Course MA2321: Michaelmas Term 2016.

Assignment 1.

To be handed in by Thursday 24th November, 2016.

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Signed:

....

Module MA2321—Analysis in Several Real Variables.
Michaelmas Term 2016.
Assignment I

1. Let q be a positive rational number, and let $f\colon [0,1] \to \mathbb{R}$ be the real-valued function on the interval $[0,1]$ defined such that $f(x) = x^q$ for all real numbers satisfying $0 \leq x \leq 1$.

   (a) Given any real number r satisfying $0 < r < 1$, and given any integer k satisfying $k > 2$, let $P_{r,k}$ denote the partition of the interval $[0,1]$ defined such that $P_{r,k} = \{x_0, x_1, x_2, \ldots, x_k\}$, where $x_0 = 0$ and $x_i = r^{k-i}$ for $i = 1, 2, \ldots, k$. Calculate the values of the upper Darboux sum $U(P_{r,k}, f)$ and the lower Darboux sum $L(P_{r,k}, f)$ for given r and k.

   (b) For each real number r satisfying $0 < r < 1$, let

   $$\alpha(r) = \lim_{k \to +\infty} L(P_{r,k}, f) \quad \text{and} \quad \beta(r) = \lim_{k \to +\infty} U(P_{r,k}, f).$$

   Determine the values of $\alpha(r)$ and $\beta(r)$ for all real numbers r satisfying $0 < r < 1$, and then determine the values of $\lim\limits_{r \to 1^-} \alpha(r)$ and $\lim\limits_{r \to 1^-} \beta(r)$.

   (c) It follows from the definition of the Riemann integral (or Riemann-Darboux integral) that

   $$L(P_{r,k}, f) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq U(P_{r,k}, f)$$

   for all real numbers r satisfying $0 < r < 1$ and for all positive integers k. It follows on taking limits as $k \to +\infty$ that

   $$\alpha(r) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq \beta(r).$$

   It then follows, on taking limits as r tends to 1 from below, that

   $$\lim_{r \to 1^-} \alpha(r) \leq \mathcal{L}\int_0^1 x^q\,dx \leq \mathcal{U}\int_0^1 x^q\,dx \leq \lim_{r \to 1^-} \beta(r).$$

   What conclusions concerning the existence and value of the Riemann integral $\int_0^1 x^q\,dx$ can you draw, in the case where q is a positive rational number, in view of the results obtained in (b)?

2. For each of the following subsets of $\mathbb{R}^3$ determine whether that subset is open in $\mathbb{R}^3$. Determine also whether the subset is closed in $\mathbb{R}^3$. Briefly justify all your answers. (Note that a subset of $\mathbb{R}^3$ that is not open in $\mathbb{R}^3$ need not be closed in $\mathbb{R}^3$, and that a subset of $\mathbb{R}^3$ that is not closed in $\mathbb{R}^3$ need not be open in $\mathbb{R}^3$: it has been a common mistake in previous years for people to wrongly assume that "not open" implies "closed", and that "not closed" implies "open".) It is suggested that you read the note at the end of the question before attempting the question.

   (a) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : x^2 - 6x + y^2 - 8y + z^2 - 10z + 46 < 0\};$$

   (b) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 < 3 \text{ and } x^2 - 2x + y^2 + z^2 < 3\};$$

   (c) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 \geq 3 \text{ or } x^2 - 2x + y^2 + z^2 \geq 3\};$$

   (d) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + y^2 + z^2 \geq 3 \text{ and } x^2 - 2x + y^2 + z^2 < 3\};$$

   (e) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : \sin(x^2 + y^2) < \cos(y^2 + z^2)\};$$

   (f) The set
   $$\{(x, y, z) \in \mathbb{R}^3 : z > 0 \text{ and } z(x^2 + y^2) = 1\}.$$

   Note: ideally you should not normally write more than three to five sentences of justification per part. If a subset of $\mathbb{R}^3$ is not open in $\mathbb{R}^3$ then this can be shown by exhibiting a point of that set with the property that no open ball of positive radius about the point in question is contained in the set. To show that a set is not closed, it suffices to exhibit a point in the complement of the set for which every open ball of positive radius about the point in question intersects the set. In order to show that a set is closed, one may either make use of general properties of closed sets or else show that the complement of the set is open. It is advisable to review the results obtained in Section 4 of the module, and deploy basic results such as the following: unions of open sets are open; finite intersections of open sets are open; preimages of open sets under continuous maps are open.