# Module MA2321: Analysis in Several Real Variables
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# Section 6

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## Contents

# 6 Euclidean Spaces, Continuity, and Open Sets

## 6.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all $n$-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents $n$-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let $\lambda$ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

**Proposition 6.1** (Schwarz's Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

**Proof** We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers $\lambda$ and $\mu$. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers $\lambda$ and $\mu$. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

**Corollary 6.2** (Triangle Inequality) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$. *Then* $|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

**Proof** Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x}+\mathbf{y}|^2 &= (\mathbf{x}+\mathbf{y}).(\mathbf{x}+\mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x}\cdot\mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 6.2) that

$$|\mathbf{z}-\mathbf{x}| \leq |\mathbf{z}-\mathbf{y}| + |\mathbf{y}-\mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

## 6.2 Convergence of Sequences in Euclidean Spaces

**Definition** A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number $\varepsilon$ satisfying $\varepsilon > 0$ there exists some positive integer $N$ such that $|\mathbf{p}-\mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j\to+\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

**Lemma 6.3** *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$*, where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$*. Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* $i$*th components of the elements of this sequence converge to* $p_i$ *for* $i = 1, 2, \ldots, n$.

**Proof** Let $x_{ji}$ and $p_i$ denote the $i$th components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim\limits_{j\to+\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all $j$. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each $i$, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let $N$ be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

**Definition** A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be a *Cauchy sequence* if and only if the following criterion is satisfied:—

> given any real number $\varepsilon$ satisfying $\varepsilon > 0$ there exists some positive integer $N$ such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

**Lemma 6.4** *A sequence of points in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

**Proof** Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$ converging to some point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some positive integer $N$ such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If $j \geq N$ and $k \geq N$ then

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by the Triangle Inequality. Thus every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Now let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then the $i$th components of the elements of this sequence constitute a Cauchy sequence of real numbers. This Cauchy sequence must converge to some real number $p_i$, by Cauchy's Criterion for Convergence (Theorem 2.7). It follows from Lemma 6.3 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. ∎

## 6.3 Continuity of Functions of Several Real Variables

**Definition** Let $X$ and $Y$ be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from $X$ to $Y$ is said to be *continuous* at a point $\mathbf{p}$ of $X$ if and only if the following criterion is satisfied:—

> given any strictly positive real number $\varepsilon$, there exists some strictly positive real number $\delta$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on $X$ if and only if it is continuous at every point $\mathbf{p}$ of $X$.

**Lemma 6.5** *Let $X$, $Y$ and $Z$ be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f: X \to Y$ and $g: Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that $f$ is continuous at some point $\mathbf{p}$ of $X$ and that $g$ is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

**Proof** Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

**Lemma 6.6** *Let $X$ and $Y$ be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a continuous function from $X$ to $Y$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $X$ which converges to some point $\mathbf{p}$ of $X$. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

**Proof** Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function $f$ is continuous at $\mathbf{p}$. Also there exists some positive integer $N$ such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let $X$ and $Y$ be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from $X$ to $Y$. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from $X$ to $\mathbb{R}$, referred to as the *components* of the function $f$.

**Proposition 6.7** *Let $X$ and $Y$ be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f: X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

**Proof** Note that the $i$th component $f_i$ of $f$ is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its $i$th coordinate $y_i$. Now any composition of continuous functions is continuous, by Lemma 6.5. Thus if $f$ is continuous at $\mathbf{p}$, then so are the components of $f$.

Conversely suppose that the components of $f$ are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let $\delta$ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function $f$ is continuous at $\mathbf{p}$, as required. ∎

**Lemma 6.8** *The functions $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x, y) = x + y$ and $m(x, y) = xy$ are continuous.*

**Proof** Let $(u, v) \in \mathbb{R}^2$. We first show that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(u, v)$. Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If $(x, y)$ is any point of $\mathbb{R}^2$ whose distance from $(u, v)$ is less than $\delta$ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence

$$|s(x, y) - s(u, v)| = |x + y - u - v| \leq |x - u| + |y - v| < 2\delta = \varepsilon.$$

This shows that $s: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(u, v)$.

Next we show that $m: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(u, v)$. Now

$$m(x, y) - m(u, v) = xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

for all points $(x, y)$ of $\mathbb{R}^2$. Thus if the distance from $(x, y)$ to $(u, v)$ is less than $\delta$ then $|x - u| < \delta$ and $|y - v| < \delta$, and hence $|m(x, y) - m(u, v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x, y) - m(u, v)| < \varepsilon$ for all points $(x, y)$ of $\mathbb{R}^2$ whose distance from $(u, v)$ is less than $\delta$. This shows that $p: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(u, v)$. ∎

**Proposition 6.9** *Let $X$ be a subset of $\mathbb{R}^n$, and let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be continuous functions from $X$ to $\mathbb{R}$. Then the functions $f + g$, $f - g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function $f/g$ is continuous.*

**Proof** Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h: X \to \mathbb{R}^2$, $s: \mathbb{R}^2 \to \mathbb{R}$ and $m: \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u, v) = u + v$ and $m(u, v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 6.7, Lemma 6.8 and Lemma 6.5 that $f + g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both $f$ and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function $r$ is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that $f/g$ is continuous. ∎

**Example** Consider the function $f: \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}^2$ defined by

$$f(x, y) = \left( \frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right).$$

The continuity of the components of the function $f$ follows from straightforward applications of Proposition 6.9. It then follows from Proposition 6.7 that the function $f$ is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$.

## 6.4 Limits of Functions of Several Real Variables

**Definition** Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p} \in \mathbb{R}^m$. The point $\mathbf{p}$ is said to be a *limit point* of the set $X$ if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of $X$ such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

It follows easily from the definition of convergence of sequences of points in Euclidean space that if $X$ is a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$ and if $\mathbf{p}$ is a point of $\mathbb{R}^m$ then the point $\mathbf{p}$ is a limit point of the set $X$ if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of $X$, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

**Definition** Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set $X$ into $n$-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set $X$, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in $X$, if and only if the following criterion is satisfied:—

> given any strictly positive real number $\varepsilon$, there exists some strictly positive real number $\delta$ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set $X$ into $n$-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set $X$, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in $X$ then we can denote this fact by writing $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

**Proposition 6.10** *Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set $X$ into $n$-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set $X$, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. Let $\tilde{X} = X \cup \{\mathbf{p}\}$, and let $\tilde{f}: \tilde{X} \to \mathbb{R}^n$ be defined such that*

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p}; \\ \mathbf{q} & \text{if } \mathbf{x} = \mathbf{p}. \end{cases}$$

*Then $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(x) = \mathbf{q}$ if and only if the function $\tilde{f}$ is continuous at $\mathbf{p}$.*

**Proof** The result follows directly on comparing the relevant definitions. ∎

**Corollary 6.11** *Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}^n$ be a function mapping the set $X$ into $n$-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set $X$ that is also a limit point of $X$. Then the function $f$ is continuous at the point $\mathbf{p}$ if and only if $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set $X$. Suppose that the point $\mathbf{p}$ is not a limit point of the set $X$. Then there exists some strictly positive real number $\delta_0$ such that $|\mathbf{x}-\mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$. The point $\mathbf{p}$ is then said to be an *isolated point* of $X$.

Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f\colon X \to \mathbb{R}^n$ mapping the set $X$ into $n$-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain $X$.

**Corollary 6.12** *Let $X$ be a subset of $m$-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be real-valued functions on $X$, and let $\mathbf{p}$ be a limit point of the set $X$. Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x}))$, $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x}))$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x}))$, and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),\\
\lim_{\mathbf{x}\to\mathbf{p}}(f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

*If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then*

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

**Proof** Let $\tilde{X} = X \cup \{\mathbf{p}\}$, and let $\tilde{f}\colon \tilde{X} \to \mathbb{R}$ and $\tilde{g}\colon \tilde{X} \to \mathbb{R}$ be defined such that

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p};\\ l & \text{if } \mathbf{x} = \mathbf{p}. \end{cases} \qquad \tilde{g}(\mathbf{x}) = \begin{cases} g(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p};\\ m & \text{if } \mathbf{x} = \mathbf{p}. \end{cases},$$

where $l = \lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $m = \lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$. Then the functions $\tilde{f}$ and $\tilde{g}$ are continuous at $\mathbf{p}$. The result therefore follows on applying Proposition 6.9. ∎

## 6.5 Open Sets in Euclidean Spaces

Let $X$ be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of $X$ and a non-negative real number $r$, the *open ball* $B_X(\mathbf{p}, r)$ in $X$ of *radius* $r$ about $\mathbf{p}$ is defined to be the subset of $X$ given by

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x}-\mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of $X$ that lie within a sphere of radius $r$ centred on the point $\mathbf{p}$.)

**Definition** Let $X$ be a subset of $\mathbb{R}^n$. A subset $V$ of $X$ is said to be *open* in $X$ if and only if, given any point $\mathbf{p}$ of $V$, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of $X$. (The criterion given above is satisfied vacuously in the case when $V$ is the empty set.)

In particular, a subset $V$ of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of $V$, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

**Example** Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where $c$ is some real number. Then $H$ is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of $H$. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point $(x, y, z)$ to the point $(u, v, w)$ is less than $\delta$ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore $H$ is an open set.

The previous example can be generalized. Given any integer $i$ between 1 and $n$, and given any real number $c_i$, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}, \qquad \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

**Example** Let $U$ be an open set in $\mathbb{R}^n$. Then for any subset $X$ of $\mathbb{R}^n$, the intersection $U \cap X$ is open in $X$. (This follows directly from the definitions.) Thus for example, let $S^2$ be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let $N$ be the subset of $S^2$ given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then $N$ is open in $S^2$, since $N = H \cap S^2$, where $H$ is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that $N$ is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to $N$, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius $\delta$ about $(0, 0, 1)$ contains points $(x, y, z)$ for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius $\delta$ about the point $(0, 0, 1)$ is not a subset of $N$.

**Lemma 6.13** *Let $X$ be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of $X$. Then, for any positive real number $r$, the open ball $B_X(\mathbf{p}, r)$ in $X$ of radius $r$ about $\mathbf{p}$ is open in $X$.*

**Proof** Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

**Lemma 6.14** *Let $X$ be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of $X$. Then, for any non-negative real number $r$, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in $X$.*

**Proof** Let $\mathbf{x}$ be a point of $X$ satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of $X$ satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

**Proposition 6.15** *Let $X$ be a subset of $\mathbb{R}^n$. The collection of open sets in $X$ has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set $X$ are both open in $X$;*

(ii) *the union of any collection of open sets in $X$ is itself open in $X$;*

(iii) *the intersection of any* finite *collection of open sets in $X$ is itself open in $X$.*

**Proof** The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set $X$. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in $X$, and let $U$ denote the union of all the open sets belonging to $\mathcal{A}$. We must show that $U$ is itself open in $X$. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set $V$ belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that $U$ is open in $X$. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of $X$ that are open in $X$, and let $V$ denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let $\delta$ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection $V$ of the sets $V_1, V_2, \ldots, V_k$ is itself open in $X$. This proves (iii). ∎

**Example** The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

**Example** The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

**Example** The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers $n$.

**Example** For each positive integer $k$, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set $V_k$ is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets $V_k$ for all positive integers $k$ is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets $V_k$ for all positive integers $k$ is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

**Lemma 6.16** *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set $U$ which contains $\mathbf{p}$, there exists some positive integer $N$ such that $\mathbf{x}_j \in U$ for all $j$ satisfying $j \geq N$.*

**Proof** Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set $U$ which contains $\mathbf{p}$, there exists some positive integer $N$ such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius $\varepsilon$ about $\mathbf{p}$ is an open set by Lemma 6.13. Therefore there exists some positive integer $N$ such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let $U$ be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius $\varepsilon$ about $\mathbf{p}$ is a subset of $U$. Thus there exists some $\varepsilon > 0$ such that $U$ contains all points $\mathbf{x}$ of $X$ that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer $N$ with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

## 6.6 Closed Sets in Euclidean Spaces

Let $X$ be a subset of $\mathbb{R}^n$. A subset $F$ of $X$ is said to be *closed* in $X$ if and only if its complement $X \setminus F$ in $X$ is open in $X$. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

**Example** The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number $c$, since the complements of these sets are open in $\mathbb{R}^3$.

**Example** Let $X$ be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of $X$. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number $r$. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in $X$. (These results follow immediately using Lemma 6.13 and Lemma 6.14 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set $X$. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of $X$ is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of $X$ is the union of the complements of those sets). The following result therefore follows directly from Proposition 6.15.

**Proposition 6.17** *Let $X$ be a subset of $\mathbb{R}^n$. The collection of closed sets in $X$ has the following properties:—*

(i) *the empty set* $\emptyset$ *and the whole set* $X$ *are both closed in* $X$*;*

(ii) *the intersection of any collection of closed sets in* $X$ *is itself closed in* $X$*;*

(iii) *the union of any* finite *collection of closed sets in* $X$ *is itself closed in* $X$*.*

**Lemma 6.18** *Let* $X$ *be a subset of* $\mathbb{R}^n$*, and let* $F$ *be a subset of* $X$ *which is closed in* $X$*. Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *be a sequence of points of* $F$ *which converges to a point* $\mathbf{p}$ *of* $X$*. Then* $\mathbf{p} \in F$*.*

**Proof** The complement $X \setminus F$ of $F$ in $X$ is open, since $F$ is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 6.16 that $\mathbf{x}_j \in X \setminus F$ for all values of $j$ greater than some positive integer $N$, contradicting the fact that $\mathbf{x}_j \in F$ for all $j$. This contradiction shows that $\mathbf{p}$ must belong to $F$, as required. ∎

## 6.7 Continuous Functions and Open Sets

Let $X$ and $Y$ be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from $X$ to $Y$. We recall that the function $f$ is continuous at a point $\mathbf{p}$ of $X$ if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{u}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{u}$ of $X$ satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $f: X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function $f$ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in $X$ and $Y$ of radius $\delta$ and $\varepsilon$ about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f: X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset $V$ of $Y$ under the map $f$, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

**Proposition 6.19** *Let* $X$ *and* $Y$ *be subsets of* $\mathbb{R}^m$ *and* $\mathbb{R}^n$*, and let* $f: X \to Y$ *be a function from* $X$ *to* $Y$*. The function* $f$ *is continuous if and only if* $f^{-1}(V)$ *is open in* $X$ *for every open subset* $V$ *of* $Y$*.*

**Proof** Suppose that $f: X \to Y$ is continuous. Let $V$ be an open set in $Y$. We must show that $f^{-1}(V)$ is open in $X$. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But $V$ is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But $f$ is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that $f$ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$.

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in $X$ for every open set $V$ in $Y$. Let $\mathbf{p} \in X$. We must show that $f$ is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_X(f(\mathbf{p}), \varepsilon)$ is an open set in $Y$, by Lemma 6.13, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in $X$ which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $f$ maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that $f$ is continuous at $\mathbf{p}$, as required. ∎

Let $X$ be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let $c$ be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in $X$, and, given real numbers $a$ and $b$ satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in $X$.

## 6.8 The Multidimensional Bolzano-Weierstrass Theorem

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be *bounded* if there exists some constant $K$ such that $|\mathbf{x}_j| \leq K$ for all $j$.

**Example** Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\, (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between $-1$ and $1$. Moreover $x_j = 0$ whenever $j$ is the square of a positive integer, $y_j = 1$ whenever $j$ is even and $z_j = 1$ whenever $j$ is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those $x_j$ for which $j$ is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and $-1$. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those $x_j$ for which $j$ is the square of an even positive integer. The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when $j$ is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer $p$.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those $x_j$ for which $j$ is equal to two raised to the power of an even positive integer. Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4),\ (x_{16}, y_{16}, z_{16}),\ (x_{64}, y_{64}, z_{64}),\ (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and it therefore follows from Lemma 6.3 that this sequence is a convergent subsequence of the given sequence of points in $\mathbb{R}^3$.

**Example** Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$. Then the first components $x_j$ for which the index $j$ is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components $y_j$ for which the index $j$ is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices $j$ for which $x_j$ is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and $y_j$ is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index $j$ for which $x_j$ and $y_j$ both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded sequence of points in $\mathbb{R}^2$.

**Theorem 6.20** *Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.*

**Proof** We prove the result by induction on the dimension $n$ of the Euclidean space $\mathbb{R}^n$ that contains the infinite sequence in question. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that the theorem is true when $n = 1$. Suppose that $n > 1$, and that every bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of elements of $\mathbb{R}^n$, and let $x_{j,i}$ denote the $i$th component of $\mathbf{x}_j$ for $i = 1, 2, \ldots, n$ and for all positive integers $j$. The induction hypothesis requires that all bounded sequences in $\mathbb{R}^{n-1}$ contain convergent subsequences. Therefore there exist real numbers $p_1, p_2, \ldots, p_{n-1}$ and an increasing sequence $m_1, m_2, m_3, \ldots$ of positive integers such that $\lim\limits_{k \to +\infty} x_{m_k,i} = p_i$ for $i = 1, 2, \ldots, n-1$. The $n$th components $x_{m_1,n}, x_{m_2,n}, x_{m_3,n}, \ldots$ of the members of the subsequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ then constitute a bounded sequence of real numbers. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers for which the sequence $x_{m_{k_1},n}, x_{m_{k_2},n}, x_{m_{k_3},n}, \ldots$ converges. Let $s_j = m_{k_j}$ for all positive integers $j$, and let $p_n = \lim\limits_{j \to +\infty} x_{m_{k_j},n} = \lim\limits_{j \to +\infty} x_{s_j,n}$. Then the sequence $x_{s_1,i}, x_{s_2,i}, x_{s_3,i}, \ldots$ converges for values of $i$ between 1 and $n-1$, because it is a subquence of the convergent sequence $x_{m_1,i}, x_{m_2,i}, x_{m_3,i}, \ldots$. Moreover $x_{s_1,n}, x_{s_2,n}, x_{s_3,n}, \ldots$ also converges. Thus the $i$th components of the infinite sequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ converge for $i = 1, 2, \ldots, n$. It then follows from Lemma 6.3 that $\lim\limits_{j \to +\infty} \mathbf{x}_{s_k} = \mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The result follows. ∎

## 6.9 The Extreme Value Theorem for Functions of Several Real Variables

**Theorem 6.21** (The Extreme Value Theorem for Continuous Functions on Closed Bounded Sets) *Let $X$ be a closed bounded set in $m$-dimensional Euclidean space, and let $f\colon X \to \mathbb{R}$ be a continuous real-valued function defined on $X$. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of $X$ such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

**Proof** We prove the result for an arbitrary continuous real-valued function $f\colon X \to \mathbb{R}$ by showing that the result holds for a related continuous function

$g: X \to \mathbb{R}$ that is known to be bounded above and below on $X$. Let $h: \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. Then the continuous function $h: \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$ (see the proof of Theorem 3.15).

Let $f: X \to \mathbb{R}$ be a continuous real-valued function on the closed bounded set $X$, and let $g: X \to \mathbb{R}$ be the continuous real-valued function defined on $X$ such that

$$g(\mathbf{x}) = h(f(\mathbf{x})) = \frac{f(\mathbf{x})}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Then $-1 \leq g(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$. The set of values of the function $g$ is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then, for each positive integer $j$, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function $g$, and therefore there exists some point $\mathbf{x}_j$ in the set $X$ for which $M - j^{-1} < g(\mathbf{x}_j) \leq M$. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is then a bounded sequence of points in $\mathbb{R}^m$, because the set $X$ is bounded. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{v}$ of $\mathbb{R}^n$. Moreover this point $\mathbf{v}$ belongs to the set $X$ because $X$ is closed (see Lemma 6.18). Now

$$M - \frac{1}{k_j} < g(\mathbf{x}_{k_j}) \leq M$$

for all positive integers $j$, and therefore $g(\mathbf{x}_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 6.6 that

$$g(\mathbf{v}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = M.$$

But $g(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. It follows that $h(f(\mathbf{x})) = g(\mathbf{x}) \leq g(\mathbf{v}) = h(f(\mathbf{v}))$ for all $\mathbf{x} \in X$. Moreover $h: \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

On applying this result with the continuous function $f$ replaced by the function $-f$, we conclude also that there exists some point $\mathbf{u}$ of $X$ such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$. The result follows. ∎

## 6.10 Uniform Continuity for Functions of Several Real Variables

**Definition** Let $X$ be a subset of $\mathbb{R}^m$. A function $f\colon X \to \mathbb{R}^n$ from $X$ to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of $X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

**Theorem 6.22** *Let $X$ be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f\colon X \to \mathbb{R}^n$ is uniformly continuous.*

**Proof** Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer $j$, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in $X$ such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since $X$ is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 6.20). Moreover $\mathbf{p} \in X$, since $X$ is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since $\lim\limits_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0$. But then the sequences $f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$ and $f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$ would converge to $f(\mathbf{p})$, since $f$ is continuous (Lemma 6.6), and thus $\lim\limits_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0$. But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$ for all $j$. We conclude therefore that there must exist some $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎