Course MA2321: Michaelmas Term 2016.

Assignment 1.

To be handed in by Thursday 24th November, 2016.

Module MA2321—Analysis in Several Real Variables.
Michaelmas Term 2016.
Assignment I

1. *Let q be a positive rational number, and let $f\colon [0,1] \to \mathbb{R}$ be the real-valued function on the interval $[0,1]$ defined such that $f(x) = x^q$ for all real numbers satisfying $0 \le x \le 1$.*

   (a) *Given any real number r satisfying $0 < r < 1$, and given any integer k satisfying $k > 2$, let $P_{r,k}$ denote the partition of the interval $[0,1]$ defined such that $P_{r,k} = \{x_0, x_1, x_2, \ldots, x_k\}$, where $x_0 = 0$ and $x_i = r^{k-i}$ for $i = 1, 2, \ldots, k$. Calculate the values of the upper Darboux sum $U(P_{r,k}, f)$ and the lower Darboux sum $L(P_{r,k}, f)$ for given r and k.*

   Let
   $$M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$$
   and
   $$m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$$
   for $i = 1, 2, \ldots, k$. Then
   $$M_1 = f(x_1) = r^{qk-q} \quad \text{and} \quad m_1 = 0,$$
   and
   $$M_i = f(x_i) = r^{qk-qi} \quad \text{and} \quad m_i = f(x_{i-1}) = r^{qk-qi+q} = r^q M_i$$
   for $2 \le i \le k$. It follows from the definitions of the Darboux upper and lower sums that
   $$\begin{aligned} U(P_{r,k}, f) &= r^{q(k-1)} r^{k-1} + \sum_{i=2}^{k} r^{qk-qi}(r^{k-i} - r^{k-i+1}) \\ &= r^{(q+1)(k-1)} + \sum_{j=0}^{k-2} r^{qj}(r^j - r^{j+1}) \\ &\qquad \text{(substituting } i = k - j) \\ &= r^{(q+1)(k-1)} + (1-r) \sum_{j=0}^{k-2} r^{(q+1)j} \\ &= r^{(q+1)(k-1)} + (1-r)\,\frac{1 - r^{(q+1)(k-1)}}{1 - r^{q+1}} \end{aligned}$$

and

$$\begin{aligned} L(P_{r,k}, f) &= \sum_{i=2}^{k} r^{qk-qi+q}(r^{k-i} - r^{k-i+1}) \\ &= r^q \sum_{j=0}^{k-2} r^{qj}(r^j - r^{j+1}) \\ &= r^q(1-r) \sum_{j=0}^{k-2} r^{(q+1)j} \\ &= r^q(1-r)\,\frac{1-r^{(q+1)(k-1)}}{1-r^{q+1}}. \end{aligned}$$

(b) *For each real number r satisfying $0 < r < 1$, let*

$$\alpha(r) = \lim_{k\to+\infty} L(P_{r,k}, f) \quad \textit{and} \quad \beta(r) = \lim_{k\to+\infty} U(P_{r,k}, f).$$

Determine the values of $\alpha(r)$ and $\beta(r)$ for all real numbers r satisfying $0 < r < 1$, and then determine the values of $\lim_{r\to1^-} \alpha(r)$ and $\lim_{r\to1^-} \beta(r)$.

For each real number r satisfying $0 < r < 1$,

$$\begin{aligned} \alpha(r) &= \lim_{k\to+\infty}\left(\frac{r^q(1-r)}{1-r^{q+1}}(1-r^{(q+1)k-1})\right) \\ &= r^q\frac{1-r}{1-r^{q+1}}(1 - \lim_{k\to+\infty} r^{(q+1)k-1}) \\ &= r^q\frac{1-r}{1-r^{q+1}} \end{aligned}$$

and

$$\begin{aligned} \beta(r) &= \lim_{k\to+\infty}\left(r^{(q+1)(k-1))} + \frac{1-r}{1-r^{q+1}}(1-r^{(q+1)k-1})\right) \\ &= \frac{1-r}{1-r^{q+1}}(1 - \lim_{k\to+\infty} r^{(q+1)k}) \\ &= \frac{1-r}{1-r^{q+1}}. \end{aligned}$$

Now it follows from basic calculus that

$$\lim_{r\to1^-} \frac{1-r^{q+1}}{1-r} = \frac{d}{dr}(r^{q+1})\bigg|_{r=1} = q+1.$$

Also $\lim_{r\to 1^-} r^q = 1$. It follows that

$$\lim_{r\to 1^-} \alpha(r) = \lim_{r\to 1^-} \beta(r) = \frac{1}{q+1}.$$

(c) *It follows from the definition of the Riemann integral (or Riemann-Darboux integral) that*

$$L(P_{r,k}, f) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq U(P_{r,k}, f)$$

for all real numbers r satisfying $0 < r < 1$ and for all positive integers k. It follows on taking limits as $k \to +\infty$ that

$$\alpha(r) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq \beta(r).$$

It then follows, on taking limits as r tends to 1 from below, that

$$\lim_{r\to 1^-} \alpha(r) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq \lim_{r\to 1^-} \beta(r).$$

What conclusions concerning the existence and value of the Riemann integral $\int_0^1 x^q\,dx$ can you draw, in the case where q is a positive rational number, in view of the results obtained in (b)?

The upper and lower integrals of x^q on $[0,1]$ satisfy

$$\mathcal{L}\int_0^1 x^q\,dx = \mathcal{U}\int_0^1 x^q\,dx = \frac{1}{q+1}.$$

Therefore the function f sending x to x^q for all $x \in [0,1]$ is Riemann-integrable on $[0,1]$, and moreover

$$\int_0^1 x^q\,dx = \frac{1}{q+1}$$

for all positive rational numbers q.

2. *For each of the following subsets of $\mathbb{R}^3$ determine whether that subset is open in $\mathbb{R}^3$. Determine also whether the subset is closed in $\mathbb{R}^3$. Briefly justify all your answers. (Note that a subset of $\mathbb{R}^3$ that is not open in $\mathbb{R}^3$ need not be closed in $\mathbb{R}^3$, and that a subset of $\mathbb{R}^3$ that is not closed in $\mathbb{R}^3$ need not be open in $\mathbb{R}^3$: it has been a common mistake in previous years for people to wrongly assume that "not open" implies "closed", and that "not closed" implies "open".) It is suggested that you read the note at the end of the question before attempting the question.*

(a) *The set*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 - 6x + y^2 - 8y + z^2 - 10z + 46 < 0\};$$

This set is the open ball of radius 2 about the point $(3, 4, 5)$, and is therefore an open set, because all open balls are open sets. The point $(3, 4, 7)$ is contained in the complement of the set, but every open ball of positive radius about this point intersects the given set. Therefore the complement of the set is not open, and therefore the given set is not closed.

(b) *The set*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 < 3 \text{ and } x^2 - 2x + y^2 + z^2 < 3\};$$

The set is the intersection of open balls of radius 2 about the points $(-1, 0, 0)$ and $(1, 0, 0)$, and is therefore open. The point $(1, 0, 0)$ is in the complement of the set, but every open ball of positive radius about this point intersects the given set (e.g., in a point $(x, 0, 0)$ for some value of x that is sufficiently close to 1 but satisfies $x < 1$). Therefore the given set is not closed.

(c) *The set*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 \geq 3 \text{ or } x^2 - 2x + y^2 + z^2 \geq 3\};$$

This set is the complement of the set considered in (b). It is therefore closed, and not open.

(d) *The set*

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 \geq 3 \text{ and } x^2 - 2x + y^2 + z^2 < 3\};$$

This set is neither open nor closed. The point $(1, 0, 0)$ belongs to the set, but any open ball of positive radius about $(1, 0, 0)$ contains points $(x, 0, 0)$ with $x < 1$ that do not belong to the set, and therefore the set is not open. The point $(3, 0, 0)$ does not belong to the set but every open ball of positive radius about this point contains points $(x, 0, 0)$ with $x < 3$ that belong to the set, and therefore the given set is not closed.

(e) *The set*

$$\{(x, y, z) \in \mathbb{R}^3 : \sin(x^2 + y^2) < \cos(y^2 + z^2)\};$$

This set is open. The function $f\colon \mathbb{R}^3 \to \mathbb{R}$ defined such that

$$f(x,y,z) = \cos(y^2+z^2) - \sin(x^2+y^2)$$

for all $(x,y,z) \in \mathbb{R}^3$ is continuous, and the given set is the preimage of the set of positive real numbers under this function, and thus is the preimage of an open set under a continuous function. The set is not closed. The point $(0, \frac{1}{2}\sqrt{\pi}, 0)$ does not belong to the set, but every open ball of positive radius about this point contains points $(0,y,0)$ where $y < \frac{1}{2}\sqrt{\pi}$, and such points do not belong to the set.

(f) *The set*

$$\{(x,y,z) \in \mathbb{R}^3 : z > 0 \ \textit{and} \ z(x^2+y^2) = 1\}.$$

This set is closed. It can be written in the form

$$\{(x,y,z) \in \mathbb{R}^3 : z \geq 0 \ \text{and} \ z(x^2+y^2) = 1\}.$$

The set $\{(x,y,z) \in \mathbb{R}^3 : z \geq 0\}$ is

$$\{(x,y,z) \in \mathbb{R}^3 : z(x^2+y^2) = 1\}$$

is also a closed set as it is the preimage of the closed set $\{1\}$ under a continuous function of x, y and z. The intersection of these two closed sets is the given set. The set is not open: the point $(1,0,1)$ belongs to the set, but no open ball of positive radius about this point is contained in the set.

***Note**: ideally you should not normally write more than three to five sentences of justification per part. If a subset of $\mathbb{R}^3$ is not open in $\mathbb{R}^3$ then this can be shown by exhibiting a point of that set with the property that no open ball of positive radius about the point in question is contained in the complement of the set. To show that a set is not closed, it suffices to exhibit a point in the complement of the set for which every open ball of positive radius about the point in question intersects the set. In order to show that a set is closed, one may either make use of general properties of closed sets or else show that the complement of the set is open. It is advisable to review the results obtained in Section 4 of the module, and deploy basic results such as the following: unions of open sets are open; finite intersections of open sets are open; preimages of open sets under continuous maps are open.*