Module MA2321: Analysis in Several Real Variables
Michaelmas Term 2015
Sections 4 and 5

D. R. Wilkins

Contents

4 Differentiation

4.1 Interior Points and Open Sets in the Real Line

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers. We say that D is an *open set* in $\mathbb{R}$ if every point of D is an interior point of D.

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

4.2 Differentiable Functions

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Suppose that the real-valued function f is defined around some real number s and is differentiable at s. Then

$$f(s+h) = f(s) + h\,\frac{f(s+h) - f(s)}{h}$$

for all real numbers h sufficiently close to zero. It follows that

$$\begin{aligned} \lim_{x \to s} f(x) &= \lim_{h \to 0} f(s+h) = \lim_{h \to 0} f(s) + \left(\lim_{h \to 0} h\right) \left(\lim_{h \to 0} \frac{f(s+h) - f(s)}{h}\right) \\ &= f(s) + 0.f'(s) = f(s), \end{aligned}$$

and therefore f is continuous at s (see Lemma 3.10). Thus differentiability implies continuity.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Let s be a real number. If $h \neq 0$ then

$$\frac{f(s+h) - f(s)}{h} = \frac{(s+h)^2 - s^2}{h} = 2s + h.$$

Therefore the function f is differentiable at s, and $f'(s) = \lim_{h \to 0}(2s + h) = 2s$.

Example Let $g\colon [0, +\infty) \to \mathbb{R}$ be the function defined by $g(x) = \sqrt{x}$, and let $s \in (0, +\infty)$. If h is any real number satisfying $h > -s$ and $h \neq 0$ then

$$\begin{aligned}\frac{g(s+h) - g(s)}{h} &= \frac{\sqrt{s+h} - \sqrt{s}}{h} = \frac{(\sqrt{s+h} - \sqrt{s})(\sqrt{s+h} + \sqrt{s})}{h(\sqrt{s+h} + \sqrt{s})} \\ &= \frac{(s+h) - s}{h(\sqrt{s+h} + \sqrt{s})} = \frac{1}{\sqrt{s+h} + \sqrt{s}}.\end{aligned}$$

Now $\lim_{h \to 0} \sqrt{s+h} = \sqrt{s}$ (since the function $x \mapsto \sqrt{x}$ is continuous at s). It follows that the function g is differentiable at s, and

$$g'(s) = \lim_{h \to 0} \frac{g(s+h) - g(s)}{h} = \frac{1}{\lim_{h \to 0}(\sqrt{s+h} + \sqrt{s})} = \frac{1}{2\sqrt{s}}.$$

Proposition 4.1 *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f + g$ and $f - g$ are differentiable at s, and*

$$(f + g)'(s) = f'(s) + g'(s), \qquad (f - g)'(s) = f'(s) - g'(s).$$

Proof It follows from Proposition 3.11 that

$$\begin{aligned}&\lim_{h \to 0} \frac{(f+g)(s+h) - (f+g)(s)}{h} \\ &\qquad = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h} + \lim_{h \to 0} \frac{g(s+h) - g(s)}{h} \\ &\qquad = f'(s) + g'(s).\end{aligned}$$

Thus the function $f + g$ is differentiable at s, and $(f + g)'(s) = f'(s) + g'(s)$. An analogous proof shows that the function $f - g$ is also differentiable at s and $(f - g)'(s) = f'(s) - g'(s)$. ∎

Proposition 4.2 (Product Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f \cdot g$ is also differentiable at s, and $(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$.*

Proof Note that

$$\begin{aligned}
&\frac{f(s+h)g(s+h)-f(s)g(s)}{h} \\
&\qquad = \frac{f(s+h)-f(s)}{h}g(s+h)+f(s)\frac{g(s+h)-g(s)}{h}.
\end{aligned}$$

Moreover $\lim_{h\to 0} g(s+h) = g(s)$ since g is differentiable, and hence continuous, at s. It follows that

$$\begin{aligned}
&\lim_{h\to 0}\frac{f(s+h)g(s+h)-f(s)g(s)}{h} \\
&\quad = \lim_{h\to 0}\frac{f(s+h)-f(s)}{h}\lim_{h\to 0} g(s+h)+f(s)\lim_{h\to 0}\frac{g(s+h)-g(s)}{h} \\
&\quad = f'(s)g(s)+f(s)g'(s).
\end{aligned}$$

Thus the function $f\cdot g$ is differentiable at s, and $(f\cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$, as required. ∎

Proposition 4.3 (Quotient Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s and that the function g is non-zero around s. Then f/g is differentiable at s, and*

$$(f/g)'(s) = \frac{f'(s)g(s)-f(s)g'(s)}{g(s)^2}.$$

Proof Note that

$$\begin{aligned}
\frac{f(s+h)}{g(s+h)}-\frac{f(s)}{g(s)} &= \frac{f(s+h)g(s)-f(s)g(s+h)}{g(s+h)g(s)} \\
&= \frac{(f(s+h)-f(s))\,g(s)-f(s)\,(g(s+h)-g(s))}{g(s)g(s+h)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
(f/g)'(s) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(s+h)}{g(s+h)}-\frac{f(s)}{g(s)}\right) \\
&= \frac{1}{g(s)^2}\left(\lim_{h\to 0}\frac{f(s+h)-f(s)}{h}g(s)-f(s)\lim_{h\to 0}\frac{g(s+h)-g(s)}{h}\right) \\
&= \frac{f'(s)g(s)-f(s)g'(s)}{g(s)^2},
\end{aligned}$$

since $\lim_{h\to 0} g(s)g(s+h) = g(s)^2 > 0$. ∎

Proposition 4.4 (Chain Rule) *Let a be some real number, let f be a real-valued function defined around a, and let g be a real-valued function defined around $f(a)$. Suppose that the function f is differentiable at a, and the function g is differentiable at $f(a)$. Then the composition function $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof Let $b = f(a)$, and let

$$R(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & \text{if } y \neq b; \\ g'(b) & \text{if } y = b. \end{cases}$$

for values of y around b. By considering separately the cases when $f(a+h) \neq f(a)$ and $f(a+h) = f(a)$, we see that

$$g(f(a+h)) - g(f(a)) = R(f(a+h))(f(a+h) - f(a)).$$

Now the function f is continuous at a, because it is differentiable at a. Also the function R is continuous at b, where $b = f(a)$, since

$$\lim_{y \to b} R(y) = \lim_{y \to b} \frac{g(y) - g(b)}{y - b} = \lim_{k \to 0} \frac{g(b+k) - g(b)}{k} = g'(b) = R(b).$$

It follows from Proposition 3.4 that the composition function $R \circ f$ is continuous at a, and therefore

$$\lim_{h \to 0} R(f(a+h)) = R(f(a)) = g'(f(a))$$

by Lemma 3.10. It follows that $g \circ f$ is differentiable at a, and

$$\begin{aligned} (g \circ f)'(a) &= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \\ &= \lim_{h \to 0} R(f(a+h)) \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = g'(f(a))f'(a), \end{aligned}$$

as required. ∎

4.3 Rolle's Theorem and the Mean Value Theorem

Theorem 4.5 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a, b]$. It therefore follows from the Extreme Value Theorem (Theorem 3.15) that there must exist real numbers u and v in the interval $[a, b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$. If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

4.4 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 4.6 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 4.5) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

4.5 Cauchy's Mean Value Theorem

We now prove a generalization of the standard Mean Value Theorem, known as *Cauchy's Mean Value Theorem.*

Theorem 4.7 (Cauchy's Mean Value Theorem) *Let f and g be real-valued functions defined on some interval $[a, b]$. Suppose that f and g are continuous on $[a, b]$ and are differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$(f(b) - f(a))\, g'(s) = (g(b) - g(a))\, f'(s).$$

In particular, if $g(b) \neq g(a)$ and the function g' is non-zero throughout (a, b), then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(s)}{g'(s)}.$$

Proof Consider the function $h\colon [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x)\,(g(b) - g(a)) - g(x)\,(f(b) - f(a))\,.$$

Then $h(a) = f(a)g(b) - g(a)f(b) = h(b)$, and the function h satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. We deduce from Rolle's Theorem (Theorem 4.5) that $h'(s) = 0$ for some s satisfying $a < s < b$. The required result then follows immediately. ∎

4.6 One-Sided Limits and Limits at Infinity

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s and l be real numbers. We say that l is the *limit* $\lim_{x \to s^+} f(x)$ *of $f(x)$ as x tends to s from above* if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $s < x < s + \delta$.

If f is a real-valued function, if $f(x)$ is defined for all real numbers x greater than but sufficiently close to some real number s, if l is a real number, and if l is the limit of $f(x)$ as x tends to s from above, then we may denote this fact by writing

$$l = \lim_{x \to s^+} f(x).$$

Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s and l be real numbers. We say that l is the *limit* $\lim_{x \to s^-} f(x)$ *of* $f(x)$ *as* x *tends to* s *from below* if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $s - \delta < x < s$.

Let $f: D \to \mathbb{R}$ be a real-valued function defined on some subset D of $\mathbb{R}$, and let s and l be real numbers. Suppose that there exists some positive real number δ_0 with the property that $x \in D$ for all real numbers x satisfying $s < x < s + \delta_0$. Then $\lim_{x \to s^+} f(x) = l$ if and only if the real number l is the limit of $f(x)$ as x tends to s in the subset $D \cap (s, +\infty)$ of D. Thus the properties of "one-sided limits" taken as a variable x tends to some given value s from above, or from below, are consequences of properties of limits in general, and thus there is no need to develop a separate theory of "one-sided limits".

Lemma 4.8 *Let* $f: D \to \mathbb{R}$ *be a real-valued function defined on a subset* D *of* $\mathbb{R}$*, let* s *be a real number that is an interior point of* $D \cup \{s\}$*, and let* l *be a real number. Then* $\lim_{x \to s} f(x) = l$ *if and only if* $\lim_{x \to s^+} f(x) = l$ *and* $\lim_{x \to s^-} f(x) = l$.

Proof It follows directly from the definition of limits that if $\lim_{x \to s} f(x) = l$ then $\lim_{x \to s^+} f(x) = l$ and $\lim_{x \to s^-} f(x) = l$. To prove the converse, suppose that $\lim_{x \to s^+} f(x) = l$ and $\lim_{x \to s^-} f(x) = l$. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $l - \varepsilon < f(x) < l + \varepsilon$ both for all real numbers x satisfying $s < x < s + \delta_1$ and also for all real numbers x satisfying $s - \delta_2 < x < s$. Let δ be the minimum of δ_1 and δ_2. Then $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $0 < |x - s| < \delta$. It follows that $\lim_{x \to s} f(x) = l$, as required.

Definition Let $f: D \to \mathbb{R}$ be a real-valued function defined on some subset D of $\mathbb{R}$, and let l be some real number. We say that l is the *limit* $\lim_{x \to +\infty} f(x)$ of $f(x)$ as $x \to +\infty$ if, given any strictly positive real number ε, there exists some real number K such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x > K$.

On comparing definitions, it we see that if $f\colon D \to \mathbb{R}$ is a real-valued function defined on a subset D of $\mathbb{R}$, where D contains all real numbers greater than some given real number, if l is a real number, then $\lim_{x\to+\infty} f(x) = l$ if and

$$\lim_{t\to 0^+} f\left(\frac{1}{t}\right) = l.$$

It follows that properties of limits taken "at infinity" can be deduced from corresponding properties of "one-sided limits" and thus follow from the general theory of limits. In particular, if f and g are real valued functions, if $f(x)$ and $g(x)$ are defined for all sufficiently large values of x, and if the limits $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ both exist, then so do the corresponding limits of the functions $f+g$, $f-g$, $f.g$ and $|f|$, and moreover

$$\begin{aligned}
\lim_{x\to+\infty}(f(x)+g(x)) &= \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x), \\
\lim_{x\to+\infty}(f(x)-g(x)) &= \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x), \\
\lim_{x\to+\infty}(f(x)g(x)) &= \lim_{x\to+\infty} f(x) \times \lim_{x\to+\infty} g(x), \\
\lim_{x\to+\infty}|f(x)| &= \left|\lim_{x\to+\infty} f(x)\right|.
\end{aligned}$$

Moreover if in addition $\lim_{x\to+\infty} g(x) \neq 0$ then

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}.$$

4.7 L'Hôpital's Rule

An important corollary of Cauchy's Mean Value Theorem is *l'Hôpital's Rule* for evaluating the limit of a quotient of two functions at a point where both functions vanish.

Proposition 4.9 (L'Hôpital's Rule for Limits from above) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ_0 such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $s < x < s + \delta_0$, and that $\lim_{x\to s^+}\frac{f'(x)}{g'(x)}$ exists (and is finite). Then $\lim_{x\to s^+}\frac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x\to s+}\frac{f(x)}{g(x)} = \lim_{x\to s+}\frac{f'(x)}{g'(x)}.$$

Proof Let $l = \lim_{x \to s^+} \dfrac{f'(x)}{g'(x)}$, and let some strictly positive real number ε be given. By choosing a sufficiently small strictly positive real number δ we can ensure that $f(x)/g(x)$ and $f'(x)/g'(x)$ are well-defined and

$$l - \varepsilon < \frac{f'(x)}{g'(x)} < l + \varepsilon$$

for all real numbers x satisfying $s < x < s + \delta$. Now $f(s) = g(s) = 0$. An application of Cauchy's Mean Value Theorem to the functions f and g on the interval $[s, x]$ therefore ensures that there exists some real number t satisfying $s < t < x$ for which

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f'(t)}{g'(t)}.$$

But then $s < t < s + \delta$. It follows that

$$l - \varepsilon < \frac{f'(t)}{g'(t)} < l + \varepsilon,$$

and therefore

$$l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon.$$

This shows that $\lim_{x \to s^+} f(x)/g(x) = l$, as required. ∎

Corollary 4.10 (L'Hôpital's Rule for Limits from below) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ_0 such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $s - \delta_0 < x < s$, and that $\lim_{x \to s^-} \dfrac{f'(x)}{g'(x)}$ exists (and is finite). Then $\lim_{x \to s^-} \dfrac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x \to s-} \frac{f(x)}{g(x)} = \lim_{x \to s-} \frac{f'(x)}{g'(x)}.$$

Proof It follows from Proposition 4.9 and the definitions of limits from above and from below that

$$\lim_{x \to s^-} \frac{f(x)}{g(x)} = \lim_{h \to 0^+} \frac{f(s-h)}{g(s-h)} = \lim_{h \to 0^+} \frac{f'(s-h)}{g'(s-h)} = \lim_{x \to s^-} \frac{f'(x)}{g'(x)},$$

as required. ∎

Proposition 4.11 (L'Hôpital's Rule) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $0 < |x - s| < \delta$, and that the limit of $f'(x)/g'(x)$ exists (and is finite) as $x \to s$. Then the limit of $f(x)/g(x)$ exists as $x \to s$, and*

$$\lim_{x\to s} \frac{f(x)}{g(x)} = \lim_{x\to s} \frac{f'(x)}{g'(x)}.$$

Proof Let $l = \lim\limits_{x\to s} \dfrac{f'(x)}{g'(x)}$. It follows from Proposition 4.9 and Corollary 4.10 that

$$\lim_{x\to s^+} \frac{f(x)}{g(x)} = \lim_{x\to s+} \frac{f'(x)}{g'(x)} = l$$

and

$$\lim_{x\to s^-} \frac{f(x)}{g(x)} = \lim_{x\to s-} \frac{f'(x)}{g'(x)} = l.$$

It then follows from Lemma 4.8 that

$$\lim_{x\to s} \frac{f(x)}{g(x)} = l,$$

as required. ∎

Example Using l'Hôpital's Rule twice, we see that

$$\lim_{x\to 2} \frac{x^3 + x^2 - 16x + 20}{x^3 - 3x^2 + 4} = \lim_{x\to 2} \frac{3x^2 + 2x - 16}{3x^2 - 6x} = \lim_{x\to 2} \frac{6x + 2}{6x - 6} = \frac{7}{3}.$$

Proposition 4.12 (L'Hôpital's Rule for Limits at Infinity) *Let f and g be differentiable real-valued functions defined for all real numbers that are greater than some given real number. Suppose that $\lim\limits_{x\to+\infty} f(x) = 0$ and $\lim\limits_{g\to+\infty} g(x) = 0$. Suppose also that there exists some real number K such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $x > K$, and that the limit of $f'(x)/g'(x)$ exists (and is finite) as $x \to +\infty$. Then the limit of $f(x)/g(x)$ exists as $x \to +\infty$, and*

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}.$$

Proof Suppose that

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = l.$$

Let $p\colon [0, 1/K) \to \mathbb{R}$ and $q\colon [0, 1/K) \to \mathbb{R}$ be defined such that $p(0) = q(0) = 0$, $p(t) = f(1/t)$ and $q(t) = g(1/t)$ for all real numbers t satisfying $0 < t < 1/K$. The requirements that $\lim\limits_{x\to+\infty} f(x) = 0$ and $\lim\limits_{g\to+\infty} g(x) = 0$ ensure that the functions p and q defined on the interval $[0, 1/K)$ are continuous at 0. Moreover

$$p'(t) = -\frac{1}{t^2} f'\left(\frac{1}{t}\right) \quad \text{and} \quad q'(t) = -\frac{1}{t^2} g'\left(\frac{1}{t}\right)$$

for all real numbers t satisfying $0 < t < 1/K$, and thus

$$\lim_{t\to0^+} \frac{p'(t)}{q'(t)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = l.$$

It follows that there exists some positive real number δ such that $l - \varepsilon < p'(t)/q'(t) < l + \varepsilon$ for all real numbers t satisfying $0 < t < \delta$. Let s be a real number satisfying $0 < s < \delta$. application of Cauchy's Mean Value Theorem shows that there exists some real number t satisfying $0 < t < s < \delta$ for which

$$\frac{p(s)}{q(s)} = \frac{p(s) - p(0)}{q(s) - q(0)} = \frac{p'(t)}{q'(t)}.$$

But then $l - \varepsilon < p(s)/q(s) < l + \varepsilon$. It follows that $\lim\limits_{s\to0^+} p(s)/q(s) = l$, and thus $\lim\limits_{x\to+\infty} f(x)/g(x) = l$, as required. ∎

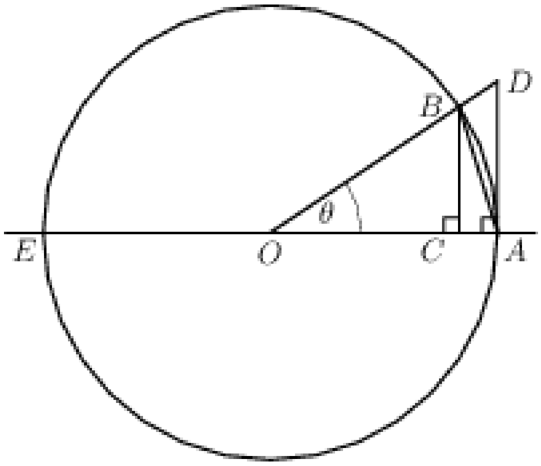
4.8 Derivatives of Trigonometrical Functions

Proposition 4.13 *Let* $\sin\colon \mathbb{R} \to \mathbb{R}$ *be the sine function whose value* $\sin\theta$*, for a given real number* θ *is the sine of an angle of* θ *radians. Then*

$$\lim_{\theta\to0} \frac{\sin\theta}{\theta} = 1.$$

Proof Let E and A be the endpoints of a diameter of a circle of unit radius, let O be the centre of the circle, and let B be a point on the circle for which the line OB makes an angle of θ radians with the line OA, where $0 < \theta < \frac{\pi}{2}$. Let C be the point on the line segment OA for which the angle OCB is a right angle, and let the line OB be produced to the point D determined so that the angle OAD is a right angle.

The sector OAB of the unit circle is by definition the region bounded by the arc AB of the circle and the radii OA and OB. Now the area of a sector



of a circle subtending at the centre an angle of θ radians is equal to the area of the circle multiplied by $\dfrac{\theta}{2\pi}$. But the area of a circle of unit radius is π. It follows that a sector of the unit circle subtending at the centre an angle of θ radians has area $\frac{1}{2}\theta$. Also the triangles OAB and OAD have heights equal to lengths of the line segments BC and AD respectively, and the definitions of the sine, cosine and tangent functions ensure that the lengths of BC and AD are $\sin\theta$ and $\tan\theta$ respectively. Also the common base OA of the triangles OAB and OAD has length one unit, because the circle has unit radius. Now, in Euclidean geometry, the area of any triangle is half the base of the triangle multiplied by the height of the triangle. Therefore

$$\begin{aligned} \text{area of triangle OAB} &= \tfrac{1}{2}\sin\theta, \\ \text{area of sector OAB} &= \tfrac{1}{2}\theta, \\ \text{area of triangle OAD} &= \tfrac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}. \end{aligned}$$

Moreover the triangle OAB is strictly contained in the sector OAB, which in turn is strictly contained in the triangle OAD. It follows that

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta},$$

for all real numbers θ satisfying $0 < \theta < \frac{\pi}{2}$, and therefore

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

for all real numbers θ satisfying $0 < \theta < \frac{\pi}{2}$. Now, given any positive real number ε, there exists some real numbeer δ satisfying $0 < \delta < \frac{\pi}{2}$ such that

$1-\varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$. (In geometrical terms, we are choosing δ so that the length of the line segment BA in the figure associated with this proof is less than ε whenever $0 < \theta < \delta$.) But then

$$1 - \varepsilon < \frac{\sin\theta}{\theta} < 1$$

whenever $0 < \theta < \delta$. These inequalities also hold when $-\delta < \theta < 0$, because the value of $\dfrac{\sin\theta}{\theta}$ is unchanged on replacing θ by $-\theta$. It follows that $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$, as required. ∎

Corollary 4.14 *Let* $\cos\colon \mathbb{R} \to \mathbb{R}$ *be the cosine function whose value* $\cos\theta$, *for a given real number* θ *is the cosine of an angle of* θ *radians. Then*

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0.$$

Proof Basic trigonometrical identities ensure that

$$1 - \cos\theta = 2\sin^2 \tfrac{1}{2}\theta \quad \text{and} \quad \sin\theta = 2\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta$$

for all real numbers θ. Therefore

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

for all real numbers θ. It follows that

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} = 0,$$

and therefore

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\sin\theta} \times \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 0 \times 1 = 0,$$

as required. ∎

Corollary 4.15 *The derivatives of the sine and cosine functions satisfy*

$$\frac{d}{dx}(\sin x) = \cos x, \quad \textit{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Proof Using standard principles of differential calculus we see that

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\,\cos h-\cos x\,\sin h-\sin x}{h} \\
&= \cos x\,\lim_{h\to 0}\frac{\sin h}{h}-\sin x\,\lim_{h\to 0}\frac{1-\cos h}{h} \\
&= \cos x, \\
\frac{d}{dx}(\cos x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\,\cos h-\sin x\,\sin h-\sin x}{h} \\
&= -\sin x\,\lim_{h\to 0}\frac{\sin h}{h}-\cos x\,\lim_{h\to 0}\frac{1-\cos h}{h} \\
&= -\sin x,
\end{aligned}$$

as required. ∎

4.9 Derivatives of Logarithmic and Exponential Functions

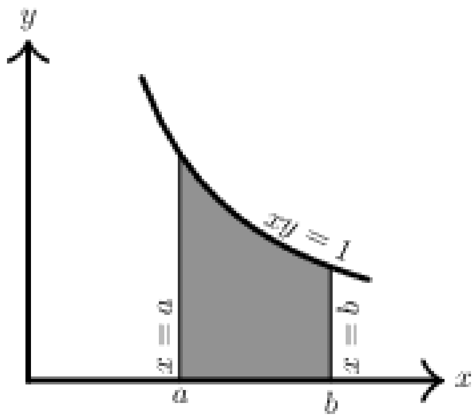
Given any real numbers a and b satisfying $a < b$, let $L(a,b)$ denote the area of the region

$$\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ \ y\ge 0\ \text{ and }\ xy\le 1\}$$

of the Euclidean plane bounded by the x-axis (i.e., the line $y = 0$), the line $x = a$, the line $y = b$ and the hyperbola $xy = 1$. (The quantity $L(a,b)$ thus denotes the area under the graph of the function sending x to $1/x$ (i.e., between the graph of that function and the x-axis) in the interval from $x = a$ and $x = b$.

Let r be a positive real number, and let $T:\mathbb{R}^2\to\mathbb{R}^2$ denote the transformation of the Euclidean plane defined such that $T(x,y) = (rx, r^{-1}x)$ for all real numbers x and y. Given any rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes, the image of that rectangle under the transformation T has the same area as the rectangle itself. It follows from this that the $T:\mathbb{R}^2\to\mathbb{R}^2$ preserves the area of any geometrical figure whose boundary can be approximated sufficiently closely by a polygonal curve with sides parallel to the coordinate axes. Now the transformation T maps the hyperbola $xy = 1$ onto itself. It therefore maps the region

$$\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ \ y\ge 0\ \text{ and }\ xy\le 1\}$$



onto the region

$$\{(x, y) \in \mathbb{R}^2 : ra \leq x \leq rb, \;\; y \geq 0 \;\text{ and }\; xy \leq 1\}.$$

It follows that $L(ra, rb) = L(a, b)$ for all strictly positive real numbers a, b and r satisfying $a < b$.

Let us define $L(a, a) = 0$ and $L(b, a) = -L(a, b)$ for all positive real numbers a and b satisfying $a < b$. Then $L(a, b) = L(ra, rb)$ for all positive real numbers a, b and r, irrespective of whether $a < b$, $a = b$ or $a > b$. Moreover $L(a, c) = L(a, b) + L(b, c)$ for all positive real numbers a, b and c.

We define $\log x = L(1, x)$ for all positive real numbers x. The real-valued function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ defined on the set $\mathbb{R}^+$ of positive real numbers is the *natural logarithm function*.

If u and v are real numbers satisfying $u < v$ then $\log v - \log u = L(u, v) > 0$, and thus $\log u < \log v$. Thus the logarithm function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ is a monotonically increasing function.

Lemma 4.16 *The natural logarithm function* $\log\colon \mathbb{R}^+ \to \mathbb{R}$ *satisfies*

$$\log(xy) = \log x + \log y.$$

for all real numbers x and y.

Proof Given real numbers a and b, let $L(a, b)$ denote the area of the region $X_{a,b}$ of the plane defined such that

$$X_{a,b} = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \;\; y \geq 0 \;\text{ and }\; xy \leq 1\}.$$

Then $L(a, c) = L(a, b) + L(b, c)$ for all positive real numbers a, b and c. It follows that

$$\log xy = L(1, xy) = L(1, x) + L(x, xy) = L(1, x) + L(1, y) = \log x + \log y,$$

as required. ∎

Lemma 4.17 *The natural logarithm function* $\log\colon \mathbb{R}^+ \to \mathbb{R}$ *satisfies*

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

for all real numbers x.

Proof Given real numbers a and b, let $L(a,b)$ denote the area of the region $X_{a,b}$ of the plane defined such that

$$X_{a,b} = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b, \ \ y \geq 0 \ \text{ and } \ xy \leq 1\}.$$

Let s be a positive real number. Then

$$\frac{\log(s+h) - \log s}{h} = \frac{1}{h} L(s, s+h)$$

for all real numbers h satisfying $h > -s$. Suppose that $h > 0$. Then

$$X_{s,h} \supset \{(x,y) \in \mathbb{R}^2 : s \leq x \leq s+h \ \text{ and } \ 0 \leq y \leq 1/(s+h)\}$$

and

$$X_{s,s+h} \subset \{(x,y) \in \mathbb{R}^2 : s \leq x \leq s+h \ \text{ and } \ 0 \leq y \leq 1/s\},$$

and therefore

$$\frac{1}{s+h} < \frac{1}{h}\, L(s, s+h) < \frac{1}{s}.$$

Taking the limit as h tends to zero from above, we find that

$$\lim_{h \to 0^+} \frac{\log(s+h) - \log s}{h} = \lim_{h \to 0^+} \frac{1}{h} L(s, s+h) = \frac{1}{s}.$$

Similarly

$$\frac{1}{s} < \frac{1}{k}\, L(s-k, s) < \frac{1}{s-k}$$

for all real numbers k satisfying $0 < k < s$, and therefore

$$\lim_{h \to 0^-} \frac{\log(s+h) - \log s}{h} = \lim_{k \to 0^+} \frac{\log s - \log(s-k)}{k} = \lim_{k \to 0^+} \frac{1}{k} L(s-k, s) = \frac{1}{s}.$$

It follows that

$$\lim_{h \to 0} \frac{\log(x+h) - \log x}{h} = \frac{1}{s}.$$

We deduce that the natural logarithm function is differentiable, and

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

for all positive real numbers x, as required. ∎

Let s be a real number satisfying $s > 1$, and let n be a positive integer. Then $\log s > 0$, $\log s^n = n \log s$ and $\log s^{-n} = -n \log s$. The Intermediate Value Theorem (Theorem 3.13) then ensures that all real numbers between $-n \log s$ and $n \log s$ belong to the range of the natural logarithm function. Now, given any real number y, we can choose n large enough to ensure that $|y| < n \log s$. It follows that there exists some positive real number x satisfying $\log x = y$. This shows that the range of the logarithm function is the set $\mathbb{R}$ of real numbers. Also $\log u < \log v$ for all real numbers u and v satisfying $u < v$. It follows that the function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ provides a one-to-one correspondence between the set $\mathbb{R}^+$ of positive real numbers and the set $\mathbb{R}$ of real numbers, and therefore there exists a well-defined function $\exp\colon \mathbb{R} \to \mathbb{R}$ whose value $\exp(t)$ at any real number t is equal to the unique positive real number s satisfying $\log s = t$. This function $\exp\colon \mathbb{R} \to \mathbb{R}$ is the *exponential function*. The range of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is the set $\mathbb{R}^+$ of positive real numbers. It follows from the definition of the exponential function that $\exp(\log x) = x$ for all positive real numbers x.

Lemma 4.18 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *is differentiable, and*

$$\frac{d}{dx}(\exp(x)) = \exp(x)$$

for all real numbers x.

Proof Let t be a real number. Then there exists some positive real number s satisfying $\log s = t$. Now the logarithm function is differentiable at s, and its derivative at s is equal to $1/s$. It follows that

$$s = \lim_{k \to 0} \frac{k}{\log(s+k) - \log s} = \lim_{u \to s} \frac{u - s}{\log u - \log s}.$$

Let some strictly positive number ε be given. Then there exists some strictly positive number η such that

$$s - \varepsilon < \frac{u - s}{\log u - \log s} < s + \varepsilon$$

for all real numbers u satisfying $s - \eta < u < s + \eta$ that are not equal to s. Now $t = \log s$, and therefore $\log(s - \eta) < t < \log(s + \eta)$. Let δ be the minimum of $\log(s+\eta) - t$ and $t - \log(s-\eta)$. Then $\delta > 0$, and, given any real number x that differs from t but satisfies the inequalities $t - \delta < x < t + \delta$, there exists some positive real number u satisfying $s - \eta < u < s + \eta$ for which $x = \log u$. Moreover $u \neq s$, because $x = \log u$, $t = \log s$ and $x \neq t$. $x \neq s$. But then $u = \exp(x)$ and $s = \exp(t)$, and therefore

$$s - \varepsilon < \frac{\exp(x) - \exp(t)}{x - t} < s + \varepsilon.$$

Thus, given any positive real number ε, there exists some positive real number δ such that

$$\exp(t) - \varepsilon < \frac{\exp(t + h) - \exp(t)}{h} < \exp(t) + \varepsilon.$$

for all real numbers h satisfying $0 < |h| < \delta$. It follows that

$$\lim_{h \to 0} \frac{\exp(t + h) - \exp(t)}{h} = \exp(t),$$

as required. ∎

4.10 Continuous Differentiability and Smoothness

Definition An *open set* in $\mathbb{R}$ is a subset D of $\mathbb{R}$ with the property that, given any element s of D, there exists some strictly positive real number δ such that every real number x satisfying $|x - s| < \delta$ belongs to the set D.

Definition Let $f\colon D \to \mathbb{R}$ be a real valued function defined on an open set D in $\mathbb{R}$. The function f is said to be *k-times continuously differentiable* (or C^k) on D if the function f itself and its first k derivatives $f', f'', \ldots, f^{(k)}$ are well-defined and continuous on D.

Definition Let $f\colon D \to \mathbb{R}$ be a real valued function defined on an open set D in $\mathbb{R}$. The function f is said to be *smooth* (or C^∞) on D if the function f itself and its derivatives $f', f'', f''', \ldots$ of all orders are well-defined and continuous on D.

Sums, differences and products of smooth functions are smooth. Also a quotient of a smooth function by another smooth function that is everywhere non-zero is itself smooth.

In particular polynomial functions are smooth, and the sine, cosine, tangent, logarithm and exponential functions are smooth where they are defined.

Lemma 4.19 *Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ be smooth functions defined over open subsets D and E of $\mathbb{R}$, where $f(D) \subset E$. Then the composition function $g \circ f\colon D \to \mathbb{R}$ is smooth.*

Proof Let $f^{(0)} = f$, $g^{(0)} = g$, $f^{(1)} = f'$, $g^{(1)} = g'$ etc., and let $\mathcal{C}$ denote the the collection of functions that either are of the form $g^{(k)} \circ f$ for some non-negative integer k or else are of the form

$$(g^{(k)} \circ f) \cdot f^{(j_1)} \cdot f^{(j_2)} \cdot \cdots \cdot f^{(j_m)}$$

for some non-negative integer k and positive integers $j_1, j_2, \ldots, j_m$. Now it follows from the Chain Rule (Proposition 4.4) and the Product Rule (Proposition 4.2) that any function belonging to this collection $\mathcal{C}$ is differentiable, and moreover the derivative of a function belonging to $\mathcal{C}$ either belongs itself to $\mathcal{C}$ or else is expressible as a sum of functions belonging to the collection $\mathcal{C}$. Thus any function expressible as a sum of functions belonging to $\mathcal{C}$ is differentiable, and its derivative is expressible as a sum of functions belonging to the collection $\mathcal{C}$. It follows that any function belonging to the collection $\mathcal{C}$ is smooth. In particular, the composition function $g \circ f$ is smooth, as required. ∎

4.11 Taylor's Theorem for Functions of One Real Variable

A subset I of $\mathbb{R}$ is an *interval* if and only if $(s, u) \subset I$ for all $s, u \in I$, where

$$(s, u) = \{x \in \mathbb{R} : s \leq x \leq u\}.$$

Thus a subset I of $\mathbb{R}$ is an interval if and only if, given real numbers s, x and u satisfying $s < x < u$ for which $s \in I$ and $u \in I$, the real number x also satisfies $x \in I$. An *open interval* is an interval that is also an open set in $\mathbb{R}$. Given real numbers c and d satisfying $c < d$, the intervals (c, d), $(c, +\infty)$ and $(-\infty, d)$ are open intervals, as is the whole real line $\mathbb{R}$. It is a straightforward exercise to verify, using the Least Upper Bound Principle, that all open intervals in $\mathbb{R}$ conform to one of the types just described.

Lemma 4.20 *Let s and h be real numbers, let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$, let $c_0, c_1, \ldots, c_{k-1}$ be real numbers, and let*

$$p(t) = f(s + th) - \sum_{n=0}^{k-1} c_n t^n.$$

for all real numbers t belonging to some open interval I for which $0 \in I$ and $1 \in I$. Then $p^{(n)}(0) = 0$ for all integers n satisfying $0 \leq n < k$ if and only if

$$c_n = \frac{h^n f^{(n)}(s)}{n!}$$

for all integers n satisfying $0 \leq n < k$.

Proof On setting $t = 0$, we find that $p(0) = f(s) - c_0$, and thus $p(0) = 0$ if and only if $c_0 = f(s)$.

Let the integer n satisfy $0 < n < k$. On differentiating the function p n times (using in particular the Chain Rule to differentiate $f(s + th)$ and its derivatives as functions of t), we find that

$$p^{(n)}(t) = h^n f^{(n)}(s + th) - \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} c_j t^{j-n}.$$

Then, on setting $t = 0$, we find that only the term with $j = n$ contributes to the value of the sum on the right hand side of the above identity, and therefore

$$p^{(n)}(0) = h^n f^{(n)}(s) - n! c_n.$$

The result follows. ∎

Theorem 4.21 (Taylor's Theorem) *Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$. Then*

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s + th)$ is defined for all $t \in I$, and let $p: I \to \mathbb{R}$ be defined so that

$$p(t) = f(s + th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

for all $t \in I$. A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$ (see Lemma 4.20). Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0, 1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$, and $q(1) = 0$. We can therefore apply Rolle's Theorem (Theorem 4.5) to the function q on the interval $[0, 1]$ to deduce the existence of some real number t_1 satisfying $0 < t_1 < 1$ for which $q'(t_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, t_1]$ to deduce the existence of some real number t_2 satisfying $0 < t_2 < t_1$ for which $q''(t_2) = 0$. By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_k < t_{k-1} < \cdots < t_1 < 1$ with the property that $q^{(n)}(t_n) = 0$ for $n = 1, 2, \ldots, k$. Let $\theta = t_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1),$$

hence

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h),$$

as required. ∎

Corollary 4.22 *Let $f\colon D \to \mathbb{R}$ be a k-times continuously differentiable function defined over an open subset D of $\mathbb{R}$ and let $s \in \mathbb{R}$. Then given any strictly positive real number ε, there exists some strictly positive real number δ such that*

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| < \varepsilon |h|^k$$

whenever $|h| < \delta$.

Proof The function f is k-times continuously differentiable, and therefore its kth derivative $f^{(k)}$ is continuous. Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ that is small enough to ensure that $s + h \in D$ and $|f^{(k)}(s+h) - f^{(k)}(s)| < k!\varepsilon$ whenever $|h| < \delta$. If h is an real number satisfying $|h| < \delta$, and if θ is a real number satisfying $0 < \theta < 1$, then $s + \theta h \in D$ and $|f^{(k)}(s+\theta h) - f^(k)(s)| < k!\varepsilon$. Now it follows from Taylor's Theorem (Theorem 4.21) that, given any real number h satisfying $|h| < \delta$ there exists some real number θ satisfying $0 < \theta < 1$ for which

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h).$$

Then

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| = \frac{|h|^k}{k!} |f^{(k)}(s+\theta h) - f^(k)(s)| < \varepsilon |h|^k,$$

as required. ∎

Corollary 4.23 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

for all real numbers x*.*

Proof The derivative of the exponential function is the exponential function itself (Lemma 4.18). It follows from Taylor's Theorem (Theorem 4.21) that

$$\exp x = \sum_{n=0}^{m} \frac{x^n}{n!} + \frac{x^{m+1}}{(m+1)!} \exp(\theta x)$$

for some real number θ satisfying $0 < \theta < 1$. It follows that

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \leq b_{m+1}(x) \exp(|x|),$$

where

$$b_n(x) = \frac{|x|^n}{n!}$$

for all real numbers x and non-negative integers n. Note that $b_n(x) \geq 0$ for all real numbers x and non-negative integers n.

Let N be some positive integer satisfying $N \geq 2|x|$. If n is a positive integer satisfying $n \geq N$ then $n + 1 > 2|x|$, and therefore

$$b_{n+1}(x) = \frac{|x|}{n+1} \times b_n(x) < \tfrac{1}{2} b_n(x).$$

It follows that $0 \leq b_n(x) < \dfrac{1}{2^{n-N}} b_N(x)$ whenever $n \geq N$, and therefore $\lim\limits_{n \to +\infty} b_n(x) = 0$. Thus

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \to 0$$

as $m \to +\infty$, and thus

$$\exp x = \lim_{m \to +\infty} \sum_{n=0}^{m} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!},$$

as required. ∎

Corollary 4.24 *The sine function* $\sin\colon \mathbb{R} \to \mathbb{R}$ *and cosine function* $\cos\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \textit{and} \quad \cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

for all real numbers x.

Proof The derivatives of the sine function are given by

$$\sin^{(2k)}(x) = (-1)^k \sin(x) \quad \text{and} \quad \sin^{(2k+1)}(x) = (-1)^k \cos(x)$$

for all positive integers k. It follows from Taylor's Theorem that, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\sin x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^{m+1} x^{2m+3}}{(2m+3)!} \cos(\theta x)$$

(The value of θ will depend on x and m.) It follows that

$$\left| \sin x - \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| \leq b_{2m+3}(x),$$

for all non-negative integers m, where $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But it was shown in the proof of Corollary 4.23 that $\lim\limits_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\sin x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Similarly the derivatives of the cosine function are given by

$$\cos^{(2k)}(x) = (-1)^k \cos(x) \quad \text{and} \quad \cos^{(2k-1)}(x) = (-1)^k \sin(x)$$

for all positive integers k. Therefore, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\cos x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k}}{(2k)!} + \frac{(-1)^{k+1} x^{2m+2}}{(2k+2)!} \cos(\theta x)$$

But then

$$\left|\cos x - \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!}\right| \leq b_{2m+2}(x),$$

where, as before, $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But $\lim_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\cos x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

as required. ∎

4.12 Real-Analytic Functions

Definition A real-valued function $f\colon D \to \mathbb{R}$ defined over an open subset D of the set $\mathbb{R}$ of real numbers is said to be *real-analytic* if, given any real number s belonging to the domain D of the function, there exists some strictly positive real number δ such that

$$f(s+h) = f(s) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(s)$$

for all real numbers h satisfying $|h| < \delta$.

It can be shown that sums, differences, products, quotients and compositions of real-analytic functions are themselves real-analytic over their domains of definition. In particular, polynomial functions and quotients of polynomial functions are real-analytic. The natural logarithm function is real-analytic over the set of positive real numbers because its derivative is real-analytic. It follows from Corollary 4.23 that the exponential function is real-analytic. and it follows from Corollary 4.24 that the sine and cosine functions are real-analytic. Inverses of real-analytic functions are real-analytic.

All real-analytic functions are smooth. However not all smooth functions are real-analytic.

4.13 Smooth Functions that are not the Sum of their Taylor Series

Let f be an infinitely differentiable real-valued function defined around some real number a. The infinite series

$$f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a)$$

is referred to as the *Taylor expansion* of the function f about a. For many functions, typically including those constructed from polynomial functions, logarithm functions, exponential functions, trigonometrical functions and their inverses, identities of the form

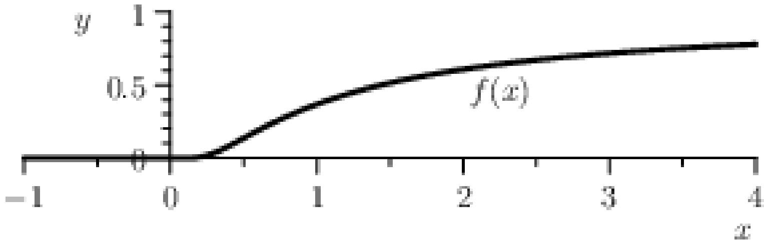
$$f(a+h) = f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a) = f(a) + \lim_{m\to+\infty} \left(\sum_{n=1}^{m} \frac{h^n}{n!} f^{(n)}(a) \right)$$

for all sufficiently small values of h. Such functions are said to be *real-analytic*. However there exist functions whose Taylor expansion about some real number a does not converge to the given function for any non-zero value of h. Such a function is the subject of the following lemma.

Proposition 4.25 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that*

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \textit{if } x > 0; \\ 0 & \textit{if } x \le 0. \end{cases}$$

Then the function $f\colon \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. In particular $f^{(k)}(0) = 0$ for all positive integers k.



Proof We show by induction on k that the function f is k times differentiable on $\mathbb{R}$ and $f^{(k)}(0) = 0$ for all positive integers k. Now it follows from standard rules for differentiating functions that

$$f^{(k)}(x) = \frac{p_k(x)}{x^{2k}} \exp\left(-\frac{1}{x}\right)$$

for all strictly positive real numbers x, where $p_1(x) = 1$ and

$$p_{k+1}(x) = x^2 p_k'(x) + (1 - 2kx) p_k(x)$$

for all k. A straightforward proof by induction shows that $p_k(x)$ is a polynomial in x of degree $k-1$ for all positive integers k with leading term $(-1)^{k-1}k!x^{k-1}$.

Now

$$\frac{d}{dt}\left(t^n e^{-t}\right) = t^{n-1}(n-t)e^{-t}$$

for all positive real numbers t. It follows that function sending each positive real number t to $t^n e^{-t}$ is increasing when $0 \leq t < n$ and decreasing when $t > n$, and therefore $t^n e^{-t} \leq M_n$ for all positive real numbers t, where $M_n = n^n e^{-n}$. It follows that

$$0 \leq \frac{1}{x^{2k+1}} \exp\left(-\frac{1}{x}\right) \leq M_{2k+2}x$$

for all positive real numbers x, and therefore

$$\lim_{h\to 0^+} \frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right) = 0.$$

It then follows that

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f^{(k)}(h)}{h} &= \lim_{h\to 0^+} \left(\frac{p_k(h)}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\
&= \lim_{h\to 0^+} p_k(h) \times \lim_{h\to 0^+} \left(\frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\
&= p_k(0) \times 0 = 0
\end{aligned}$$

for all positive integers k.

Now

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)}{h} = 0 = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}.$$

It follows that the function f is differentiable at zero, and $f'(0) = 0$.

Suppose that the function $f(x)$ is k-times differentiable at zero for some positive integer k, and that $f^{(k)}(0) = 0$. Then

$$\lim_{h\to 0^+} \frac{f^{(k)}(h)-f^{(k)}(0)}{h} = \lim_{h\to 0^+} \frac{f^{(k)}(h)}{h} = 0 = \lim_{h\to 0^-} \frac{f^{(k)}(h)-f^{(k)}(0)}{h}.$$

It then follows that the function $f^{(k)}$ is differentiable at zero, and moreover the derivative $f^{(k+1)}(0)$ of this function at zero is equal to zero. The function f is thus $(k+1)$-times differentiable at zero.

It now follows by induction on k that $f^{(k)}(x)$ exists for all positive integers k and real numbers x, and moreover

$$f^{(k)}(x) = \begin{cases} \dfrac{p_k(x)}{x^{2k}} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

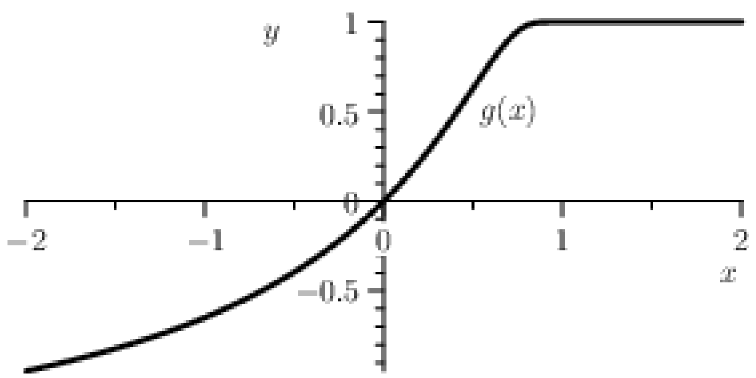
The function $f\colon \mathbb{R} \to \mathbb{R}$ is thus a smooth function, as required. ∎

Remark Note that the function $f\colon \mathbb{R} \to \mathbf{R}$ defined in the statement of Lemma 4.25 has a well-defined Taylor expansion about $x = 0$. Moreover all the terms of this Taylor expansion are zero, and therefore the Taylor expansion of f converges to the zero function. This function therefore provides an example of a function where the Taylor expansion is well-defined but does not converge to the given function.

Corollary 4.26 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that*

$$g(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \text{if } x < 0; \\ 1 & \text{if } x \geq 1. \end{cases}$$

Then the function $g\colon \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. Moreover the function g is a strictly increasing function on $\{x \in \mathbb{R} : x < 1\}$, and $g(0) = 0$.

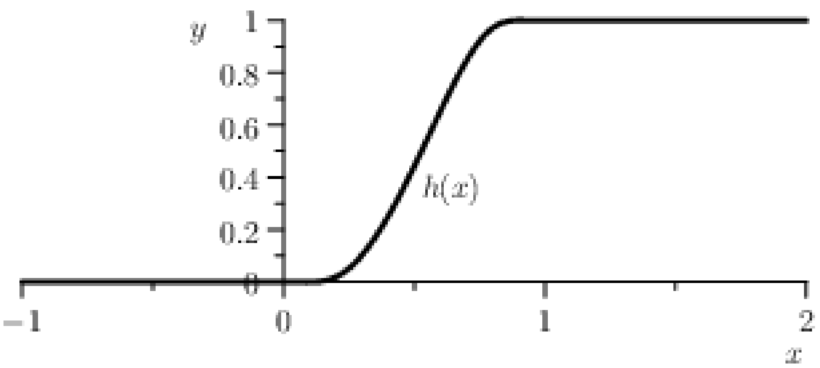


Proof Let $f\colon \mathbb{R} \to \mathbb{R}$ be the real-valued function defined on the set $\mathbb{R}$ of real numbers so that

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Now
$$-\frac{x}{1-x} = 1 - \frac{1}{1-x}$$
for all real numbers x. It follows from the definition of the functions f and g that $g(x) = 1 - ef(1-x)$ for all real numbers x, where $e = \exp(1)$. Now Proposition 4.25 ensures that the function f is smooth on $\mathbb{R}$. It follows that the function g is also smooth on $\mathbb{R}$. Also $g(0) = 0$. Now $f(1-x)$ is a strictly decreasing function of x on $\{x \in \mathbb{R} : x < 1\}$. It follows that the function g is strictly increasing on that set, as required. ∎

Corollary 4.27 *Let $h: \mathbb{R} \to \mathbb{R}$ be defined such that $h(x) = g(f(x)/f(1))$ for all real numbers x, where*
$$\begin{aligned} f(x) &= \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \\ g(x) &= \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \text{if } x < 0; \\ 1 & \text{if } x \geq 1. \end{cases} \end{aligned}$$
Then the function $h: \mathbb{R} \to \mathbb{R}$ is smooth, $h(x) = 0$ whenever $x \leq 0$, $h(1) = 1$ whenever $x \geq 1$, and $h(x)$ is a strictly increasing function of x when restricted to the interval $\{x \in \mathbb{R} : 0 < x < 1\}$.



Proof The function h is a composition of smooth functions, and is therefore smooth (see Lemma 4.19). If $x \leq 0$ then $h(x) = g(f(0)) = g(0) = 0$. If $x \geq 1$ then $f(x)/f(1) \geq 1$ and therefore $h(x) = 1$. The function sending a real number x satisfying $0 < x < 1$ to $f(x)/f(1)$ is strictly increasing on the interval $(0, 1)$ and maps that interval into itself. Also the function g is strictly increasing on the interval $(0, 1)$. Thus the function h restricted to the interval $(0, 1)$ is a composition of two strictly increasing functions, and is thus itself strictly increasing, as required. ∎

4.14 Historical Note

Representation of functions as sums of infinite series have been known to mathematicians for centuries. The standard representation of the sine and cosine and arctangent functions was known to, and presumably discovered by, Madhava of Sangramagrama (c. 1340–c. 1425), whose work gave impetus to the flourishing of the study of astronomy and mathematics in Kerala, in southern India. The theory of infinite series was extensively developed in Western Europe in the 17th century, with Isaac Newton (1642–1726/7) being particularly active in the field. Isaac Newton's manuscript on the *Method of fluxions and infinite series* was completed in 1671, and was posthumously published in 1736.

In 1797, Joseph-Louis Lagrange published his *Théorie des fonctions analytiques*. One of the primary aims of this book was to develop an approach to the principles of differential and integral calculus taking as its starting point the principle that functions of a real variable studied by mathematicians could be represented around a particular value through an infinite series expansion, so that, in particular, an analytic function $f(x)$ defined for values of x close to some given value s could be represented through an infinite series expansion of the form

$$f(s+h) = \sum_{n=0}^{+\infty} a_n h^n$$

for all sufficiently small values of the increment h. Lagrange defined the derivative of such a function f to be the function $f'(x)$ whose infinite series expansion takes the form

$$f'(s+h) = \sum_{n=1}^{+\infty} n a_n h^{n-1}.$$

Lagrange intended that his theory of analytic functions would supply an approach to the foundations of calculus that required neither "infinitesimal quantities" nor the use of limits.

In 1830, William Rowan Hamilton published a paper in the *Transactions of the Royal Irish Academy* entitled *On the Error of a received Principle of Analysis, respecting Functions which vanish with their Variables.* In this paper, Hamilton pointed that the function whose value at x, for non-zero real numbers x, is $e^{-x^{-2}}$ cannot be expressed around zero as the sum of a power series. The following year Hamilton published a note in the *Transactions of the Royal Irish Academy* to put on record the fact that, prior to Hamilton's earlier paper, Cauchy had published a paper citing this same function as an example of a function whose derivatives at zero of all orders are all equal to zero though the function itself takes non-zero values at non-zero values of its argument.

These examples demonstrated that the theory of calculus could not be founded on the assumption that all functions relevant to mathematical analysis could be represented as sums of power series in the neighbourhood of any value at which they are defined. Accordingly mathematicians in the nineteenth century returned to the approach of justifying the basic principles of differential and integral calculus on the theory of limits and quadratures. A theory of limits had already been employed by Isaac Newton, using the terminology of *prime and ultimate ratios*. However the concept of limit employed by Newton was only applicable to variable geometrical quantities that approached their limiting values monotonically. The Newton version of the limit concept was not applicable to functions such as $x \sin\left(\frac{1}{x}\right)$ which oscillates round zero as the value of x approaches zero from above, but nevertheless can be made to approximate to zero to within any given margin of error, provided that the value of x is sufficiently close to zero. The theory of limits was accordingly generalized and further developed in the nineteenth century by mathematicians such as Bolzano (1781–1848) and Cauchy (1789–1857) to cover such situations. The generalized concept of limit developed by Bolzano and Cauchy proved to be more appropriate to serve as the basis for defining the basic concepts and proving the basic theorems that justify the principles of calculus. The definitive treatment of mathematical analysis was provided by Karl Weierstrass (1815–1897), whose lectures at Berlin established the standard approach to the foundations of real and complex analysis through the use of “epsilon-delta” definitions and proofs, together with the systematic use of standard theorems such as the Bolzano-Weierstrass Theorem.

5 The Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral.*

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *lower sum* (or *lower Darboux sum*) $L(P, f)$ and the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ are defined by

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

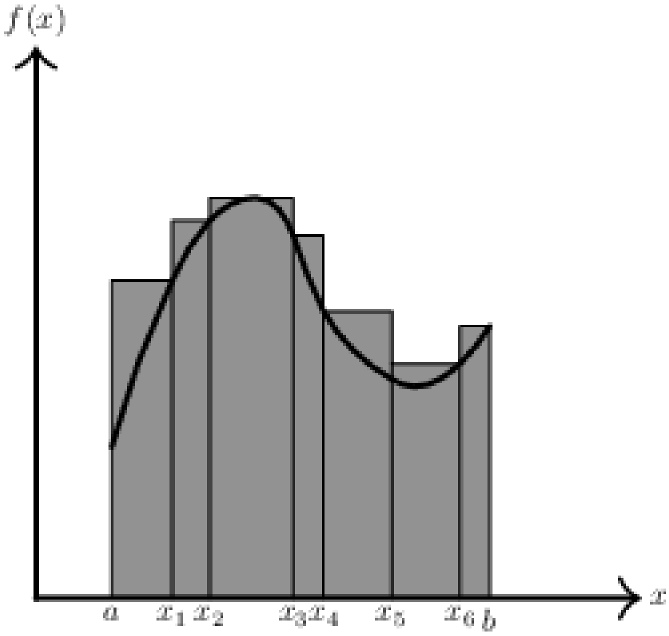
for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

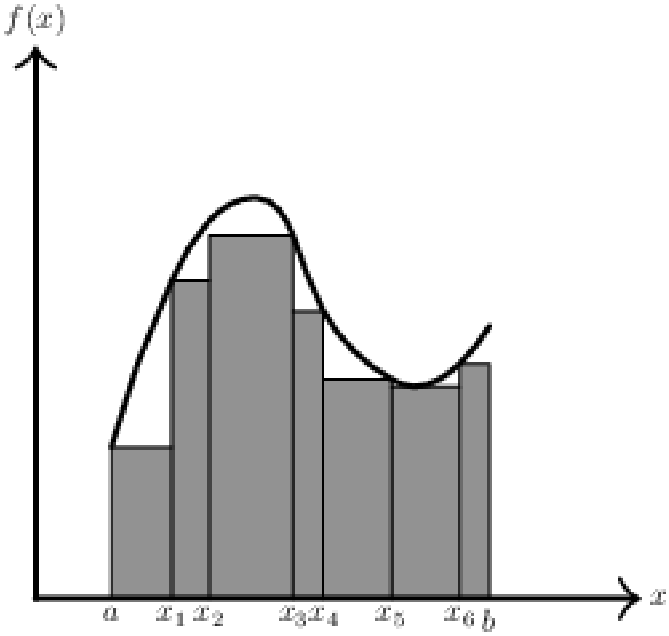
$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &\equiv \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &\equiv \sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

(i.e., $\mathcal{U}\int_a^b f(x)\,dx$ is the infimum of the values of $U(P, f)$ and $\mathcal{L}\int_a^b f(x)\,dx$ is the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$). If

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx$$



The upper sum $U(P, f)$



The lower sum $L(P, f)$

then the function f is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a, b]$, and the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a, b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x) = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

Definition Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 5.1 *Let R be a refinement of some partition P of $[a, b]$. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f\colon [a, b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^{n} L(R_i, f)$ and $U(R, f) = \sum_{i=1}^{n} U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 5.2 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 5.1 that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \leq U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx + d$, where $c \geq 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Thus

$$\begin{aligned} L(P_n, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d - \frac{c}{n}\right) \\ &= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\ U(P_n, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d\right) \\ &= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}. \end{aligned}$$

But $L(P_n, f) \leq \mathcal{L}\int_a^b f(x)\,dx \leq \mathcal{U}\int_a^b f(x)\,dx \leq U(P_n, f)$ for all n. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0,1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0,1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0,1]$.

It can be shown that sums and products of Riemann-integrable functions are themselves Riemann-integrable.

Proposition 5.3 *Let f be a bounded real-valued function on the interval $[a, c]$. Suppose that f is Riemann-integrable on the intervals $[a, b]$ and $[b, c]$, where $a < b < c$. Then f is Riemann-integrable on $[a, c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let Q and R be any partitions of the intervals $[a, b]$ and $[b, c]$ respectively. These partitions combine to give a partition $Q \cup R$ of the interval $[a, c]$: thus if $Q = \{a, x_1, \ldots, x_{n-1}, b\}$ and $R = \{b, u_1, \ldots, u_{m-1}, c\}$, where

$$a < x_1 < x_2 < \cdots < x_{n-1} < b < u_1 < u_2 < \cdots < u_{m-1} < c,$$

then $Q \cup R = \{a, x_1, \ldots, x_{n-1}, b, u_1, \ldots, u_{m-1}, c\}$. Clearly the lower and upper sums of f satisfy $L(Q, f) + L(R, f) = L(Q \cup R, f)$ and $U(Q, f) + U(R, f) = U(Q \cup R, f)$. It follows that

$$L(Q, f) + L(R, f) \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Taking the supremum of the left hand side of this inequality over all partitions Q of $[a, b]$ and all partitions R of $[b, c]$, we deduce that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Similarly $U(Q, f) + U(R, f) \geq \mathcal{U} \int_a^c f(x)\,dx$, and hence

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \geq \mathcal{U} \int_a^c f(x)\,dx.$$

But $\mathcal{L} \int_a^c f(x)\,dx \leq \mathcal{U} \int_a^c f(x)\,dx$ by Lemma 5.2. It follows that

$$\mathcal{L} \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \mathcal{U} \int_a^c f(x)\,dx,$$

as required. ∎

5.1 Integrability of Monotonic functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a, b] \to \mathbb{R}$ defined on the closed bounded interval $[a, b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a, b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a, b] \to \mathbb{R}$ is said to be *monotonic* on $[a, b]$ if either it is non-decreasing on $[a, b]$ or else it is non-increasing on $[a, b]$.

Proposition 5.4 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a, b]$ is Riemann-integrable on $[a, b]$.*

Proof Let $f\colon [a, b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a, b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$, and therefore the function f is bounded on $[a, b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a, b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$. Then the maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P, f)$ and $L(P, f)$ of f for the partition P satisfy

$$U(P, f) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \quad \text{and} \quad L(P, f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$

Moreover $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned} U(P, f) - L(P, f) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})(x_i - x_{i-1}) \\ &< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But $\mathcal{U}\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx$. It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f\colon [a, b] \to \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ∎

Corollary 5.5 *Let a and b be real numbers satisfing $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof The result follows immediately on applying the results of Proposition 5.3 and Proposition 5.4. ∎

Remark The result and proof of Proposition 5.4 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

5.2 Integrability of Continuous functions

Theorem 5.6 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a, b]$. It follows from the Extreme Value Theorem (Theorem 3.15) that f is bounded above and below on the interval $[a, b]$.

Let some strictly positive real number ε be given. It follows from Proposition 3.16 that the function f is uniformly continuous on the interval $[a, b]$, and therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$. Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \leq x \leq x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \leq m_i \leq M_i \leq f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Therefore

$$\begin{aligned} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) &\leq L(P, f) \leq U(P, f) \\ &\leq \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a), \end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P, and hence

$$0 \leq \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \leq U(P, f) - L(P, f) \leq 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

5.3 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 5.6). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 5.7 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a,b)$. Now the function f is continuous at x, where $a < x < b$. Thus, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(t) - f(x)| < \frac{1}{2}\varepsilon$ for all $t \in [a,b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt.$$

But if $0 < |h| < \delta$ and $x + h \in [a,b]$ then $\left|\int_x^{x+h} (f(t) - f(x))\,dt\right| \leq \frac{1}{2}\varepsilon|h|$, and thus

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

It follows that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function on a closed interval $[a,b]$. We say that f is *continuously differentiable* on $[a,b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$f'(a) = \lim_{h\to 0^+}\frac{f(a+h) - f(a)}{h}, \qquad f'(b) = \lim_{h\to 0^-}\frac{f(b+h) - f(b)}{h}$$

exist at the endpoints of $[a,b]$, and the function f' is continuous on $[a,b]$.

If $f\colon [a,b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a,b]$ then the one-sided derivatives of F at the endpoints of $[a,b]$ exist, and

$$\lim_{h\to 0^+}\frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h\to 0^-}\frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 5.8 *Let f be a continuously differentiable real-valued function on the interval $[a,b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\,dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\,dt\right) = 0$$

for all x satisfing $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 4.6) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b - a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 5.9 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(t)\frac{dg(x)}{dx}\,dx = f(b)g(b) - f(a)g(a) - \int_a^b \frac{df(x)}{dx}g(x)\,dx.$$

Proof This result follows from Corollary 5.8 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - \frac{df(x)}{dx}g(x).$$ ∎

Corollary 5.10 (Integration by Substitution) *Let $u\colon [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\,dx = \int_a^b f(u(t))\frac{du(t)}{dt}\,dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt}dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\,dy,$$

and $H'(s) = f(s)$ for all $s \in [a,b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a,b)$. On applying the Mean Value Theorem (Theorem 4.6) to the function $F - G$ on the interval $[a,b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $F(b) = G(b) = H(d)$, which yields the required identity. ∎

5.4 Interchanging Limits and Integrals, Uniform Convergence

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x)$ converges for all $x \in [a, b]$. We wish to determine whether or not

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. Now

$$\lim_{j \to +\infty} \int_0^1 f_j(x)\, dx = \lim_{j \to +\infty} \left(\frac{j}{j+1} - \frac{j}{2j+1} \right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $jx^j \to 0$ as $j \to +\infty$. To verify this, choose u satisfying $x < u < 1$. Then $0 \leq (j+1)u^{j+1} \leq nu^j$ for all positive integers j satisfying $j > u/(1-u)$. Therefore there exists some constant B with the property that $0 \leq nu^j \leq B$ for all positive integers j. But then $0 \leq jx^j \leq B(x/u)^j$ for all positive integers j, and $(x/u)^j \to 0$ as $j \to +\infty$. Therefore $jx^j \to 0$ as $j \to +\infty$, as claimed. It follows that

$$\lim_{j \to +\infty} f_j(x) = \left(\lim_{j \to +\infty} jx^j \right) \left(\lim_{j \to +\infty} (1 - x^j) \right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_j(1) = 0$ for all positive integers j. We conclude that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$, which is what we set out to show.

We now introduce the concept of *uniform convergence.* Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a, b]$, the identity

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a, b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_j) is said to converge *uniformly* to a function f on D as $j \to +\infty$ if and only if the following criterion is satisfied:

given any strictly positive real number ε, there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all positive integers j satisfying $j \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each positive integer j, let $M_j = \sup\{f_j(x) : x \in D\}$. We claim that $M_j \to 0$ as $j \to +\infty$. To prove this, let some strictly positive real number ε be given. Then there exists some positive integer N such that $|f_j(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $j \geq N$. Thus if $j \geq N$ then $M_j \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_j \to 0$ as $j \to +\infty$, as claimed.

Example Let $(f_j : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. We have already shown (in an earlier example) that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. However a straightforward exercise in Calculus shows that the maximum value attained by the function f_j is $j/4$ (which is attained at $x = 1/2^{\frac{1}{j}}$), and $j/4 \to +\infty$ as $j \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \ldots$ does not converge uniformly to the zero function on the interval $[0, 1]$.

Proposition 5.11 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let some strictly positive real number ε be given. If j is chosen sufficiently large then $|f(x) - f_j(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_j \to f$ uniformly on D as $j \to +\infty$. It then follows from the continuity of f_j that there exists some strictly positive real number δ such that $|f_j(x) - f_j(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$|f(x) - f(s)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(s)| + |f_j(s) - f(s)| < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon$$

whenever $|x - s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 5.12 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a, b]$ to some continuous real-valued function f. Then*

$$\lim_{j \to +\infty} \int_a^b f_j(x)\, dx = \int_a^b f(x)\, dx.$$

Proof Let some strictly positive real number ε. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b-a) < \varepsilon$. Then there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon_0$ for all $x \in [a,b]$ and $j \geq N$, since the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to f on $[a,b]$. Now

$$-\int_a^b |f_j(x) - f(x)|\,dx \leq \int_a^b f_j(x)\,dx - \int_a^b f(x)\,dx \leq \int_a^b |f_j(x) - f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f_j(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_j(x) - f(x)|\,dx \leq \varepsilon_0(b-a) < \varepsilon,$$

whenever $j \geq N$. The result follows. ∎

5.5 Integrals over Unbounded Intervals

We define integrals over unbounded intervals by appropriate limiting processes. Given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $[a, +\infty)$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. Similarly, given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. If f is bounded and Riemann integrable over each closed bounded interval in $\mathbb{R}$ then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty, b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit exists.

Remark Using techniques of complex analysis, it can be shown that

$$\lim_{b \to +\infty} \int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

However it can also be shown that

$$\int_0^b \frac{|\sin x|}{x}\,dx \to +\infty \text{ as } b \to +\infty.$$

Therefore, in the standard theory of the Riemann integral, the integral of the function $(\sin x)/x$ on the interval $[0, +\infty)$ is defined, and $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

There is an alternative theory of integration, due to Lebesgue, which is generally more powerful. All bounded Riemann-integrable functions on a closed bounded interval are Lebesgue-integrable on that interval. But a real-valued function f on a "measure space" is Lebesgue-integrable if and only if $|f|$ is Lebesgue-integrable on that measure space. Let $f\colon [0, +\infty) \to \mathbb{R}$ be the real-valued function defined such that $f(0) = 1$ and $f(x) = (\sin x)/x$ for all positive real numbers x. Then the function $|f|$ is neither Riemann-integrable nor Lebesgue-integrable on $[0, +\infty)$. It follows that the function f itself is not Lebesgue-integrable on $[0, +\infty)$. But, as we have remarked, the theory of the Riemann integral assigns a value of $\frac{\pi}{2}$ to $\int\limits_0^{+\infty} f(x)\,dx$.