Module MA2321: Analysis in Several Real Variables
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Section 9

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Contents

9 Topologies, Compactness, and the Multidimensional Heine-Borel Theorem

9.1 Open Sets in Subsets of Euclidean Spaces

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A subset U of X is said to be *open in* X if, given any point $\mathbf{u}$ of U, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

Lemma 9.1 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$. Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 6.13), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 6.15). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \in V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

9.2 Topological Spaces

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

It follows from Proposition 6.15 that if X is a subset of n-dimensional Euclidean space then the collection of subsets of X that are open in X is a topology on X. We refer to this topology as the *usual topology* on X. A subset U of X is open with respect to the usual topology on X if and only if, given any point $\mathbf{u}$ of U, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Any subset of a Euclidean space is a Hausdorff space. Indeed let X be a subset of a Euclidean space $\mathbb{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of X. Let $\delta = \frac{1}{2}|\mathbf{x} - \mathbf{y}|$. Then the open balls of radius δ about the points $\mathbf{x}$ and $\mathbf{y}$ are open sets in X containing $\mathbf{x}$ and $\mathbf{y}$ respectively whose intersection is the empty set.

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A. (It is a straightforward exercise to verify that the topological space axioms are satisfied.) The topology τ_A on A is referred to as the *subspace topology* on A.

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Euclidean space $\mathbb{R}^n$ of dimension n is a topological space with the usual topology. It follows from Lemma 9.1 that the usual topology on any subset X of $\mathbb{R}^n$ is the subspace topology on that subset.

Definition A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

It follows from Proposition 6.19 that the definition of continuity for functions between topological spaces generalizes the standard definition of continuity for functions between subsets of Euclidean spaces.

It is an easy exercise to prove from the definition of continuity for functions between topological spaces that any composition of continuous functions is continuous.

Let $f: X \to Y$ be a continuous function between topological spaces X and Y. Then $f^{-1}(G)$ is closed in X for all closed sets G in Y. Indeed if G is a closed set in Y then the complement $Y \setminus G$ of Y in G is an open set in Y. The continuity of $f: X \to Y$ ensures that $f^{-1}(Y \setminus G)$ is closed in X. But it is straightforward to verify that $f^{-1}(Y \setminus G) = X - f^{-1}(G)$. It follows that $f^{-1}(G)$ is closed in X.

Definition Let X and Y be topological spaces. A function $h: X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: Y \to X$),
- the function $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h: X \to Y$ from X to Y.

If $h: X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

9.3 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 9.2 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

Lemma 9.3 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 9.2 that A is compact, as required. ∎

Lemma 9.4 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 9.5 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof For each positive integer m let

$$U_m = \{x \in X : -m < f(x) < m\}.$$

Then $U_m = f^{-1}((-m, m))$, where $(-m, m)$ is the open interval in $\mathbb{R}$ consisting of all real numbers t that satisfy $-m < t < m$. It follows from the definition of continuity for functions between topological space that U_m is open in X for all positive integers k. Now, given any point x of X, there exists some positive integer m such that $-m < f(x) < m$. It follows that the open sets $U_1, U_2, U_3, \ldots$ cover the compact space X. The definition of compactness ensures the existence of a finite subcover $U_{m_1}, U_{m_2}, \ldots, U_{m_k}$, where $m_1, m_2, \ldots, m_k$ are positive integers. Let M be the maximum of $m_1, m_2, \ldots, m_k$. Then $-M < f(x) < M$ for all $x \in X$. The result follows. ∎

Proposition 9.6 *Let $f: X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 9.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x) - m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 9.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

9.4 Compact Subsets of Euclidean Spaces

Proposition 9.7 *Let A be a compact subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then A is closed and bounded in $\mathbb{R}^n$.*

Proof The function that sends each $\mathbf{x} \in A$ to $|\mathbf{x}|$ is a continuous function on A. Every continuous function on a compact topological space is bounded (Lemma 9.5). It follows that there exists a real number M such that $|\mathbf{x}| < M$ for all $\mathbf{x}$ in A. Thus the set A is bounded.

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to A, and let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{p})$. The function f is continuous on $\mathbb{R}^n$. It therefore follows from Proposition 9.6

that there is a point $\mathbf{q}$ of A such that $f(\mathbf{x}) \geq f(\mathbf{q})$ for all $\mathbf{x} \in A$, since A is compact. Now $f(\mathbf{q}) > 0$, since $\mathbf{q} \neq \mathbf{p}$. Let δ satisfy $0 < \delta \leq f(\mathbf{q})$. Then the open ball of radius δ about the point $\mathbf{p}$ is contained in the complement of A, since $f(\mathbf{x}) < f(\mathbf{q})$ for all points $\mathbf{x}$ of this open ball. It follows that the complement of A is an open set in $\mathbb{R}^n$, and thus A itself is closed in $\mathbb{R}^n$. ∎

We shall prove the converse of Proposition 9.7. The proof will make use of the following proposition.

Proposition 9.8 *Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property. Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then $B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ of points of X with the property that, for all positive integers j, the open ball $B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$. The sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) that there would exist a convergent subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$. Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 6.18). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 9.8 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\operatorname{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\operatorname{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \le K$ for all $\mathbf{x}, \mathbf{y} \in A$.

A *hypercube* in n-dimensional Euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \le x_i \le u_i + l\},$$

where l is a positive constant that is the length of the edges of the hypercube and $(u_1, u_2, \ldots, u_n)$ is the point in $\mathbb{R}^n$ at which the Cartesian coordinates of points in the hypercube attain their minimum values. The diameter of a hypercube with edges of length l is $l\sqrt{n}$.

Lemma 9.9 *Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the* $\operatorname{diam}(A_i) < \delta$ *for $i = 1, 2, \ldots, s$ and*

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof The set X is bounded, and therefore there exists some positive real number M such that that if $(x_1, x_2, \ldots, x_n) \in X$ then $-M \le x_j \le M$ for

$j = 1, 2, \ldots, n$. Choose k large enough to ensure that $2M/k < \delta_L/\sqrt{n}$. Then the large hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -M \leq x_j \leq M \text{ for } j = 1, 2, \ldots, n\}$$

can be subdivided into k^n hypercubes with edges of length l, where $l = 2M/k$. Each of the smaller hypercubes is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_j \leq x_j \leq u_j + l \text{ for } j = 1, 2, \ldots, n\},$$

where $(u_1, u_2, \ldots, u_n)$ is the corner of the hypercube at which the Cartesian coordinates have their minimum values. If $\mathbf{p}$ is a point belonging to such a small hypercube, then all points of the hypercube lie within a distance $l\sqrt{n}$ of the point $\mathbf{p}$. It follows that the small hypercube is wholly contained within the open ball $B_{\mathbb{R}^n}(\mathbf{p}, \delta_L)$ of radius δ about the point $\mathbf{p}$.

Now the number of small hypercubes resulting from the subdivision is finite. Let $H_1, H_2, \ldots, H_s$ be a listing of the small hypercubes that intersect the set X, and let $A_i = H_i \cap X$. Then $\operatorname{diam}(H_i) \leq \sqrt{n}l < \delta_L$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

as required. ∎

Theorem 9.10 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof It follows from Proposition 9.15 that a compact subset of $\mathbb{R}^n$ is both closed and bounded. We must prove the converse.

Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 9.8 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 9.9 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎

9.5 Compact Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality.*

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function* d, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$ defined above.

Definition Let (X, d) be a metric space. Given a point x of X and $r \geq 0$, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined by

$$B_X(x, r) = \{x' \in X : d(x', x) < r\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some $\delta > 0$ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Let (X, d) be a metric space. Then $\emptyset$ and X itself are open subsets of X. Every union of open subsets in X is itself an open set in X. Also any finite intersection of open sets in X is an open set in X. (The proof of these results is a straightforward generalization of the proof of Proposition 6.15).

Lemma 9.11 *Any open ball in a metric space is an open set.*

Proof Let X be a metric space with distance function d let x be a point of X, and let r be a positive real number. If $y \in B_X(x,r)$ and if $z \in B_X(y,\delta)$, where $B_X(x,r)$ and $B_X(y,\delta)$ are the open balls of radius r and δ about the points x and y respectively, then

$$d(z,x) \leq d(z,y) + d(y,x) < d(y,x) + \delta.$$

But $d(y,x) < r$. It follows that if $0 < \delta < r - d(y,x)$ then $B_X(y,\delta) \subset B_X(x,r)$. ∎

Lemma 9.12 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\delta = \frac{1}{2}d(x,y)$. Then $x \in B_X(x,\delta)$ and $y \in B_X(y,\delta)$. Moreover $B_X(x,\delta) \cap B_X(y,\delta) = \emptyset$. Indeed were there to exist some point z in the intersection of $B_X(x,\delta) \cap B_X(y,\delta) = \emptyset$ then $d(x,y) \leq d(x,y) + d(y,z) < 2\delta$; but this contradicts the choice of δ. The balls $B_X(x,\delta)$ and $B_X(y,\delta)$ are open in X (Lemma 9.11). The result follows. ∎

The following definition of continuity for functions between metric spaces generalizes that for functions of a real or complex variable.

Proposition 9.13 *Let X and Y be metric spaces with distance functions d_X and d_Y respectively. Then one can prove that a function $f: X \to Y$ from X to Y is continuous (in accordance with the definition of continuity for functions between topological spaces) if and only if, given any point x of X and given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x' of X satisfying $d_X(x,x') < \delta$.*

The proof of Proposition 9.13 this result is a straightforward generalization of the proof of Proposition 6.19.

Lemma 9.14 *Let X be a metric space with distance function d, and let p be a point of X. Let $f_p: X \to \mathbb{R}$ be the function defined such that $f_p(x) = d(x,p)$ for all $x \in X$. Then the function f_p is continuous on X. Moroever $|f_p(x) - f_p(y)| \leq d(x,y)$ for all $x, y \in X$.*

Proof Let x and y be points of X. Then

$$f_p(x) = d(x,p) \leq d(x,y) + d(y,p) = f_p(y) + d(x,y)$$

and therefore $f_p(x) - f_p(y) \leq d(x,y)$. Interchanging x and y, we find that $f_p(y) - f_p(x) \leq d(x,y)$. It follows that $|f_p(x) - f_p(y)| \leq d(x,y)$ for all $x, y \in X$. The required result then follows on applying Lemma 9.14. ∎

Proposition 9.15 *Let A be a compact subset of a metric space X. Then A is closed in X.*

Proof Let p be a point of X that does not belong to A, and let $f_p(x) = d(x,p)$, where d is the distance function on X. It follows from Proposition 9.6 that there is a point a_0 of A such that $f_p(a) \geq f_p(a_0)$ for all $a \in A$, since A is compact. Now $f_p(a_0) > 0$, since $a_0 \neq p$. Let δ satisfy $0 < \delta \leq f(a_0)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f_p(x) < f_p(a_0)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X. ∎

Let X be a metric space with distance function d. Given a closed subset A of X, we denote by $d(x,A)$ the greatest lower bound on the distances from x to the points of the set A. Thus

$$d(x,A) = \inf\{d(x,a) : a \in A\}.$$

Lemma 9.16 *Let X be a metric space with distance function d, let A be a closed set in X, and let $f_A: X \to \mathbb{R}$ be defined so that*

$$f_A(x) = d(x,A) = \inf\{d(x,a) : a \in A\}.$$

Then the function f_A is continuous on X, and

$$A = \{x \in X : f_A(x) = 0\}.$$

Moreover $|f_A(x) - f_A(y)| \leq d(x,y)$ for all $x,y \in A$.

Proof Let x and y be points of X. Then $d(x,a) \geq f_A(x)$ and $d(y,a) \geq f_A(y)$ for all $x \in A$. Let some positive real number ε be given. Then there exist points p and q of A such that $d(x,p) < f_A(x) + \varepsilon$ and $d(y,q) < f_A(y) + \varepsilon$. Then

$$f_A(x) \leq d(x,q) \leq d(x,y) + d(y,q) \leq f_A(y) + d(x,y) + \varepsilon.$$

It follows from this that $f_A(x) - f_A(y) < d(x,y) + \varepsilon$ for all positive real numbers ε, and therefore $f_A(x) - f_A(y) \leq d(x,y)$. Similarly $f_A(y) - f_A(x) \leq d(x,y)$. Thus $|f_A(x) - f_A(y)| < d(x,y)$ for all $x,y \in X$. It follows from Lemma 9.14 that the function $f_A: X \to \mathbb{R}$ is continuous. If $x \in A$ then $0 \leq f_A(x) \leq d(x,x)$, and $d(x,x) = 0$, and therefore $f_A(x) = 0$. If $x \notin A$ then there exists some positive real number δ such that the open ball of radius δ about the point A is contained in the complement of A and therefore $f_A(x) \geq \delta > 0$. Therefore A point x of X belongs to the subset A if and only if $f_A(x) = 0$. The result follows. ∎

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x, y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x, y)$ as x and y range over all points of A.)

Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists some non-negative real number K with the property that $d(x, y) \leq K$ for all $x, y \in A$.

Definition Let X be a metric space with distance function d. The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of X is defined so that

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

Lemma 9.17 (Lebesgue Covering Lemma) *Let (X, d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ_L such that every subset of X whose diameter is less than δ_L is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

1st Proof The open cover $\mathcal{U}$ of X has a finite subcover, because X is compact. Therefore there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to the open cover $\mathcal{U}$ which covers X. Let $A_i = X \setminus V_i$ for $i = 1, 2, \ldots, k$, let

$$f_i(x) = d(x, A_i) = \inf\{d(x, a) : a \in A_i\}.$$

for $i = 1, 2, \ldots, k$, and let

$$F(x) = \sum_{i=1}^{k} f_i(x) = \sum_{i=1}^{k} d(x, A_i).$$

It follows from Lemma 9.16 that each function f_i is a continuous function on X. Therefore the function $F\colon X \to \mathbb{R}$ is a continuous real-valued function on X.

Given any point x of X there exists some integer i between 1 and k for which $x \in V_i$. Then $x \notin A_i$. It follows from Lemma 9.16 that $f_i(x) > 0$. Therefore $F(x) > 0$. Thus $F(x)$ is strictly positive for all $x \in X$. It follows from Proposition 9.6 that there exists some point u of X with the property that $F(x) \geq F(u)$ for all $x \in X$. Let δ_L be a positive real number for which $k\delta_L < F(u)$.

Let $g(x) = \text{maximum}(f_1(x), f_2(x), \ldots, f_k(x)$ for all $x \in X$. Then $k\delta_L < F(u) \leq F(x) \leq kg(x)$ for all $x \in X$. Therefore, given any point x in X, there exists some integer i between 1 and k for which $f_i(x) > \delta_L$. But then $d(x, A_i) > \delta_L$, and therefore the open ball $B_X(x, \delta_L)$ of radius δ_L about the point x is wholly contained in the open set V_i. Now any non-empty subset of X of diameter less than δ_L is contained within $B_X(x, \delta)$ for any $x \in L$. Therefore every subset of X of diameter less than δ_L is wholly contained within one of the open sets belonging to the open cover $\mathcal{U}$, as required. ∎

2nd Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta_L > 0$ be given by

$$\delta_L = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ_L. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta_L + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Definition Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ_L such that every subset of X whose diameter is less than δ_L is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.

The Lebesgue Covering Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f\colon X \to Y$ be a function from X to Y. The function f

is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 9.18 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f: X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 9.17) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f: X \to Y$ is uniformly continuous, as required. ∎

Definition A metric space X with distance function d is said to be *totally bounded* if and only if, given any positive real number δ, there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that $\operatorname{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Lemma 9.9 ensures that every bounded subset of n-dimensional Euclidean space is totally bounded.

Lemma 9.19 *Let X be a metric space that is totally bounded. Suppose that every open cover of X has a Lebesgue number. Then X is compact.*

Proof Let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L that is a Lebesgue number for this open cover. There then exists a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X such that $\operatorname{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

because X is totally bounded. The definition of Lebesgue numbers then ensures that, for each integer i between 1 and k, there exists an open set V_i belonging to the open cover $\mathcal{V}$ such that $A_i \subset V_i$. Then

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_k.$$

Thus the open cover $\mathcal{V}$ has a finite subcover. This proves that X is compact, as required. ∎

Remark The proof of Lemma 9.19 is an obvious generalization of part of the proof of the multidimensional Heine-Borel Theorem (Theorem 9.10) given above.

Definition A metric space X is said to be *sequentially compact* if every sequence of points in X has a convergent subsequence.

The multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) and Lemma 6.18 together ensure that every closed bounded subset of a Euclidean space is sequentially compact.

Proposition 9.20 *Let X be a sequentially compact metric space. Then, given any open cover of X, there exists a Lebesgue number for that open cover.*

Proposition 9.8 is a special case of Proposition 9.20, and the proof of the latter proposition is an obvious generalization of that of the former.

Let X be a metric space with distance function d An infinite sequence $x_1, x_2, x_3, \ldots$ of points in X is said to be a *Cauchy sequence* if, given any positive real number ε, there exists some positive integer N such that $d(x_j, x_k) < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

It can be shown that the three following conditions on a metric space are equivalent:—

(i) the metric space is compact;

(ii) the metric space is sequentially compact;

(iii) the metric space is complete and totally bounded;

(iv) the metric space is totally bounded and, given any open cover of the space, there is a Lebesgue number for that open cover.

9.6 Norms on a Finite-Dimensional Vector Space

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

Any normed vector space $(X, \|.\|)$ is a metric space with distance function d defined so that $d(x, y) = \|x - y\|$ for all $x, y \in X$.

In addition to the Euclidean norm, the norms on $\mathbb{R}^n$ include the norms $\|.\|_1$ and $\|.\|_{\text{sup}}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\text{sup}} = \text{maximum}(|x_1|, |x_2|, \ldots, |x_n|).$$

Definition Let X and Y be normed vector spaces. A linear transformation $T: X \to Y$ is said to be *bounded* if there exists some non-negative real number C with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If T is bounded, then the smallest non-negative real number C with this property is referred to as the *operator norm* of T, and is denoted by $\|T\|$.

A linear transformation between normed vector spaces is continuous if and only if it is bounded.

Definition Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.

Suppose that norms $\|.\|$ and $\|.\|_*$ be equivalent norms on a real vector space X. Then there exist positive constants C and C_* such that $\|x\|_* \leq C\|x\|$ and $\|x\| \leq C_*\|x\|_*$ for all $x \in X$. Let V be a subset of X that is open with respect to the norm $\|.\|_*$, and let $p \in V$. Then there exists a positive real number δ small enough to ensure that

$$\{x \in X : \|x - p\|_* < C\delta\} \subset V.$$

Then

$$\{x \in X : \|x - p\| < \delta\} \subset V.$$

It follows that if V is open in the topology generated by the $\|.\|_*$ norm then it is also open in the topology generated by the $\|.\|$ norm. Conversely if V is open in the topology generated by the $\|.\|$ norm then it is also open in the topology generated by the $\|.\|_*$ norm. Thus if norms $\|.\|$ and $\|.\|_*$ are equivalent, then they generate the same topology on X.

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 9.21 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous with respect to the topology generated by the Euclidean norm on $\mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Using Schwarz' Inequality, we see that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= \left\| \sum_{j=1}^n (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j - y_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x} - \mathbf{y}|, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and $|\mathbf{x} - \mathbf{y}|$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$, defined so that

$$|\mathbf{x} - \mathbf{y}| = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}.$$

Also $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$, since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq C|\mathbf{x} - \mathbf{y}|,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ with respect to the topology generated by the Euclidean norm on $\mathbb{R}^n$. ∎

Theorem 9.22 *Any two norms on $\mathbb{R}^n$ are equivalent.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. Also the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous (Lemma 9.21). Also it follows from the Extreme Value Theorem (Theorem 6.21) that any continuous real-valued function on a closed bounded subset of Euclidean space attains both its maximum and minimum values on that subset. Therefore there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda| \, \|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. ∎

Let X be a finite-dimensional real vector space. Then X is isomorphic to $\mathbb{R}^n$, where n is the dimension of X. It follows immediately from Theorem 9.22 and that all norms on X are equivalent and therefore generate the same topology on X. This result does not generalize to infinite-dimensional vector spaces.