a peak index. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 is not a peak index (because j_2 is greater than j_1 and j_1 in turn is greater than every peak index). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 1.2. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

1.11 The Definition of Continuity for Functions of a Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $f: D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be *continuous* at s if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Lemma 1.4 *Let $f: D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$, and let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers belonging to D. Suppose that $x_j \to s$ as $j \to +\infty$, where $s \in D$, and that f is continuous at s. Then $f(x_j) \to f(s)$ as $j \to +\infty$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then there exists some positive integer N such that $|x_j - s| < \delta$ for all j satisfying $j \geq N$. Thus $|f(x_j) - f(s)| < \varepsilon$ whenever $j \geq N$. Hence $f(x_j) \to f(s)$ as $j \to +\infty$. ∎

1.12 The Intermediate Value Theorem

Theorem 1.5 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f: [a, b] \to \mathbb{R}$ be a continuous function* *defined on the interval $[a, b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a, b]$ for which $f(s) = c$.*

Proof If $f(a) = c$ then we may take $s = a$, and if $f(b) = c$ then we may take $s = b$.

It remains to consider cases where $f(a) < c < f(b)$ or $f(a) > c > f(b)$. In the case where $f(a) < c < f(b)$ let $g\colon [a, b] \to \mathbb{R}$ be defined such that $g(x) = f(x) - c$. In the case where $f(a) > c > f(b)$ let $g\colon [a, b] \to \mathbb{R}$ be defined such that $g(x) = c - f(x)$. In both cases the function g is a continuous function on $[a, b]$ defined so that $g(a) < 0$ and $g(b) > 0$, and in both cases we must prove the existence of a real number s belonging to the interval $[a, b]$ for which $g(s) = 0$.

Let

$$S = \{x \in [a, b] : g(x) \leq 0\}.$$

Then $a \in S$, and $x \leq b$ for all $x \in S$. The set S is thus non-empty and bounded above, and therefore there exists a least upper bound $\sup S$ for the set S. Let $s = \sup S$.

Now it follows from Lemma 1.1 that there exists an infinite sequence $x_1, x_2, x_3, \ldots$ such that $x_j \in S$ for all positive integers j and $\lim\limits_{j \to +\infty} x_j = s$. Now $g(x_j) \leq 0$ for all positive integers j (because $x_j \in S$). Moreover the continuity of the function g ensures that $g(s) = \lim\limits_{j \to +\infty} g(x_j)$. It follows that $g(s) \leq 0$. Moreover $s < b$ (because $g(b) > 0$), and therefore there exists an infinite sequence $y_1, y_2, y_3, \ldots$ such that $s < y_j \leq b$ for all positive integers j and $\lim\limits_{j \to +\infty} y_j = s$. (Indeed we could take $y_j = s + (b - s)/j$ for all positive integers j.) Now $g(y_j) > 0$ for all positive integers j (because $y_j \notin S$), and $g(s) = \lim\limits_{j \to +\infty} g(y_j)$, and therefore $g(s) \geq 0$. We have now shown that both $g(s) \leq 0$ and $g(s) \geq 0$. It follows that $g(s) = 0$, and thus $f(s) = c$, as required. ∎

1.13 The Extreme Value Theorem

Proposition 1.6 *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then there exists a positive constant M with the property that $|f(x)| \leq M$ for all $x \in [a, b]$.*

Proof Let S be the set consisting of those real numbers t satisfying $a \leq t \leq b$ for which the function f is bounded on $[a, t]$. A real number t therefore belongs to the set S if and only if $a \leq t \leq b$ and also there exists some positive real number K_t with the property that $|f(x)| \leq K_t$ for all $x \in [a, t]$. Now $a \in S$ and $t \leq b$ for all $t \in S$. Thus set S is non-empty and bounded above. It follows from the Least Upper Bound Principle that the set S has a least upper bound $\sup S$. Let $s = \sup S$. Then $s \in [a, b]$.

Now the function f is continuous at s. Therefore there exists some positive real number δ such that $|f(x)| \leq |f(s)| + 1$ for all whenever $x \in [a, b]$ and $s - \delta < x < s + \delta$. Also $s - \delta$ is not an upper bound for the set S and therefore there exists some element t of S satisfying $s - \delta < t \leq s$. There then exists some positive real number K_t with the property that $|f(x)| \leq K_t$ for all $x \in [a, t]$.

Let $M = \max(K_t, |f(s)|+1)$. Then $|f(x)| \leq M$ for all $x \in [a, b]$ satisfying $x < s + \delta$, and therefore $x \in S$ for all $x \in [a, b]$ satisfying $x < s + \delta$. If it were the case that $s < b$ then s would not be an upper bound for the set S, contradicting the definition of s as the least upper bound of S. Therefore $s = b$. It follows that $|f(x)| \leq M$ for all $x \in [a, b]$. Thus the function f is bounded on $[a, b]$, as required. ∎

Theorem 1.7 (The Extreme Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then there exist real numbers u and v belonging to the interval $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.*

Proof It follows from Proposition 1.6 that the set

$$\{f(x) : x \in [a, b]\}$$

is bounded above and below. This set is also non-empty. It follows that there exist real numbers M and m such that

$$M = \sup\{f(x) : x \in [a, b]\} \quad \text{and} \quad m = \inf\{f(x) : x \in [a, b]\}.$$

If it were the case that $f(x) < M$ for all $x \in [a, b]$ then there would exist a well-defined function $g\colon [a, b] \to \mathbb{R}$ satisfying

$$g(x) = \frac{1}{M - f(x)}$$

for all $x \in [a, b]$. This function would not be bounded, because, given any positive constant K, there would exist $x \in [a, b]$ for which $f(x) > M - 1/K$ and $g(x) > K$. The existence of such a function g would contradict the result of Proposition 1.6. Therefore there must exist $v \in [a, b]$ with the property that $f(x) \leq f(v)$ for all $x \in [a, b]$.

Similarly there cannot exist any continuous function $h\colon [a,b] \to \mathbb{R}$ with the property that

$$h(x) = \frac{1}{f(x) - m}$$

for all $x \in [a,b]$, and therefore there must exist $u \in [a,b]$ with the property that $f(u) \leq f(x)$ for all $x \in [a,b]$. This completes the proof. ∎

1.14 Uniform Continuity

Definition A function $f\colon D \to \mathbb{R}$ is said to be *uniformly continuous* over a subset D of $\mathbb{R}$ if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in [a,b]$ satisfying $|u - v| < \delta$. (where δ does not depend on u or v).

A continuous function defined over a subset D of $\mathbb{R}$ is not necessarily uniformly continuous on D. (One can verify for example that the function sending a non-zero real number x to $1/x$ is not uniformly continuous on the set of all non-zero real numbers.) However we show that continuity does imply uniform continuity when $D = [a,b]$ for some real numbers a and b satisfying $a < b$.

Theorem 1.8 *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on a closed bounded interval $[a,b]$. Then the function f is uniformly continuous on $[a,b]$.*

Proof Let some strictly positive real number ε be given. Suppose that there did not exist any strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$. Then, for each positive integer j, there would exist values u_j and v_j in the interval $[a,b]$ such that $|u_j - v_j| < 1/j$ and $|f(u_j) - f(v_j)| \geq \varepsilon$. But the sequence $u_1, u_2, u_3, \ldots$ would be bounded (since $a \leq u_j \leq b$ for all j) and thus would possess a convergent subsequence $u_{k_1}, u_{k_2}, u_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 1.3).

Let $l = \lim\limits_{j\to+\infty} u_{k_j}$. Then $l = \lim\limits_{j\to+\infty} v_{k_j}$ also, since $\lim\limits_{j\to+\infty} (v_{k_j} - u_{k_j}) = 0$. Moreover $a \leq l \leq b$. It follows from the continuity of f that $\lim\limits_{j\to+\infty} f(u_{k_j}) = \lim\limits_{j\to+\infty} f(v_{k_j}) = f(l)$ (see Lemma 1.4). Thus $\lim\limits_{j\to+\infty} \bigl(f(u_{k_j}) - f(v_{k_j})\bigr) = 0$. But this is impossible, since u_j and v_j have been chosen so that $|f(u_j) - f(v_j)| \geq \varepsilon$ for all positive integers j. We conclude therefore that there must exist some strictly positive real number δ with the required property. ∎

1.15 Historical Note on the Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techniques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second. Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convergence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.

2 The Mean Value Theorem

2.1 Interior Points and Open Sets in the Real Line

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers. We say that D is an *open set* in $\mathbb{R}$ if every point of D is an interior point of D.

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

2.2 Differentiable Functions of a Single Real Variable

We recall basic results of the theory of differentiable functions.

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Let s be some real number, and let f and g be real-valued functions defined around s that are differentiable at s. The basic rules of differential calculus then ensure that the functions $f+g$, $f-g$ and $f \cdot g$ are differentiable at s (where

$$(f+g)(x) = f(x)+g(x), \quad (f-g)(x) = f(x)-g(x) \quad \text{and} \quad (f.g)(x) = f(x)g(x)$$

for all real numbers x at which both $f(x)$ and $g(x)$ are defined), and

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

$$(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s) \quad (\textit{Product Rule}).$$

If moreover $g(s) \neq 0$ then the function f/g is differentiable at s (where $(f/g)(x) = f(x)/g(x)$ where both $f(x)$ and $g(x)$ are defined), and

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2} \quad (\textit{Quotient Rule}).$$

Moreover if h is a real-valued function defined around $f(s)$ which is differentiable at $f(s)$ then the composition function $h \circ f$ is differentiable at $f(s)$ and

$$(h \circ f)'(s) = h'(f(s))f'(s) \quad (\textit{Chain Rule}).$$

Derivatives of some standard functions are as follows:—

$$\frac{d}{dx}(x^m) = mx^{m-1}, \quad \frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x,$$

$$\frac{d}{dx}(\exp x) = \exp x, \quad \frac{d}{dx}(\log x) = \frac{1}{x} \quad (x > 0).$$

2.3 Rolle's Theorem

Theorem 2.1 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a,b]$. It therefore follows from the Extreme Value Theorem that there must exist real numbers u and v in the interval $[a,b]$ with the property that $f(u) \leq$

$f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$ (see Theorem 1.7). If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

2.4 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 2.2 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 2.1) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

2.5 Concavity and the Second Derivative

Proposition 2.3 *Let s and h be real numbers, and let f be a twice differentiable real-valued function defined on some open interval containing s and $s + h$. Then there exists a real number θ satisfying $0 < \theta < 1$ for which*

$$f(s + h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s + \theta h).$$

Proof Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s+th)$ is defined for all $t \in I$, and let $q: I \to \mathbb{R}$ be defined so that

$$q(t) = f(s+th) - f(s) - thf'(s) - t^2(f(s+h) - f(s) - hf'(s)).$$

for all $t \in I$. Differentiating, we find that

$$q'(t) = hf'(s+th) - hf'(s) - 2t(f(s+h) - f(s) - hf'(s))$$

and

$$q''(t) = h^2 f''(s+th) - 2(f(s+h) - f(s) - hf'(s)).$$

Now $q(0) = q(1) = 0$. It follows from Rolle's Theorem, applied to the function q on the interval $[0,1]$, that there exists some real number φ satisfying $0 < \varphi < 1$ for which $q'(\varphi) = 0$.

Then $q'(0) = q'(\varphi) = 0$, and therefore Rolle's Theorem can be applied to the function q' on the interval $[0, \varphi]$ to prove the existence of some real number θ satisfying $0 < \theta < \varphi$ for which $q''(\theta) = 0$. Then

$$0 = q''(\theta) = h^2 f''(s+\theta h) - 2(f(s+h) - f(s) - hf'(s)).$$

Rearranging, we find that

$$f(s+h) = f(s) + hf'(s) + \tfrac{1}{2}h^2 f''(s+\theta h),$$

as required. ∎

Corollary 2.4 *Let $f: (s-\delta_0, s+\delta_0)$ be a twice-differentiable function throughout some open interval $(s-\delta_0, s+\delta_0)$ centred on a real number s. Suppose that $f''(s+h) > 0$ for all real numbers h satisfying $|h| < \delta_0$. Then*

$$f(s+h) \geq f(s) + hf'(s)$$

for all real numbers h satisfying $|h| < \delta_0$.

It follows from Corollary 2.4 that if a twice-differentiable function has positive second derivative throughout some open interval, then it is concave upwards throughout that interval. In particular the function has a local minimum at any point of that open interval where the first derivative is zero and the second derivative is positive.

Corollary 2.5 *Let $f: D \to \mathbb{R}$ be a twice-differentiable function defined over a subset D of $\mathbb{R}$, and let s be a point in the interior of D. Suppose that $f'(s) = 0$ and that $f''(x) > 0$ for all real numbers x belonging to some sufficiently small neighbourhood of x. Then s is a local minimum for the function f.*

3 The Riemann Integral in One Real Variable

3.1 Darboux Sums and the Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral.*

A *partition* P of an interval $[a,b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a,b]$, the *lower sum* (or *lower Darboux sum*) $L(P,f)$ and the *upper sum* (or *upper Darboux sum*) $U(P,f)$ of f for the partition P of $[a,b]$ are defined by

$$L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$.

Clearly $L(P,f) \leq U(P,f)$. Moreover $\sum\limits_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

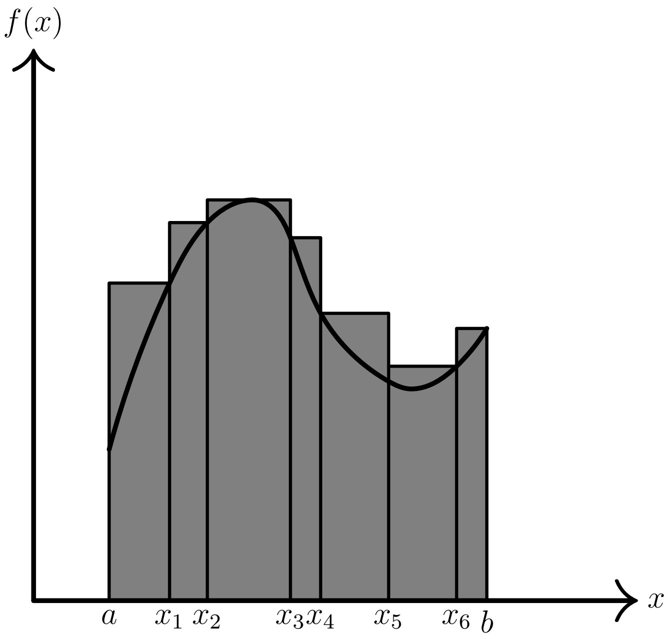
$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a),$$

for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a,b]$.

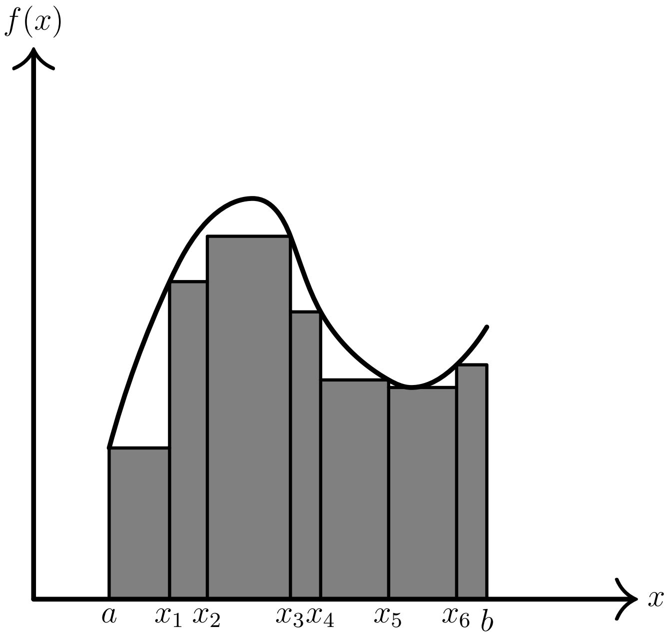
Definition Let f be a bounded real-valued function on the interval $[a,b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a,b]$ are defined by

$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &= \inf\left\{U(P,f) : P \text{ is a partition of } [a,b]\right\}, \\ \mathcal{L}\int_a^b f(x)\,dx &= \sup\left\{L(P,f) : P \text{ is a partition of } [a,b]\right\}. \end{aligned}$$

The definition of upper and lower integrals thus requires that $\mathcal{U}\int_a^b f(x)\,dx$ be the infimum of the values of $U(P,f)$ and that $\mathcal{L}\int_a^b f(x)\,dx$ be the supremum of the values of $L(P,f)$ as P ranges over all possible partitions of the interval $[a,b]$.



The upper sum $U(P, f)$



The lower sum $L(P, f)$

Definition A bounded function $f: [a, b] \to \mathbb{R}$ on a closed bounded interval $[a, b]$ is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a, b]$ if

$$\mathcal{U} \int_a^b f(x)\, dx = \mathcal{L} \int_a^b f(x)\, dx,$$

in which case the *Riemann integral* $\int_a^b f(x)\, dx$ (or *Darboux integral*) of f on $[a, b]$ is defined to be the common value of $\mathcal{U} \int_a^b f(x)\, dx$ and $\mathcal{L} \int_a^b f(x)\, dx$.

When $a > b$ we define

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x)\, dx = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

Definition Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 3.1 *Let R be a refinement of some partition P of $[a, b]$. Then*

$$L(R, f) \geq L(P, f) \quad \text{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f: [a, b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a,b]$ there exists a partition R of $[a,b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 3.2 *Let f be a bounded real-valued function on the interval $[a,b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a,b]$, and let R be a common refinement of P and Q. It follows from Lemma 3.1 that $L(P,f) \leq L(R,f) \leq U(R,f) \leq U(Q,f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P,f) \leq U(Q,f)$ as P ranges over all possible partitions of the interval $[a,b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q,f)$ for all partitions Q of $[a,b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a,b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0,1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0,1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$\begin{aligned}
L(P_n, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d - \frac{c}{n}\right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d\right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}$$

It follows that

$$\lim_{n\to+\infty} L(P_n, f) = \frac{c}{2} + d$$

and

$$\lim_{n\to+\infty} U(P_n, f) = \frac{c}{2} + d$$

Now $L(P_n, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_n, f)$ for all positive integers n. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0,1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0,1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0,1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0,1]$.

3.2 Basic Properties of the Riemann Integral

Lemma 3.3 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the lower and upper Riemann integrals of f and $-f$ are related by the identities*

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= -\mathcal{L}\int_a^b f(x)\,dx, \\ \mathcal{L}\int_a^b (-f(x))\,dx &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b,$$

and let

$$\begin{aligned} m_i &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}. \end{aligned}$$

Then the lower and upper sums of f for the partition P are given by the formulae

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Now

$$\begin{aligned} & \sup\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ & \qquad = \; -\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = -m_i, \\ & \inf\{-f(x) : x_{i-1} \leq x \leq x_i\} \\ & \qquad = \; -\sup\{f(x) : x_{i-1} \leq x \leq x_i\} = -M_i \end{aligned}$$

It follows that

$$\begin{aligned} U(P, -f) &= \sum_{i=1}^{n} (-m_i)(x_i - x_{i-1}) = -L(P, f), \\ L(P, -f) &= \sum_{i=1}^{n} (-M_i)(x_i - x_{i-1}) = -U(P, f). \end{aligned}$$

We have now shown that

$$U(P, -f) = -L(P, f) \quad \text{and} \quad L(P, -f) = -U(P, f)$$

for all partitions P of the interval $[a, b]$. Applying the definition of the upper and lower integrals, we see that

$$\begin{aligned} \mathcal{U}\int_a^b (-f(x))\,dx &= \inf\{U(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \inf\{-L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{L}\int_a^b f(x)\,dx \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{L}\int_a^b (-f(x))\,dx &= \sup\{L(P, -f) : P \text{ is a partition of } [a, b]\} \\ &= \sup\{-U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\inf\{U(P, f) : P \text{ is a partition of } [a, b]\} \\ &= -\mathcal{U}\int_a^b f(x)\,dx. \end{aligned}$$

This completes the proof. ∎

Lemma 3.4 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, and let P be a partition of the interval $[a,b]$. Then the lower sums of the functions f, g and $f+g$ satisfy*

$$L(P, f+g) \geq L(P,f) + L(P,g),$$

and the upper sums of these functions satisfy

$$U(P, f+g) \leq U(P,f) + U(P,g).$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\ L(P,g) &= \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}), \\ L(P,f+g) &= \sum_{i=1}^{n} m_i(f+g)(x_i - x_{i-1}), \end{aligned}$$

where

$$\begin{aligned} m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(g) &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(f+g) &= \inf\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\} \end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \geq m_i(f) \quad \text{and} \quad g(x) \geq m_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \geq m_i(f) + m_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $m_i(f) + m_i(g)$ is a lower bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The greatest lower bound for this set is $m_i(f+g)$. Therefore

$$m_i(f+g) \geq m_i(f) + m_i(g).$$

It follows that

$$\begin{aligned}
L(P, f+g) &= \sum_{i=1}^{n} m_i(f+g)(x_i - x_{i-1}) \\
&\geq \sum_{i=1}^{n} (m_i(f) + m_i(g))(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}) \\
&= L(P, f) + L(P, g).
\end{aligned}$$

An analogous argument applies to upper sums. Now

$$\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\
U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}),
\end{aligned}$$

where

$$\begin{aligned}
M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(g) &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(f+g) &= \sup\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}
\end{aligned}$$

for $i = 1, 2, \ldots, n$.

Now

$$f(x) \leq M_i(f) \quad \text{and} \quad g(x) \leq M_i(g).$$

for all $x \in [x_{i-1}, x_i]$. Adding, we see that

$$f(x) + g(x) \leq M_i(f) + M_i(g)$$

for all $x \in [x_{i-1}, x_i]$, and therefore $M_i(f) + M_i(g)$ is an upper bound for the set

$$\{f(x) + g(x) : x_{i-1} \leq x \leq x_i\}.$$

The least upper bound for this set is $M_i(f+g)$. Therefore

$$M_i(f+g) \leq M_i(f) + M_i(g).$$

It follows that

$$\begin{aligned} U(P, f+g) &= \sum_{i=1}^{n} M_i(f+g)(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i(f) + M_i(g))(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}) + \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}) \\ &= U(P, f) + U(P, g). \end{aligned}$$

This completes the proof that

$$L(P, f+g) \geq L(P, f) + L(P, g)$$

and

$$U(P, f+g) \leq U(P, f) + U(P, g). \quad \blacksquare$$

Proposition 3.5 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$. Then the functions $f+g$ and $f-g$ are Riemann-integrable on $[a,b]$, and moreover*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

and

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

Proof Let some strictly positive real number ε be given. The definition of Riemann-integrability and the Riemann integral ensures that there exist partitions P and Q of $[a,b]$ for which

$$L(P, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$L(Q, g) > \int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Let the partition R be a common refinement of the partitions P and Q. Then

$$L(R,f) \geq L(P,f) \quad \text{and} \quad L(R,g) \geq L(P,g).$$

Applying Lemma 3.4, and the definition of the lower Riemann integral, we see that

$$\begin{aligned}
\mathcal{L}\int_a^b (f(x)+g(x))\,dx & \\
&\geq L(R,f+g) \geq L(R,f)+L(R,g) \\
&\geq L(P,f)+L(Q,g) \\
&> \left(\int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon\right) + \left(\int_a^b g(x)\,dx - \tfrac{1}{2}\varepsilon\right) \\
&> \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon
\end{aligned}$$

We have now shown that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx > \int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \varepsilon$$

for all strictly positive real numbers ε. However the quantities of

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx, \qquad \int_a^b f(x)\,dx \quad \text{and} \quad \int_a^b g(x)\,dx$$

have values that have no dependence whatsoever on the value of ε. It follows that

$$\mathcal{L}\int_a^b (f(x)+g(x))\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We can deduce a corresponding inequality involving the upper integral of $f+g$ by replacing f and g by $-f$ and $-g$ respectively (Lemma 3.3). We find that

$$\begin{aligned}
\mathcal{L}\int_a^b (-f(x)-g(x))\,dx &\geq \int_a^b (-f(x))\,dx + \int_a^b (-g(x))\,dx \\
&= -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathcal{U}\int_a^b (f(x)+g(x))\,dx &= -\mathcal{L}\int_a^b (-f(x)-g(x))\,dx \\
&\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

Combining the inequalities obtained above, we find that

$$\begin{aligned}
&\int_a^b f(x)\,dx + \int_a^b g(x)\,dx \\
&\qquad \le \mathcal{L}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \le \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\
&\qquad \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

The quantities at the left and right hand ends of this chain of inequalities are equal to each other. It follows that

$$\begin{aligned}
\mathcal{L}\int_a^b (f(x)+g(x))\,dx &= \mathcal{U}\int_a^b (f(x)+g(x))\,dx \\
&= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.
\end{aligned}$$

Thus the function $f+g$ is Riemann-integrable on $[a,b]$, and

$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Then, replacing g by $-g$, we find that

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

as required. ∎

Proposition 3.6 *Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function on a closed bounded interval $[a,b]$, where a and b are real numbers satisfying $a \le b$. Then the function f is Riemann-integrable on $[a,b]$ if and only if, given any positive real number ε, there exists a partition P of $[a,b]$ with the property that*

$$U(P,f) - L(P,f) < \varepsilon.$$

Proof First suppose that $f\colon [a,b] \to \mathbb{R}$ is Riemann-integrable on $[a,b]$. Let some positive real number ε be given. Then

$$\int_a^b f(x)\,dx$$

is equal to the common value of the lower and upper integrals of the function f on $[a,b]$, and therefore there exist partitions Q and R of $[a,b]$ for which

$$L(Q,f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon$$

and

$$U(R,f) < \int_a^b f(x)\,dx + \tfrac{1}{2}\varepsilon.$$

Let P be a common refinement of the partitions Q and R. Now

$$L(Q,f) \leq L(P,f) \leq U(P,f) \leq U(R,f).$$

(see Lemma 3.1). It follows that

$$U(P,f) - L(P,f) \leq U(R,f) - L(Q,f) < \varepsilon.$$

Now suppose that $f\colon [a,b] \to \mathbb{R}$ is a bounded function on $[a,b]$ with the property that, given any positive real number ε, there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Let $\varepsilon > 0$ be given. Then there exists a partition P of $[a,b]$ for which $U(P,f) - L(P,f) < \varepsilon$. Now it follows from the definitions of the upper and lower integrals that

$$L(P,f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P,f),$$

and therefore

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < U(P,f) - L(P,f) < \varepsilon.$$

Thus the difference between the values of the upper and lower integrals of f on $[a,b]$ must be less than every strictly positive real number ε, and therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx.$$

This completes the proof. ∎

Let u and v be real numbers. Then

$$|u| \leq |u-v| + |v| \quad \text{and} \quad |v| \leq |u-v| + |u|,$$

and therefore $|u| - |v| \leq |u-v|$. Interchanging u and v, and using the identity $|u-v| = |v-u|$, we see that $|v| - |u| \leq |u-v|$. It follows from this that

$$\Bigl||u| - |v|\Bigr| \leq |u-v|$$

for all real numbers u and v.

Lemma 3.7 *Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}. \end{aligned}$$

Then

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

for all $u, v \in X$.

Proof Let $u, v \in X$. Then either $f(v) \geq f(u)$ or $f(u) \geq f(v)$. In the case where $f(v) \geq f(u)$ the inequalities $m_X(f) \leq f(u) \leq f(v) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. In the case where $f(u) \geq f(v)$ the inequalities $m_X(f) \leq f(v) \leq f(u) \leq M_X(f)$ ensure that $|f(v) - f(u)| \leq M_X(f) - m_X(f)$. The result follows. ∎

Lemma 3.8 *Let $f\colon X \to \mathbb{R}$ be a bounded real-valued function defined on a non-empty set X, and let*

$$\begin{aligned} M_X(f) &= \sup\{f(x) : x \in X\}, \\ M_X(|f|) &= \sup\{|f(x)| : x \in X\}, \\ m_X(f) &= \inf\{f(x) : x \in X\}, \\ m_X(|f|) &= \inf\{|f(x)| : x \in X\}. \end{aligned}$$

Then

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f).$$

Proof Let δ be a positive real number. Then there exist $u, v \in X$ such that

$$m_X(|f|) \leq |f(u)| < m_X(|f|) + \delta$$

and

$$M_X(|f|) - \delta < |f(v)| \leq M_X(|f|).$$

Then

$$|f(v)| - |f(u)| > M_X(|f|) - m_X(|f|) - 2\delta.$$

But

$$|f(v)| - |f(u)| \leq |f(v) - f(u)|,$$

and

$$|f(v) - f(u)| \leq M_X(f) - m_X(f)$$

(see Lemma 3.7). It follows that

$$\begin{aligned} M_X(|f|) - m_X(|f|) - 2\delta &< |f(v)| - |f(u)| \leq |f(v) - f(u)| \\ &\leq M_X(f) - m_X(f). \end{aligned}$$

But the values of $M_X(|f|) - m_X(|f|)$ and $M_X(f) - m_X(f)$ are independent of δ, where $\delta > 0$. It follows that

$$M_X(|f|) - m_X(|f|) \leq M_X(f) - m_X(f),$$

as required. ∎

Lemma 3.9 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, let $|f|\colon [a, b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$, and let P be a partition of the interval $[a, b]$. Then the Darboux sums $U(P, f)$ and $L(P, f)$ of the function f on $[a, b]$ and the Darboux sums $U(P, |f|)$ and $L(P, |f|)$ of the function $|f|$ on $[a, b]$ satisfy the inequality*

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f).$$

Proof Let P be a partition of $[a, b]$, and let

$$P = \{x_0, x_1, x_2, \ldots, x_n\},$$

where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

and let

$$\begin{aligned} M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i(|f|) &= \sup\{|f(x)| : x_{i-1} \leq x \leq x_i\}, \\ m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ m_i(|f|) &= \inf\{|f(x)| : x_{i-1} \leq x \leq x_i\} \end{aligned}$$

for $i = 1, 2, \ldots, n$. It follows from Lemma 3.8 that

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

for $i = 1, 2, \ldots, n$. Now the Darboux sums of the functions f and $|f|$ for the partition P are defined by the identities

$$\begin{aligned}
L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\
L(P, |f|) &= \sum_{i=1}^{n} m_i(|f|)(x_i - x_{i-1}), \\
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, |f|) &= \sum_{i=1}^{n} M_i(|f|)(x_i - x_{i-1}).
\end{aligned}$$

It follows that

$$\begin{aligned}
U(P, |f|) - L(P, |f|) &= \sum_{i=1}^{n} (M_i(|f|) - m_i(|f|))(x_i - x_{i-1}) \\
&\leq \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) \\
&= U(P, f) - L(P, f),
\end{aligned}$$

as required. ∎

Proposition 3.10 *Let $f\colon [a, b] \to \mathbb{R}$ be a bounded Riemann-integrable function on a closed interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$, and let $|f|\colon [a, b] \to \mathbb{R}$ be the function defined such that $|f|(x) = |f(x)|$ for all $x \in [a, b]$. Then the function $|f|$ is Riemann-integrable on $[a, b]$, and*

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Proof Let some positive real number ε be given. It follows from Proposition 3.6 that there exists a partition P of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon.$$

It then follows from Lemma 3.9 that

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon.$$

Proposition 3.6 then ensures that the function $|f|$ is Riemann-integrable on $[a, b]$.

Now $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx,$$

as required. ∎

Let X be a non-empty set, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on X. We denote by $f \cdot g\colon X\mathbb{R}$ the product function defined such that We denote by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.

Lemma 3.11 *Let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be bounded real-valued functions defined on a non-empty set X, let K be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in X$, and let*

$$\begin{aligned}
M_X(f) &= \sup\{f(x) : x \in X\},\\
M_X(g) &= \sup\{g(x) : x \in X\},\\
M_X(f \cdot g) &= \sup\{f(x)g(x) : x \in X\},\\
m_X(f) &= \inf\{f(x) : x \in X\},\\
m_X(g) &= \inf\{g(x) : x \in X\},\\
m_X(f \cdot g) &= \inf\{f(x)g(x) : x \in X\}.
\end{aligned}$$

Then

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Proof Let u and v be elements of the set X. Then

$$f(v)g(v) - f(u)g(u) = (f(v) - f(u))g(v) + f(u)(g(v) - g(u)),$$

and therefore

$$\begin{aligned}
&|f(v)g(v) - f(u)g(u)|\\
&\quad\leq |f(v) - f(u)|\,|g(v)| + |f(u)|\,|g(v) - g(u)|,\\
&\quad\leq K\Big(|f(v) - f(u)| + |g(v) - g(u)|\Big).
\end{aligned}$$

Now $|f(v) - f(u)| \leq M_X(f) - m_X(f)$ and $|g(v) - g(u)| \leq M_X(g) - m_X(g)$ and (see Lemma 3.7). Therefore

$$|f(v)g(v) - f(u)g(u)| \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big).$$

Now, given any positive real number δ, elements u and v of X can be chosen so that

$$m_X(f \cdot g) \leq f(u)g(u) < m_X(f \cdot g) + \delta$$

and

$$M_X(f \cdot g) - \delta < f(v)g(v) \leq M_X(f \cdot g).$$

Then

$$f(v)g(v) - f(u)g(u) > M_X(f \cdot g) - m_X(f \cdot g) - 2\delta.$$

It follows that

$$M_X(f \cdot g) - m_X(f \cdot g) - 2\delta < K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big)$$

for all positive real numbers δ, and therefore

$$M_X(f \cdot g) - m_X(f \cdot g) \leq K\Big(M_X(f) - m_X(f) + M_X(g) - m_X(g)\Big),$$

as required. ∎

Lemma 3.12 *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed interval $[a,b]$, where a and b are real numbers satisfying $a \leq b$, let K be a positive real number with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a,b]$, and let P be a partition of the interval $[a,b]$. Then the Darboux sums $U(P,f)$, $U(P,g)$, $U(P,f \cdot g)$, $L(P,f)$, $L(P,g)$ and $L(P,f \cdot g)$ of the functions f, g and $f \cdot g$ on $[a,b]$ satisfy the inequality*

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq \; K\Big(U(P,f) - L(P,f) + U(P,g) - L(P,g)\Big). \end{aligned}$$

Proof Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then

$$
\begin{aligned}
U(P, f) &= \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}), \\
U(P, g) &= \sum_{i=1}^{n} M_i(g)(x_i - x_{i-1}), \\
U(P, f \cdot g) &= \sum_{i=1}^{n} M_i(f \cdot g)(x_i - x_{i-1}), \\
L(P, f) &= \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}), \\
L(P, g) &= \sum_{i=1}^{n} m_i(g)(x_i - x_{i-1}), \\
L(P, f \cdot g) &= \sum_{i=1}^{n} m_i(f \cdot g)(x_i - x_{i-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
M_i(f) &= \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(g) &= \sup\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
M_i(f \cdot g) &= \sup\{f(x)g(x) : x_{i-1} \leq x \leq x_i\} \\
m_i(f) &= \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\
m_i(g) &= \inf\{g(x) : x_{i-1} \leq x \leq x_i\}, \\
m_i(f \cdot g) &= \inf\{f(x)g(x) : x_{i-1} \leq x \leq x_i\}.
\end{aligned}
$$

for $i = 1, 2, \ldots, n$.

Now it follows from Lemma 3.11 that

$$
M_i(f \cdot g) - m_i(f \cdot g) \leq K\Big(M_i(f) - m_i(f) + M_i(g) - m_i(g)\Big).
$$

for $i = 1, 2, \ldots, n$. The required inequality therefore holds on multiplying both sides of the inequality above by $x_i - x_{i-1}$ and summing over all integers between 1 and n. ∎

Proposition 3.13 *Let $f\colon [a, b] \to \mathbb{R}$ and $g\colon [a, b] \to \mathbb{R}$ be bounded Riemann-integrable functions on a closed bounded interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Then the function $f \cdot g$ is Riemann-integrable on $[a, b]$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in [a, b]$.*

Proof The functions f and g are bounded on $[a, b]$, and therefore there exists some positive real number K with the property that $|f(x)| \leq K$ and $|g(x)| \leq K$ for all $x \in [a, b]$.

Let some positive real number ε be given. It follows from Proposition 3.6 that there exist partitions Q and R of the closed interval $[a, b]$ for which

$$U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

Let P be a common refinement of the partitions Q and R. It follows from Lemma 3.1 that

$$U(P, f) - L(P, f) \leq U(Q, f) - L(Q, f) < \frac{\varepsilon}{2K}$$

and

$$U(P, g) - L(P, g) \leq U(R, g) - L(R, g) < \frac{\varepsilon}{2K}.$$

It then follows from Proposition 3.12 that

$$\begin{aligned} & U(P, f \cdot g) - L(P, f \cdot g) \\ & \quad \leq K\Big(U(P, f) - L(P, f) + U(P, g) - L(P, g)\Big) \\ & \quad < \varepsilon \end{aligned}$$

We have thus shown that, given any positive real number ε, there exists a partition P of the closed bounded interval $[a, b]$ with the property that

$$U(P, f \cdot g) - L(P, f \cdot g) < \varepsilon.$$

It follows from Proposition 3.6 that the product function $f \cdot g$ is Riemann-integrable, as required. ∎

Proposition 3.14 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let some positive real number ε be given. The function f is Riemann-integrable on the interval $[a, b]$ and therefore there exists a partition Q of $[a, b]$

such that the lower Darboux sum $L(Q, f)$ of f on $[a, b]$ with respect to the partition Q of $[a, b]$ satisfies

$$L(Q, f) > \int_a^b f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Similarly there exists a partition R of $[b, c]$ of $[a, b]$ such that the lower Darboux sum $L(Q, f)$ of f on $[b, c]$ with respect to the partition R of $[b, c]$ satisfies

$$L(R, f) > \int_b^c f(x)\,dx - \tfrac{1}{2}\varepsilon.$$

Now the partitions Q and R combine to give a partition P of the interval $[a, c]$, where $P = Q \cup R$. Indeed $Q = \{u_0, u_1, \ldots, u_m\}$, where $u_0, u_1, \ldots, u_m$ are real numbers satisfying

$$a = u_0 < u_1 < u_2 < \cdots u_{m-1} < u_m = b,$$

and $R = \{v_0, v_1, \ldots, v_n\}$, where $v_0, v_1, \ldots, v_n$ are real numbers satisfying

$$b = v_0 < v_1 < v_2 < \cdots v_{n-1} < v_n = c.$$

Then

$$P = \{a, u_1, u_2, \ldots, u_{m-1}, b, v_1, v_2, \ldots, v_{n-1}, c\}.$$

It follows directly from the definition of Darboux lower sums that

$$L(P, f) = L(Q, f) + L(R, f).$$

The choice of the partitions Q and R then ensures that

$$L(P, f) > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

The lower Riemann integral $\mathcal{L}\displaystyle\int_a^c f(x)\,dx$ is by definition the least upper bound of the lower Darboux sums of f on the interval $[a, c]$. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx > \int_a^b f(x)\,dx + \int_b^c f(x)\,dx - \varepsilon.$$

Moreover this inequality holds for all values of the positive real number ε. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx \geq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Applying this result with the function f replaced by $-f$ yields the inequality

$$\mathcal{L}\int_a^c (-f(x))\,dx \geq -\int_a^b f(x)\,dx - \int_b^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c (-f(x))\,dx = -\mathcal{U}\int_a^c f(x)\,dx$$

(see Lemma 3.3). It follows that

$$\mathcal{U}\int_a^c f(x)\,dx \leq \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

But

$$\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx.$$

It follows that

$$\mathcal{L}\int_a^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

The result follows. ∎

3.3 Integrability of Monotonic Functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a,b] \to \mathbb{R}$ defined on the closed bounded interval $[a,b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a,b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a,b] \to \mathbb{R}$ is said to be *monotonic* on $[a,b]$ if either it is non-decreasing on $[a,b]$ or else it is non-increasing on $[a,b]$.

Proposition 3.15 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a,b]$ is Riemann-integrable on $[a,b]$.*

Proof Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$.

The maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Now $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned} & U(P, f) - L(P, f) \\ = \; & \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ < \; & \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But

$$\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx.$$

It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f\colon [a, b] \to \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ∎

Corollary 3.16 *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof The result follows immediately on applying the results of Proposition 3.14 and Proposition 3.15. ∎

Remark The result and proof of Proposition 3.15 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

3.4 Integrability of Continuous functions

Theorem 3.17 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a, b]$. Then f is bounded above and below on the interval $[a, b]$, and moreover $f\colon [a, b] \to \mathbb{R}$ is uniformly continuous on $[a, b]$. (These results follow from Theorem 1.7 and Theorem 1.8.) Therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$.

Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) \\
&\qquad \leq \quad L(P, f) \leq U(P, f) \\
&\qquad \leq \quad \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a),
\end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P.

We have now shown that

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

3.5 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 3.17). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 3.18 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let some strictly positive real number ε be given, and let ε_0 be a real number chosen so that $0 < \varepsilon_0 < \varepsilon$. (For example, one could choose $\varepsilon_0 = \frac{1}{2}\varepsilon$.) Now the function f is continuous at x, where $a < x < b$. It follows that there exists some strictly positive real number δ such that

$$f(x) - \varepsilon_0 \leq f(t) \leq f(x) + \varepsilon_0$$

for all $t \in [a, b]$ satisfying $x - \delta < t < x + \delta$.

Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Then

$$\begin{aligned} F(x+h) &= \int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt \\ &= F(x) + \int_x^{x+h} f(t)\,dt \end{aligned}$$

whenever $x + h \in [a, b]$. Also

$$\frac{1}{h}\int_x^{x+h} f(x)\,dt = \frac{f(x)}{h}\int_x^{x+h} dt = f(x),$$

because $f(x)$ is constant as t varies between x and $x+h$. It follows that

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} (f(t)-f(x))\,dt$$

whenever $x+h \in [a,b]$. But if $0 < |h| < \delta$ and $x+h \in [a,b]$ then

$$-\varepsilon_0 \leq f(t) - f(x) \leq \varepsilon_0$$

for all real numbers t belonging to the closed interval with endpoints x and $x+h$, and therefore

$$-\varepsilon_0|h| \leq \int_x^{x+h} (f(t)-f(x))\,dt \leq \varepsilon_0|h|.$$

It follows that

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \leq \varepsilon_0 < \varepsilon$$

whenever $x+h \in [a,b]$ and $0 < |h| < \delta$. We conclude that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$\begin{aligned} f'(a) &= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \\ f'(b) &= \lim_{h\to 0^-} \frac{f(b+h)-f(b)}{h} \end{aligned}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h\to 0^+} \frac{F(a+h)-F(a)}{h} = f(a), \qquad \lim_{h\to 0^-} \frac{F(b+h)-F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 3.19 *Let f be a continuously differentiable real-valued function on the interval $[a,b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfying $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 2.2) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 3.20 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(x)\,\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x)\,\frac{df(x)}{dx}\,dx.$$

Proof This result follows from Corollary 3.19 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - g(x)\,\frac{df(x)}{dx}. \quad ∎$$

Corollary 3.21 (Integration by Substitution) *Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\, dy,$$

and $H'(s) = f(s)$ for all $s \in [a, b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a, b)$. On applying the Mean Value Theorem (Theorem 2.2) to the function $F - G$ on the interval $[a, b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $H(d) = F(b) = G(b)$, which yields the required identity. ∎

3.6 Interchanging Limits and Integrals

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x)$ converges for all $x \in [a, b]$. We wish to determine whether or not

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. Now

$$\lim_{j \to +\infty} \int_0^1 f_j(x)\, dx = \lim_{j \to +\infty} \left(\frac{j}{j+1} - \frac{j}{2j+1} \right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $jx^j \to 0$ as $j \to +\infty$. Now

$$\lim_{j \to +\infty} \frac{j+1}{j} = 1.$$

It follows that

$$\lim_{j \to +\infty} \frac{(j+1)x}{j} = x < 1,$$

Let r be chosen so that $x < r < 1$. Then there exists some positive integer N such that

$$\frac{(j+1)x^{j+1}}{jx^j} = \frac{(j+1)x}{j} \leq r$$

whenever $j \geq N$. Then $0 \leq (j+1)x^{j+1} \leq rjx^j$ whenever $j \geq N$. Let $B = Nx^N$. Then $0 \leq jx^j \leq Br^{j-N}$ whenever $j \geq N$, and therefore $jx^j \to 0$ as $j \to +\infty$. It follows that

$$\lim_{j\to+\infty} f_j(x) = \left(\lim_{j\to+\infty} jx^j\right)\left(\lim_{j\to+\infty}(1-x^j)\right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_j(1) = 0$ for all positive integers j. We conclude that $\lim\limits_{j\to+\infty} f_j(x) = 0$ for all $x \in [0,1]$, which is what we set out to show.

3.7 Uniform Convergence

We now introduce the concept of *uniform convergence*. Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a,b]$, the identity

$$\lim_{j\to+\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j\to+\infty} f_j(x)\right) dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a,b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_j) is said to converge *uniformly* to a function f on D as $j \to +\infty$ if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all positive integers j satisfying $j \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each positive integer j, let $M_j = \sup\{f_j(x) : x \in D\}$. We claim that $M_j \to 0$ as $j \to +\infty$.

To prove this, let some strictly positive real number ε be given. Then there exists some positive integer N such that $|f_j(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $j \geq N$. Thus if $j \geq N$ then $M_j \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_j \to 0$ as $j \to +\infty$, as claimed.

Example Let $(f_j : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. We have already shown (in an earlier example) that $\lim_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. However a straightforward exercise in calculus shows that the maximum value attained by the function f_j is $j/4$ (which is attained at $x = 1/2^{\frac{1}{j}}$), and $j/4 \to +\infty$ as $j \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \ldots$ does not converge uniformly to the zero function on the interval $[0, 1]$.

Proposition 3.22 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let some strictly positive real number ε be given. If j is chosen sufficiently large then $|f(x) - f_j(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_j \to f$ uniformly on D as $j \to +\infty$. It then follows from the continuity of f_j that there exists some strictly positive real number δ such that $|f_j(x) - f_j(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$\begin{aligned} |f(x) - f(s)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(s)| + |f_j(s) - f(s)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon \end{aligned}$$

whenever $|x - s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 3.23 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a, b]$ to some continuous real-valued function f. Then*

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b f(x)\, dx.$$

Proof Let some strictly positive real number ε. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b - a) < \varepsilon$. Then there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon_0$ for all $x \in [a, b]$ and $j \geq N$, since the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to f on $[a, b]$. Now

$$\left| \int_a^b (f_j(x) - f(x))\, dx \right| \leq \int_a^b |f_j(x) - f(x)|\, dx$$

for all positive integers j (see Proposition 3.10). It follows that

$$\left| \int_a^b f_j(x)\, dx - \int_a^b f(x)\, dx \right| \leq \int_a^b |f_j(x) - f(x)|\, dx \leq \varepsilon_0(b - a) < \varepsilon$$

whenever $j \geq N$. The result follows. ∎

3.8 Integrals over Unbounded Intervals

We define integrals over unbounded intervals by appropriate limiting processes. Given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $[a, +\infty)$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. Similarly, given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined.

If f is bounded and Riemann integrable over each closed bounded interval in $\mathbb{R}$ then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{\substack{a\to-\infty \\ b\to+\infty}} \int_a^b f(x)\,dx,$$

provided that this limit exists.

Remark Using techniques of complex analysis, it can be shown that

$$\lim_{b\to+\infty} \int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

However it can also be shown that

$$\int_0^b \frac{|\sin x|}{x}\,dx \to +\infty \text{ as } b \to +\infty.$$

Therefore, in the standard theory of the Riemann integral, the integral of the function $(\sin x)/x$ on the interval $[0, +\infty)$ is defined, and $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

There is an alternative theory of integration, due to Lebesgue, which is generally more powerful. All bounded Riemann-integrable functions on a closed bounded interval are Lebesgue-integrable on that interval. But a real-valued function f on a "measure space" is Lebesgue-integrable if and only if $|f|$ is Lebesgue-integrable on that measure space. Let $f\colon [0, +\infty) \to \mathbb{R}$ be the real-valued function defined such that $f(0) = 1$ and $f(x) = (\sin x)/x$ for all positive real numbers x. Then the function $|f|$ is neither Riemann-integrable nor Lebesgue-integrable on $[0, +\infty)$. It follows that the function f itself is not Lebesgue-integrable on $[0, +\infty)$. But, as we have remarked, the theory of the Riemann integral assigns a value of $\frac{\pi}{2}$ to $\int_0^{+\infty} f(x)\, dx$.