Course MA2321: Michaelmas Term 2016.
Worked Solutions to Assignment II.

Module MA2321—Analysis in Several Real Variables.
Michaelmas Term 2016.
Assignment II

1. (a) *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be defined such that* $f(x,y) = \min(|x|,|y|)$ *for all* $(x,y) \in \mathbb{R}^2$*. Is* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *continuous at* $(0,0)$*? Is* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *differentiable at* $(0,0)$*?*

   The function f is continuous at $(0,0)$. Inded $|f(x,y)| \leq \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let some positive real number ε be given. If $|(x,y)| < \varepsilon$ then $|f(x,y)| < \varepsilon$. Thus the definition of continuity is satisfied at $(x,y) = 0$.

   The function f is not differentiable at $(0,0)$. Note that

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0.$$

   If it were the case that the function were differentiable at zero, then the derivative of the function at $(0,0)$ would be determined by the above partial derivatives, and would therefore be zero. It would then follow that

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

   Suppose that $x = y = t$. Then $f(x,y) = |t|$ and $\sqrt{x^2+y^2} = \sqrt{2}\,t$. It follows that

$$\lim_{t\to 0+} \frac{f(t,t)}{\sqrt{t^2+t^2}} = \frac{1}{\sqrt{2}}.$$

   Thus it cannot be the case that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0$. Therefore the function f is not differentiable at $(0,0)$.

   (b) *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *be defined such that* $f(x,y) = \min(x^2,y^2)$ *for all* $(x,y) \in \mathbb{R}^2$*. Is* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *continuous at* $(0,0)$*? Is* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *differentiable at* $(0,0)$*?*

   This function is continuous and differentiable at $(0,0)$. Note that $f(x,y) \leq x^2+y^2$ for all $(x,y) \in \mathbb{R}^2$, and therefore

$$\frac{|f(x,y)|}{x^2+y^2} \leq \sqrt{x^2+y^2}$$

for all $(x, y) \in \mathbb{R}^2$. It follows that

$$\lim_{(x,y)\to(0,0)} \frac{|f(x,y)|}{\sqrt{x^2+y^2}} = 0.$$

It then follows from the definition of differentiability that that function f is differentiable at $(0, 0)$, and its derivative at $(0, 0)$ is zero. Differentiability implies continuity. The function f is thus continuous at $(0, 0)$.

2. *In this problem let S^2 denote the 2-dimensional sphere in $\mathbb{R}^3$, defined so that*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Given a point $\mathbf{r}$ on S^2 with components (x, y, z), where $x^2+y^2+z^2 = 1$, we denote by $T_{\mathbf{r}}S^2$ the tangent space to S^2 at $\mathbf{r}$, defined so that

$$\begin{aligned} T_{\mathbf{r}}S^2 &= \{\mathbf{b} \in \mathbb{R}^3 : \mathbf{b} \,.\, \mathbf{r} = 0\} \\ &= \{(u, v, w) \in \mathbb{R}^3 : ux + vy + wz = 0\}. \end{aligned}$$

Let

$$X = \{(x, y, z) \in \mathbb{R}^3 : -1 < z < 1\}$$

and let $\varphi^+ \colon X \to \mathbb{R}^2$ and $\varphi^- \colon X \to \mathbb{R}^2$ be defined so that

$$\varphi^+(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

and

$$\varphi^-(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z} \right) = \varphi^+(x, y, -z).$$

(a) *Let $\mathbf{r}$ be a point of X, where $\mathbf{r} = (x, y, z)$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^3$, where $\mathbf{b} = (u, v, w)$. Determine the components of the vector $(D\varphi^+)_{\mathbf{r}}\mathbf{b}$ and $(D\varphi^-)_{\mathbf{r}}\mathbf{b}$, where $(D\varphi^+)_{\mathbf{r}}$ and $(D\varphi^-)_{\mathbf{r}}$ denote the derivatives of the maps φ_* and φ_- at the point $\mathbf{r}$.*

Let φ_1^+ and φ_2^+ denote the components of φ. Then

$$\varphi_1^+(X, y, z) = \frac{x}{1-z} \quad \text{and} \quad \varphi_2^+(X, y, z) = \frac{y}{1-z}.$$

Representing the linear transformation $D\varphi^+$ by its Jacobian matrix, we find that

$$(D\varphi^+)_{\mathbf{r}} = \begin{pmatrix} \dfrac{\partial \varphi_1^+}{\partial x} & \dfrac{\partial \varphi_1^+}{\partial y} & \dfrac{\partial \varphi_1^+}{\partial z} \\ \dfrac{\partial \varphi_2^+}{\partial x} & \dfrac{\partial \varphi_2^+}{\partial y} & \dfrac{\partial \varphi_2^+}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{1-z} & 0 & \dfrac{x}{(1-z)^2} \\ 0 & \dfrac{1}{1-z} & \dfrac{y}{(1-z)^2} \end{pmatrix},$$

and therefore

$$(D\varphi^+)_{\mathbf{r}}(u,v,w) = \left(\frac{u}{1-z} + \frac{xw}{(1-z)^2}, \frac{v}{1-z} + \frac{yw}{(1-z)^2}, \right).$$

Similarly

$$(D\varphi^-)_{\mathbf{r}} = \begin{pmatrix} \dfrac{1}{1+z} & 0 & \dfrac{-x}{(1+z)^2} \\ 0 & \dfrac{1}{1+z} & \dfrac{-y}{(1+z)^2} \end{pmatrix},$$

and therefore

$$(D\varphi^-)_{\mathbf{r}}(u,v,w) = \left(\frac{u}{1+z} - \frac{xw}{(1+z)^2}, \frac{v}{1+z} - \frac{yw}{(1+z)^2}, \right).$$

(b) *Let (s,t) be a point of $\mathbb{R}^2$, where $(s,t) \neq (0,0)$. Determine the Cartesian coordinates of the unique point $\mathbf{r}$ of $X \cap S^2$ for which $\varphi^+(\mathbf{r}) = (s,t)$, and determine the Cartesian coordinates of $\varphi^-(\mathbf{r})$. Hence determine a formula for the unique map*

$$\psi \colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$$

characterized by the property that

$$\psi(\varphi^+(\mathbf{r})) = \varphi^-(\mathbf{r})$$

for all $\mathbf{r} \in X \cap S^2$. [Hint: express $s^2 + t^2$ as a function of the components of $\mathbf{r}$.]

We must find (x,y,z) satisfying $x^2 + y^2 + z^2 = 1$ for which $\varphi^+(x,y,z) = (s,t)$. Thus we require that

$$s = \frac{x}{1-z} \quad \text{and} \quad t = \frac{y}{1-z}.$$

then

$$s^2 + t^2 = \frac{x^2 + y^2}{(1-z)^2} = \frac{1 - z^2}{(1-z)^2} = \frac{1+z}{1-z}.$$

Then

$$\begin{aligned} & s^2+t^2-z(s^2+t^2)=1+z \\ \Rightarrow\quad & s^2+t^2-1=z(s^2+t^2+1) \\ \Rightarrow\quad & z=\frac{s^2+t^2-1}{s^2+t^2+1} \end{aligned}$$

Then

$$1-z=\frac{2}{s^2+t^2+1},$$

and therefore

$$(x,y,z)=\left(\frac{2s}{s^2+t^2+1},\frac{2t}{s^2+t^2+1},\frac{s^2+t^2-1}{s^2+t^2+1}\right)$$

Then

$$1+z=\frac{2(s^2+t^2)}{s^2+t^2+1},$$

and therefore

$$\psi(s,t)=\varphi^+(x,y,z)=\left(\frac{x}{1+z},\frac{y}{1+z}\right)=\left(\frac{s}{s^2+t^2},\frac{t}{s^2+t^2}\right).$$

(c) Let $(s,t)=\varphi^+(\mathbf{r})$, where $\mathbf{r}=(x,y,z)$, and let $(p,q)\in\mathbb{R}^2$. Determine the unique element (u,v,w) of the tangent space $T_\mathbf{r}S^2$ to S^2 at $\mathbf{r}$ for which $(D\varphi^+)_\mathbf{r}(u,v,w)=(p,q)$. (Note that $(u,v,w)\in T_\mathbf{r}S^2$ if and only if $ux+vy+wz=0$.)

We require that

$$p=\frac{u}{1-z}+\frac{xw}{(1-z)^2},\qquad q=\frac{v}{1-z}+\frac{yw}{(1-z)^2}$$

and

$$xu+yv+zw=0.$$

Then

$$\begin{aligned} xp+yq &= \frac{xu+yv}{1-z}+\frac{(x^2+y^2)w}{(1-z)^2}=-\frac{zw}{1-z}+\frac{(1-z^2)w}{(1-z)^2} \\ &= -\frac{zw}{1-z}+\frac{(1+z)w}{1-z}=\frac{w}{1-z} \end{aligned}$$

Thus

$$w=(1-z)(xp+yq),$$

and therefore

$$(1-z)p = u + x(xp+yq), \quad (1-z)q = v + y(xp+yq).$$

Thus

$$\begin{aligned} u &= (1-x^2-z)p - xyq, \\ v &= (1-y^2-z)q - xyp, \\ w &= (1-z)(xp+yq). \end{aligned}$$

We now express u, v and w in terms of s, t, p and q. Now

$$s^2+t^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z}.$$

It follows that

$$1+s^2+t^2 = \frac{2}{1-z}$$

and thus

$$1-z = \frac{2}{1+s^2+t^2}$$

Also $x = (1-z)s$ and $y(1-z)t$. It follows that

$$\begin{aligned} u &= \left(\frac{2}{1+s^2+t^2} - \frac{4s^2}{(1+s^2+t^2)^2}\right)p - \frac{4st}{(1+s^2+t^2)^2}q \\ &= \left(\frac{2+2t^2-2s^2}{(1+s^2+t^2)^2}\right)p - \frac{4st}{(1+s^2+t^2)^2}q \\ v &= \left(\frac{2}{1+s^2+t^2} - \frac{4t^2}{(1+s^2+t^2)^2}\right)q - \frac{4st}{(1+s^2+t^2)^2}p \\ &= \left(\frac{2+2s^2-2t^2}{(1+s^2+t^2)^2}\right)q - \frac{4st}{(1+s^2+t^2)^2}p \\ w &= \frac{4(sp+tq)}{(1+s^2+t^2)^2}. \end{aligned}$$

(d) *Determine the* 2×2 *matrix that represents the derivative* $(D\psi)_{(s,t)}$ *of* ψ *at a point* (s,t) *of* $\mathbb{R}^2 \setminus \{(0,0)\}$.

The smooth map $\psi \backslash \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ has Cartesian components ψ_1 and ψ_2, where

$$\psi_1(s,t) = \frac{s}{s^2+t^2}, \quad \psi_2(s,t) = \frac{t}{s^2+t^2}$$

for all $(s,t) \in \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$. A direct computation shows that

$$
\begin{aligned}
(D\psi)_{(s,t)} &= \begin{pmatrix} \dfrac{\partial \psi_1(s,t)}{\partial s} & \dfrac{\partial \psi_1(s,t)}{\partial t} \\ \dfrac{\partial \psi_2(s,t)}{\partial s} & \dfrac{\partial \psi_2(s,t)}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} \dfrac{t^2 - s^2}{(s^2+t^2)^2} & \dfrac{-2st}{(s^2+t^2)^2} \\ \dfrac{-2st}{(s^2+t^2)^2} & \dfrac{s^2 - t^2}{(s^2+t^2)^2} \end{pmatrix}
\end{aligned}
$$