Course MA2321: Michaelmas Term 2016.

Assignment II.

To be handed in by Friday 27th January, 2017.

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....

Module MA2321—Analysis in Several Real Variables.
Michaelmas Term 2016.
Assignment II

1. (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that $f(x,y) = \min(|x|,|y|)$ for all $(x,y) \in \mathbb{R}^2$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous at $(0,0)$? Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ differentiable at $(0,0)$?

   (b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined such that $f(x,y) = \min(x^2,y^2)$ for all $(x,y) \in \mathbb{R}^2$. Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ continuous at $(0,0)$? Is $f\colon \mathbb{R}^2 \to \mathbb{R}$ differentiable at $(0,0)$?

2. In this problem let S^2 denote the 2-dimensional sphere in $\mathbb{R}^3$, defined so that

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Given a point $\mathbf{r}$ on S^2 with components (x,y,z), where $x^2+y^2+z^2 = 1$, we denote by $T_{\mathbf{r}}S^2$ the tangent space to S^2 at $\mathbf{r}$, defined so that

$$\begin{aligned} T_{\mathbf{r}}S^2 &= \{\mathbf{b} \in \mathbb{R}^3 : \mathbf{b} \,.\, \mathbf{r} = 0\} \\ &= \{(u,v,w) \in \mathbb{R}^3 : ux + vy + wz = 0\}. \end{aligned}$$

Let

$$X = \{(x,y,z) \in \mathbb{R}^3 : -1 < z < 1\}$$

and let $\varphi^+\colon X \to \mathbb{R}^2$ and $\varphi^-\colon X \to \mathbb{R}^2$ be defined so that

$$\varphi^+(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and

$$\varphi^-(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) = \varphi^+(x,y,-z).$$

   (a) Let $\mathbf{r}$ be a point of X, where $\mathbf{r} = (x,y,z)$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^3$, where $\mathbf{b} = (u,v,w)$. Determine the components of the vector $(D\varphi^+)_{\mathbf{r}}\mathbf{b}$ and $(D\varphi^-)_{\mathbf{r}}\mathbf{b}$, where $(D\varphi^+)_{\mathbf{r}}$ and $(D\varphi^-)_{\mathbf{r}}$ denote the derivatives of the maps φ_* and φ_- at the point $\mathbf{r}$.

   (b) Let (s,t) be a point of $\mathbb{R}^2$, where $(s,t) \neq (0,0)$. Determine the Cartesian coordinates of the unique point $\mathbf{r}$ of $X \cap S^2$ for which $\varphi^+(\mathbf{r}) = (s,t)$, and determine the Cartesian coordinates of $\varphi^-(\mathbf{r})$. Hence determine a formula for the unique map

$$\psi\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2 \setminus \{(0,0)\}$$

characterized by the property that

$$\psi(\varphi^+(\mathbf{r})) = \varphi^-(\mathbf{r})$$

for all $\mathbf{r} \in X \cap S^2$. [Hint: express $s^2 + t^2$ as a function of the components of $\mathbf{r}$.]

(c) Let $(s,t) = \varphi^+(\mathbf{r})$, where $\mathbf{r} = (x,y,z)$, and let $(p,q) \in \mathbb{R}^2$. Determine the unique element (u,v,w) of the tangent space $T_{\mathbf{r}}S^2$ to S^2 at $\mathbf{r}$ for which $(D\varphi^+)_{\mathbf{r}}(u,v,w) = (p,q)$. (Note that $(u,v,w) \in T_{\mathbf{r}}S^2$ if and only if $ux + vy + wz = 0$.)

(d) Determine the 2×2 matrix that represents the derivative $(D\psi)_{(s,t)}$ of ψ at a point (s,t) of $\mathbb{R}^2 \setminus \{(0,0)\}$.