Module MA2321: Analysis in Several Real Variables
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Sections 1, 2 and 3

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1 Ordered Fields and the Real Number System

1.1 Sets

A *set* is a collection of objects. These objects are referred to as the *elements* of the set. One can specify a set by enclosing a list of suitable objects within braces. Thus, for example, $\{1, 2, 3, 7\}$ denotes the set whose elements are the numbers 1, 2, 3 and 7. If x is an element of some set X then we denote this fact by writing $x \in X$. Conversely, if x is not an element of the set X then we write $x \notin X$. We denote by $\emptyset$ the *empty set*, which is defined to be the set with no elements.

We denote by $\mathbb{N}$ the set $\{1, 2, 3, 4, 5 \ldots\}$ of all *positive integers* (also known as *natural numbers*), and we denote by $\mathbb{Z}$ the set

$$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

of all *integers* (or 'whole numbers'). We denote by $\mathbb{Q}$ the set of *rational numbers* (i.e., numbers of the form p/q where p and q are integers and $q \neq 0$), and we denote be $\mathbb{R}$ and $\mathbb{C}$ the sets of real numbers and complex numbers respectively.

If X and Y are sets then the *union* $X \cup Y$ of X and Y is defined to be the set of all elements that belong either to X or to Y (or to both), the *intersection* $X \cap Y$ of X and Y is defined to be the set of all elements that belong to both X and Y, and the *difference* $X \setminus Y$ of X and Y is defined to be the set of all elements that belong to X but do not belong to Y. Thus, for example, if

$$X = \{2, 4, 6, 8\}, \qquad Y = \{3, 4, 5, 6, 7\}$$

then

$$X \cup Y = \{2, 3, 4, 5, 6, 7, 8\}, \qquad X \cap Y = \{4, 6\},$$
$$X \setminus Y = \{2, 8\}, \qquad Y \setminus X = \{3, 5, 7\}.$$

If X and Y are sets, and if every element of X is also an element of Y then we say that X is a *subset* of Y, and we write $X \subset Y$. We use the notation $\{y \in Y : P(y)\}$ to denote the subset of a given set Y consisting of all elements y of Y with some given property $P(y)$. Thus for example $\{n \in \mathbb{Z} : n > 0\}$ denotes the set of all integers n satisfying $n > 0$ (i.e., the set $\mathbb{N}$ of all positive integers).

1.2 Rational and Irrational Numbers

Rational numbers are numbers that can be expressed as fractions of the form p/q, where p and q are integers (i.e., 'whole numbers') and $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. Operations of addition, subtraction, multiplication and division are defined on $\mathbb{Q}$ in the usual manner. In addition the set of rational numbers is ordered.

There are however certain familiar numbers which cannot be represented in the form p/q, where p and q are integers. These include $\sqrt{2}$, $\sqrt{3}$, π and e. Such numbers are referred to as *irrational numbers*. The irrationality of $\sqrt{2}$ is an immediate consequence of the following famous result, which was discovered by the Ancient Greeks.

Proposition 1.1 *There do not exist non-zero integers p and q with the property that $p^2 = 2q^2$.*

Proof Let us suppose that there exist non-zero integers p and q with the property that $p^2 = 2q^2$. We show that this leads to a contradiction. Without loss of generality we may assume that p and q are not both even (since if both p and q were even then we could replace p and q by $p/2^k$ and $q/2^k$ respectively, where k is the largest positive integer with the property that 2^k divides both p and q). Now $p^2 = 2q^2$, hence p^2 is even. It follows from this that p is even (since the square of an odd integer is odd). Therefore $p = 2r$ for some integer r. But then $2q^2 = 4r^2$, so that $q^2 = 2r^2$. Therefore q^2 is even, and hence q is even. We have thus shown that both p and q are even. But this contradicts our assumption that p and q are not both even. This contradiction shows that there cannot exist integers p and q with the property that $p^2 = 2q^2$, and thus proves that $\sqrt{2}$ is an irrational number. ∎

This result shows that the rational numbers are not sufficient for the purpose of representing lengths arising in familiar Euclidean geometry. Indeed consider the right-angled isosceles triangle whose short sides are q units long. Then the hypotenuse is $\sqrt{2}q$ units long, by Pythagoras' Theorem. Proposition 1.1 shows that it is not possible to find a unit of length for which the two short sides of this right-angled isosceles triangle are q units long and the hypotenuse is p units long, where both p and q are integers. We must therefore enlarge the system of rational numbers to obtain a number system which contains irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, π and e, and which is capable of representing the lengths of line segments and similar quantities arising in geometry and physics. The rational and irrational numbers belonging to this number system are known as *real numbers*.

1.3 Ordered Fields

An *ordered field* $\mathbb{F}$ consists of a set $\mathbb{F}$ on which are defined binary operations $+$ of addition and $\times$ of multiplication, together with an ordering relation $<$, where these binary operations and ordering relation satisfy the following axioms:—

1. if u and v are elements of $\mathbb{F}$ then their sum $u+v$ is also a element of $\mathbb{F}$;
2. (*the Commutative Law for addition*) $u + v = v + u$ for all elements u and v of $\mathbb{F}$;
3. (*the Associative Law for addition*) $(u + v) + w = u + (v + w)$ for all elements u, v and w of $\mathbb{F}$;
4. there exists an element of $\mathbb{F}$, denoted by 0, with the property that $u + 0 = x = 0 + u$ for all elements u of $\mathbb{F}$;
5. for each element u of $\mathbb{F}$ there exists some element $-u$ of $\mathbb{F}$ with the property that $u + (-u) = 0 = (-u) + u$;
6. if u and v are elements of $\mathbb{F}$ then their product $u \times v$ is also a element of $\mathbb{F}$;
7. (*the Commutative Law for multiplication*) $u \times v = v \times u$ for all elements u and v of $\mathbb{F}$;
8. (*the Associative Law for multiplication*) $(u \times v) \times w = u \times (v \times w)$ for all elements u, v and w of $\mathbb{F}$,
9. there exists an element of $\mathbb{F}$, denoted by 1, with the property that $u \times 1 = u = 1 \times u$ for all elements u of $\mathbb{F}$, and moreover $1 \neq 0$,
10. for each element u of $\mathbb{F}$ satisfying $u \neq 0$ there exists some element u^{-1} of $\mathbb{F}$ with the property that $u \times u^{-1} = 1 = u^{-1} \times u$,
11. (*the Distributive Law*) $u \times (v + w) = (u \times v) + (u \times w)$ for all elements u, v and w of $\mathbb{F}$,
12. (*the Trichotomy Law*) if u and v are elements of $\mathbb{F}$ then one and only one of the three statements $u < v$, $u = v$ and $u < v$ is true,
13. (*transitivity of the ordering*) if u, v and w are elements of $\mathbb{F}$ and if $u < v$ and $v < w$ then $u < w$,
14. if u, v and w are elements of $\mathbb{F}$ and if $u < v$ then $u + w < v + w$,

15. if u and v are elements of $\mathbb{F}$ which satisfy $0 < u$ and $0 < v$ then $0 < u \times v$,

The operations of subtraction and division are defined on an ordered field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $u - v = u + (-v)$ for all elements u and v of $\mathbb{F}$, and moreover $u/v = uv^{-1}$ provided that $v \neq 0$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Let D be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let D be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above that characterize ordered fields are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid in any ordered field whatsoever, and in particular would be valid were the system of real numbers replaced by the system of rational numbers.) We require in addition the following axiom:—

> *the Least Upper Bound Axiom*: given any non-empty set D of real numbers that is bounded above, there exists a real number $\sup D$ that is the least upper bound for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.4 Remarks on the Existence of Least Upper Bounds

We present an argument here that is intended to show that if the system of real numbers has all the properties that one would expect it to possess, then it must satisfy the Least Upper Bound Axiom.

Let $\mathbb{F}$ be an ordered field that contains the field $\mathbb{Q}$ of rational numbers. The set $\mathbb{Z}$ is a subset of $\mathbb{Q}$. Thus $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{F}$, and therefore $\mathbb{Z} \subset \mathbb{F}$.

Definition Let $\mathbb{F}$ be an ordered field that contains the field of rational numbers. The field $\mathbb{F}$ is said to satisfy the *Axiom of Archimedes* if, given any element x of $\mathbb{F}$, there exists some integer n satisfying $n \geq x$.

The Axiom of Archimedes excludes the possibility of "infinitely large" elements of the ordered field $\mathbb{F}$. Given that all real numbers should be representable in decimal arithmetic, any real number must be less than some positive integer. Thus we expect the system of real numbers to satisfy the Axiom of Archimedes.

Lemma 1.2 *Let $\mathbb{F}$ be an ordered field that satisfies the Axiom of Archimedes. Then, given any element x of $\mathbb{F}$ satisfying $x > 0$, there exists some positive integer n such that $x > \dfrac{1}{n} > 0$.*

Proof The Axiom of Archimedes ensures the existence of a positive integer n satisfying $n > \dfrac{1}{x}$. Then

$$n - \frac{1}{x} > 0 \quad \text{and} \quad \frac{x}{n} = x \times \frac{1}{n} > 0,$$

and therefore

$$x - \frac{1}{n} = \left(n - \frac{1}{x}\right) \times \frac{x}{n} > 0,$$

and thus $x > \dfrac{1}{n}$, as required. ∎

Now let $\mathbb{F}$ be an ordered field containing as a subfield the field $\mathbb{Q}$ of rational numbers. We suppose also that $\mathbb{F}$ satisfies the Axiom of Archimedes. Let D be a subset of $\mathbb{F}$ which is bounded above. The Axiom of Archimedes then ensures that there exists some integer that is an upper bound for the set D. It follows from this that there exists some integer m that is the largest integer that is *not* an upper bound for the set D. Then m is not an upper bound for D, but $m + 1$ is. Let

$$E = \{x \in \mathbb{F} : x \geq 0 \quad \text{and} \quad m + x \in D\}.$$

Then E is non-empty and $x \leq 1$ for all $x \in E$. Suppose that there exists a least upper bound $\sup E$ in $\mathbb{F}$ for the set E. Then $m + \sup E$ is a least upper bound for the set D, and thus $\sup D$ exists, and $\sup D = m + \sup E$. Thus, in order to show that every non-empty subset of D that is bounded above has a least upper bound, it suffices to show this for subsets D of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$.

Now let $\mathbb{F}$ be an ordered field containing the field $\mathbb{Q}$ of rational numbers that satisfies the Axiom of Archimedes, and let D be a subset of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$. Then, for each positive integer m, let

u_m denote the largest non-negative integer for which $u_m \times (10)^{-m}$ is *not* an upper bound for the set D. Then $0 \leq u_m < (10)^m$ and $(u_m + 1)(10)^{-m}$ is an upper bound for the set D. Thus if there were to exist a least upper bound s for the set D, then s would have to satisfy

$$\frac{u_m}{(10)^m} < s \leq \frac{u_m}{(10)^m} + \frac{1}{(10)^m}$$

for $m = 1, 2, 3, \ldots$. Now if $m > 1$ then definitions of u_m and u_{m-1} ensure that $(10u_{m-1}) \times (10)^{-m}$ is not an upper bound for the set D but $(10u_{m-1} + 10) \times (10)^{-m}$ is an upper bound for the set D. It follows that

$$10u_{m-1} \leq u_m < 10u_{m-1} + 10.$$

Let $d_1 = u_1$, and let $d_m = u_m - 10u_{m-1}$ for all integers m satisfying $m > 1$. Then d_m is an integer satisfying $0 \leq d_m < 10$ for $m = 1, 2, 3, \ldots$, and

$$\frac{u_m}{(10)^m} = \frac{d_m}{(10)^m} + \frac{u_{m-1}}{(10)^{m-1}}.$$

It follows that

$$\frac{u_m}{(10)^m} = \sum_{k=1}^{m} \frac{d_k}{(10)^k}.$$

Any least upper bound t for the set D would therefore have to satisfy the inequalities

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} < t \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}$$

for all positive integers m.

Now suppose that every well-formed decimal expansion determines a corresponding element of the ordered field $\mathbb{F}$. Assuming this, we conclude that there must exist some element s of the ordered field $\mathbb{F}$ whose decimal expansion takes the form

$$0.d_1\, d_2\, d_3\, d_4\, d_5, \ldots$$

The basic properties of decimal expansions then ensure that

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} \leq s \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}.$$

Let ε be an element of $\mathbb{F}$ satisfying $\varepsilon > 0$. Then, because the ordered field $\mathbb{F}$ is required to satisfy the Axiom of Archimedes, a positive integer m can be chosen large enough to ensure that $0 < (10)^{-m} < \varepsilon$. Then

$$s - \varepsilon < \sum_{k=1}^{m} \frac{d_k}{(10)^k} = \frac{u_m}{(10)^m},$$

and therefore $s - \varepsilon$ cannot be an upper bound for the set D. Also

$$s + \varepsilon > \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m} = \frac{u_m}{(10)^m} + \frac{1}{(10)^m},$$

and therefore $s + \varepsilon$ is an upper bound for the set D. We see therefore if s is an element of $\mathbb{F}$ satisfying $0 \leq s \leq$, and if s is determined by the decimal expansion whose successive decimal digits are $d_1, d_2, d_3, \ldots$, where these digits are determined by D as described above, then $s - \varepsilon$ cannot be an upper bound for the set D for any $\varepsilon > 0$, but $s + \varepsilon$ must be an upper bound for the set D for all $\varepsilon > 0$.

Now if there were to exist any element x of D satisfying $x > s$, then we could obtain a contradiction on choosing $\varepsilon \in \mathbb{F}$ such that $0 < \varepsilon < x - s$. It follows that $x \leq s$ for all $x \in D$, and thus s is an upper bound for the set D. But if $\varepsilon > 0$ then $s - \varepsilon$ is not an upper bound for the set D. Therefore s must be the least upper bound for the set D.

This analysis shows that if $\mathbb{F}$ is an ordered field, containing the field of rational numbers, that satisfies the Axiom of Archimedes, and if every decimal expansion determines a corresponding element of $\mathbb{F}$ then every non-empty subset of $\mathbb{F}$ that is bounded above must have a least upper bound. The ordered field $\mathbb{F}$ must therefore satisfy the Least Upper Bound Axiom.

This justifies the characterization of the field $\mathbb{R}$ of real numbers as an ordered field that satisfies the Least Upper Bound Axiom.

1.5 Intervals

Given real numbers a and b satisfying $a \leq b$, we define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

If $a < b$ then we define

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

For each real number c, we also define

$$[c, +\infty) = \{x \in \mathbb{R} : c \leq x\}, \qquad (c, +\infty) = \{x \in \mathbb{R} : c < x\},$$

$$(-\infty, c] = \{x \in \mathbb{R} : x \leq c\}, \qquad (-\infty, c) = \{x \in \mathbb{R} : x < c\}.$$

All these subsets of $\mathbb{R}$ are referred to as *intervals*. An *interval* I may be defined as a non-empty set of real numbers with the following property: if s,

t and u are real numbers satisfying $s < t < u$ and if s and u both belong to the interval I then t also belongs to the interval I. Using the Least Upper Bound Axiom, one can prove that every interval in $\mathbb{R}$ is either one of the intervals defined above, or else is the whole of $\mathbb{R}$.

1.6 The Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techiques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second. Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convegence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.

2 Infinite Sequences of Real Numbers

2.1 Convergence

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim_{j \to +\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l - \varepsilon < x_j < l + \varepsilon$ for all positive integers j satisfying $j \geq N$.

Example A straightforward application of the definition of convergence shows that $1/j \to 0$ as $j \to +\infty$. Indeed suppose that we are given any strictly positive real number ε. If we pick some positive integer N large enough to satisfy $N > 1/\varepsilon$ then $|1/j| < \varepsilon$ for all positive integers j satisfying $j \geq N$, as required.

Example We show that $(-1)^j/j^2 \to 0$ as $j \to +\infty$. Indeed, given any strictly positive real number ε, we can find some positive integer N satisfying $N^2 > 1/\varepsilon$. If $j \geq N$ then $|(-1)^j/j^2| < \varepsilon$, as required.

Example The infinite sequence $x_1, x_2, x_3, \ldots$ defined by $x_j = j$ is not convergent. To prove this formally, we suppose that it were the case that $\lim_{j \to +\infty} x_j = l$ for some real number l, and derive from this a contradiction. On setting $\varepsilon = 1$ (say) in the formal definition of convergence, we would deduce that there would exist some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $x_j < l + 1$ for all $j \geq N$, which is impossible. Thus the sequence cannot converge.

Example The infinite sequence $u_1, u_2, u_3, \ldots$ defined by $u_j = (-1)^j$ is not convergent. To prove this formally, we suppose that it were the case that $\lim_{j \to +\infty} u_j = l$ for some real number l. On setting $\varepsilon = \frac{1}{2}$ in the criterion for convergence, we would deduce the existence of some positive integer N such that $|u_j - l| < \frac{1}{2}$ for all $j \geq N$. But then

$$|u_j - u_{j+1}| \leq |u_j - l| + |l - u_{j+1}| < \tfrac{1}{2} + \tfrac{1}{2} = 1$$

for all $j \geq N$, contradicting the fact that $u_j - u_{j+1} = \pm 2$ for all j. Thus the sequence cannot converge.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 2.1 *Every convergent sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number l. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $l - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $l + 1$. ∎

Proposition 2.2 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum, difference and product of these sequences are convergent, and*

$$\begin{aligned}
\lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j y_j) &= \left(\lim_{j \to +\infty} x_j \right) \left(\lim_{j \to +\infty} y_j \right).
\end{aligned}$$

If in addition $y_j \neq 0$ for all positive integers j and $\lim_{j\to+\infty} y_j \neq 0$, then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j\to+\infty} \frac{x_j}{y_j} = \frac{\lim_{j\to+\infty} x_j}{\lim_{j\to+\infty} y_j}.$$

Proof Throughout this proof let $l = \lim_{j\to+\infty} x_j$ and $m = \lim_{j\to+\infty} y_j$.

First we prove that $x_j + y_j \to l + m$ as $j \to +\infty$. Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (l+m)| < \varepsilon$ whenever $j \geq N$. Now $x_j \to l$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - l| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - l| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - m| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (l+m)| &= |(x_j - l) + (y_j - m)| \leq |x_j - l| + |y_j - m| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to l + m$ as $j \to +\infty$.

Let c be some real number. We show that $cy_j \to cm$ as $j \to +\infty$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|y_j - m| < \varepsilon/|c|$ whenever $j \geq N$. But then $|cy_j - cm| = |c||y_j - m| < \varepsilon$ whenever $j \geq N$. Thus $cy_j \to cm$ as $j \to +\infty$.

If we combine this result, for $c = -1$, with the previous result, we see that $-y_j \to -m$ as $j \to +\infty$, and therefore $x_j - y_j \to l - m$ as $j \to +\infty$.

Next we show that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences, and if $u_j \to 0$ and $v_j \to 0$ as $j \to +\infty$, then $u_j v_j \to 0$ as $j \to +\infty$. Let some strictly positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. We deduce that $u_j v_j \to 0$ as $j \to +\infty$.

We can apply this result with $u_j = x_j - l$ and $v_j = y_j - m$ for all positive integers j. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{j\to+\infty} (u_j v_j) = \lim_{j\to+\infty} (x_j y_j - x_j m - l y_j + lm) \\ &= \lim_{j\to+\infty} (x_j y_j) - m \lim_{j\to+\infty} x_j - l \lim_{j\to+\infty} y_j + lm = \lim_{j\to+\infty} (x_j y_j) - lm. \end{aligned}$$

Thus $x_j y_j \to lm$ as $j \to +\infty$.

Next we show that if $w_1, w_2, w_3, \ldots$ is an infinite sequence of non-zero real numbers, and if $w_j \to 1$ as $j \to +\infty$ then $1/w_j \to 1$ as $j \to +\infty$. Let some strictly positive real number ε be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some positive integer N such that $|w_j - 1| < \varepsilon_0$ whenever $j \geq N$. Thus if $j \geq N$ then $|w_j - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < w_j < \frac{3}{2}$. But then

$$\left| \frac{1}{w_j} - 1 \right| = \left| \frac{1 - w_j}{w_j} \right| = \frac{|w_j - 1|}{|w_j|} < 2|w_j - 1| < \varepsilon.$$

We deduce that $1/w_j \to 1$ as $j \to +\infty$.

Finally suppose that $\lim\limits_{j \to +\infty} x_j = l$ and $\lim\limits_{j \to +\infty} y_j = m$, where $m \neq 0$. Let $w_j = y_j/m$. Then $w_j \to 1$ as $j \to +\infty$, and hence $1/w_j \to 1$ as $j \to +\infty$. We see therefore that $m/y_j \to 1$, and thus $1/y_j \to 1/m$, as $j \to +\infty$. The result we have already obtained for products of sequences then enables us to deduce that $x_j/y_j \to l/m$ as $j \to +\infty$. ∎

Example We shall show that if $s_j \to 2$ as $j \to +\infty$, where $s_j = \dfrac{6j^2 - 4j}{3j^2 + 7}$ for all positive integers j. Now neither $6j^2 - 4j$ nor $3j^2 + 7$ converges to any (finite) limit as $j \to +\infty$; and therefore we cannot directly apply the result in Proposition 2.2 concerning the convergence of the quotient of two convergent sequences. However on dividing both the numerator and the denominator of the fraction defining s_j by j^2, we see that

$$s_j = \frac{6j^2 - 4j}{3j^2 + 7} = \frac{6 - \dfrac{4}{j}}{3 + \dfrac{7}{j^2}}.$$

Moreover $6 - \dfrac{4}{j} \to 6$ and $3 + \dfrac{7}{j^2} \to 3$ as $j \to +\infty$, and therefore, on applying Proposition 2.2, we see that

$$\lim_{j \to +\infty} \frac{6j^2 - 4j}{3j^2 + 7} = \lim_{j \to +\infty} \frac{6 - \dfrac{4}{j}}{3 + \dfrac{7}{j^2}} = \frac{\lim\limits_{j \to +\infty} \left(6 - \dfrac{4}{j} \right)}{\lim\limits_{j \to +\infty} \left(3 + \dfrac{7}{j^2} \right)} = \frac{6}{3} = 2.$$

2.2 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 2.3 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Example Let $x_1 = 2$ and

$$x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j}$$

for all positive integers j. Now

$$x_{j+1} = \frac{x_j^2 + 2}{2x_j} \text{ and } x_{j+1}^2 = x_j^2 - (x_j^2 - 2) + \left(\frac{x_j^2 - 2}{2x_j}\right)^2 = 2 + \left(\frac{x_j^2 - 2}{2x_j}\right)^2.$$

It therefore follows by induction on j that $x_j > 0$ and $x_j^2 > 2$ for all positive integers j. But then $x_{j+1} < x_j$ for all j, and thus the sequence $x_1, x_2, x_3, \ldots$ is decreasing and bounded below. It follows from Theorem 2.3 that this sequence converges to some real number α. Also $x_j > 1$ for all positive integers j (since $x_j > 0$ and $x_j^2 > 2$), and therefore $\alpha \geq 1$. But then, on applying Proposition 2.2, we see that

$$\alpha = \lim_{j \to +\infty} x_{j+1} = \lim_{j \to +\infty} \left(x_j - \frac{x_j^2 - 2}{2x_j}\right) = \alpha - \frac{\alpha^2 - 2}{2\alpha}.$$

Thus $\alpha^2 = 2$, and so $\alpha = \sqrt{2}$.

2.3 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

2.4 The Bolzano-Weierstrass Theorem

Proposition 2.4 *Let $x_1, x_2, x_3, \ldots$ be a bounded infinite sequence of real numbers. Then there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$.*

First Proof The infinite sequence $(x_j : j \in \mathbb{N})$ is bounded, and therefore there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j. For each real number s let

$$Q_s = \{j \in \mathbb{N} : x_j > s\}.$$

Then $Q_s = \emptyset$ whenever $s \geq B$, and $Q_s = \mathbb{N}$ whenever $s < A$.

Let S be the set consisting of all real numbers s for which the corresponding set Q_s is infinite. Then $s \notin S$ whenever $s \geq B$, and $s \in S$ whenever $s < A$. It follows that the set S is a non-empty subset of $\mathbb{R}$ that is bounded above by B. The Least Upper Bound Principle therefore ensures that the set S has a well-defined least upper bound. Let c be the least upper bound bound of the set S.

Let some strictly positive real number ε be given. Let v satisfy $c < v < c + \varepsilon$. Then $v \notin S$, because c is an upper bound for the set S, and therefore the set Q_v is a finite subset of $\mathbb{N}$. Also $c - \varepsilon$ is not an upper bound for the set S, because c is the least upper bound for this set, and therefore there exists some element u of S satisfying $c - \varepsilon < u \leq c$. Then Q_u is an infinite subset of $\mathbb{N}$. It follows that the complement $Q_u \setminus Q_v$ of Q_v in Q_u is a subset of $\mathbb{N}$ with infinitely many elements.

Now

$$Q_u \setminus Q_v = \{j \in \mathbb{N} : x_j > u\} \setminus \{j \in \mathbb{N} : x_j > v\} = \{j \in \mathbb{N} : u < x_j \leq v\}.$$

Thus $c - \varepsilon < u < x_j \leq v < c + \varepsilon$ for all $j \in Q_u \setminus Q_v$. Therefore the number of positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$ must be infinite, as required. ∎

Theorem 2.5 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

First Proof Let $x_1, x_2, x_3, \ldots$ be an bounded infinite sequence of real numbers. It follows from Proposition 2.4 that there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$. There then exists some positive integer k_1 such that $c - 1 < x_{k_1} < c + 1$.

Now suppose that positive integers $k_1, k_2, \ldots, k_m$ have been determined such that $k_1 < k_2 < \cdots < k_m$ and

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for $j = 1, 2, \ldots, m$. The interval

$$\left\{x \in \mathbb{R} : c - \frac{1}{m+1} < x < c + \frac{1}{m+1}\right\}$$

must then contain infinitely many members of the original sequence, and therefore there exists some positive integer k_{m+1} for which $k_m < k_{m+1}$ and

$$c - \frac{1}{m+1} < x_{k_{m+1}} < c + \frac{1}{m+1}.$$

Thus we can construct in this fashion a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ of the original sequence with the property that

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for all positive integers j. This subsequence then converges to c. The given sequence therefore has a convergent subsequence, as required. ∎

Second Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers, and let

$$S = \{j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j\}$$

(i.e., S is the set of all positive integers j with the property that a_j is greater than or equal to all the succeeding members of the sequence).

First let us suppose that the set S is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the manner in which the set S was defined that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 2.3 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S is finite. Choose a positive integer j_1 which is greater than every positive integer belonging to S. Then j_1 does not belong to S. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 does not belong to S (since j_2 is greater than j_1 and j_1 is greater than every positive integer belonging to S). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 2.3. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

2.5 Cauchy's Criterion for Convergence

Definition A sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - x_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 2.6 *Every Cauchy sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence. Then there exists some positive integer N such that $|x_j - x_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|x_j| \leq |x_N| + 1$ whenever $j \geq N$. Therefore $|x_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|x_1|, |x_2|, \ldots, |x_{N-1}|$ and $|x_N| + 1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence*.

Theorem 2.7 (Cauchy's Criterion for Convergence) *An infinite sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences are Cauchy sequences. Let $x_1, x_2, x_3, \ldots$ be a convergent sequence of real numbers, and let $l = \lim_{j \to +\infty} x_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - l| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|x_j - l| < \frac{1}{2}\varepsilon$ and $|x_k - l| < \frac{1}{2}\varepsilon$, and hence

$$|x_j - x_k| = |(x_j - l) - (x_k - l)| \leq |x_j - l| + |x_k - l| < \varepsilon.$$

Thus the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $x_1, x_2, x_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 2.6. The sequence $x_1, x_2, x_3, \ldots$ therefore has a convergent subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.5). Let $l = \lim_{j \to +\infty} x_{k_j}$. We claim that the sequence $x_1, x_2, x_3, \ldots$ itself converges to l.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - x_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|x_{k_m} - l| < \frac{1}{2}\varepsilon$. Then

$$|x_j - l| \leq |x_j - x_{k_m}| + |x_{k_m} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $x_j \to l$ as $j \to +\infty$, as required. ∎

3 Continuity for Functions of a Real Variable

3.1 The Definition of Continuity for Functions of a Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $f: D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be *continuous* at s if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Example Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function f is not continuous at 0. To prove this formally we note that when $0 < \varepsilon \leq 1$ there does not exist any strictly positive real number δ with the property that $|f(x) - f(0)| < \varepsilon$ for all x satisfying $|x| < \delta$ (since $|f(x) - f(0)| = 1$ for all $x > 0$).

Example Let $g: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We show that this function is not continuous at 0. Suppose that ε is chosen to satisfy $0 < \varepsilon < 1$. No matter how small we choose the strictly positive real number δ, we can always find $x \in \mathbb{R}$ for which $|x| < \delta$ and $|g(x) - g(0)| \geq \varepsilon$. Indeed, given any strictly positive real number δ, we can choose some integer j large enough to ensure that $0 < x_j < \delta$, where x_j satisfies $1/x_j = (4j+1)\pi/2$. Moreover $g(x_j) = 1$. This shows that the criterion defining the concept of continuity is not satisfied at $x = 0$.

Example Let $h: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h(x) = \begin{cases} 3x \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that the function h is continuous at 0. To prove this, we must apply the definition of continuity directly. Let some strictly positive real number ε be given. If $\delta = \frac{1}{3}\varepsilon$ then $|h(x)| \leq 3|x| < \varepsilon$ for all real numbers x satisfying $|x| < \delta$, as required.

Lemma 3.1 *Let D be a subset of $\mathbb{R}$, let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be real-valued functions on D, and let $s \in D$. Suppose that the functions f and g are continuous at s. Then so is the function $f+g$, where $(f+g)(x) = f(x)+g(x)$ for all $x \in D$.*

Proof Suppose that $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are continuous at s, where $s \in D$. We show that $f + g$ is continuous at s. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(x) - f(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta_1$, and $|g(x)-g(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x-s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $|x - s| < \delta$ then

$$|f(x) + g(x) - (f(s) + g(s))| \leq |f(x) - f(s)| + |g(x) - g(s)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

showing that $f + g$ is continuous at s, as required. ∎

Lemma 3.2 *Let D be a subset of $\mathbb{R}$, let $f: D \to \mathbb{R}$ be a real-valued function on D, let c be a real number, and let $s \in D$. Suppose that the function f is continuous at s. Then so is the function cf, where $(cf)(x) = cf(x)$ for all $x \in D$.*

Proof If $c = 0$ then the function cf is the zero function, and is therefore continuous. We may therefore restrict attention to the case where $c \neq 0$.

Let some strictly positive real number ε be given, and let $\varepsilon_0 = \varepsilon/|c|$. Then $\varepsilon_0 > 0$, and the continuity of f at s then ensures the existence of some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon_0$ whenever $x \in D$ satisfies $|x - s| < \delta$. But then

$$|cf(x) - cf(s)| = |c|\,|f(x) - f(s)| < |c|\varepsilon_0 = \varepsilon$$

whenever $x \in D$ satisfies $|x - s| < \delta$. This shows that the function cf is continuous as s, as required. ∎

Lemma 3.3 *Let D be a subset of $\mathbb{R}$, let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be real-valued functions on D, and let $s \in D$. Suppose that the functions f and g are continuous at s. Then so is the function $f \cdot g$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in D$.*

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(x)-f(s)| < \sqrt{\varepsilon}$ whenever $x \in D$ satisfies $|x - s| < \delta_1$ and $|g(x) - g(s)| < \sqrt{\varepsilon}$ whenever $x \in D$ satisfies

$|x-s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $|x-s| < \delta$ then

$$|(f(x) - f(s)(g(x) - g(s))| = |(f(x) - f(s)| \, |(g(x) - g(s))| < \sqrt{\varepsilon} \times \sqrt{\varepsilon} = \varepsilon.$$

But

$$\begin{aligned} (f(x) - f(s)(g(x) - g(s)) &= f(x)g(x) - f(s)g(x) - g(s)f(x) + f(s)g(s) \\ &= h(x) - h(s), \end{aligned}$$

where $h: D \to \mathbb{R}$ is the real-valued function on D defined such that

$$h(x) = f(x)g(x) - f(s)g(x) - g(s)f(x)$$

for all $x \in D$. It follows that $|h(x) - h(s)| < \varepsilon$ whenever $x \in D$ satisfies $|x - s| < \delta$. We conclude from this that the function $h: D \to \mathbb{R}$ is continuous at s. Now

$$f(x)g(x) = h(x) + f(s)g(x) + g(s)f(x).$$

It therefore follows from Lemma 3.1 and Lemma 3.2 that the function $f \cdot g$ is continuous as s, as required. ∎

Proposition 3.4 *Let $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{R}$ satisfying $f(D) \subset E$. Let s be an element of D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Then the composition $g \circ f$ of f and g is continuous at s.*

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(u) - g(f(s))| < \varepsilon$ for all $u \in E$ satisfying $|u - f(s)| < \eta$. But then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \eta$ for all $x \in D$ satisfying $|x - s| < \delta$. Thus if $|x - s| < \delta$ then $|g(f(x)) - g(f(s))| < \varepsilon$. Hence $g \circ f$ is continuous at s. ∎

Lemma 3.5 *Let $f: D \to \mathbb{R}$ be a function defined on a subset D of $\mathbb{R}$, and let s be an element of D. Suppose that $f(x) \neq 0$ for all $x \in D$ and that the function f is continuous at s for some $s \in D$. Then the function $1/f$ is also continuous at s, where $(1/f)(x) = 1/f(x)$ for all $x \in D$.*

Proof Let $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined such that $r(t) = 1/t$ for all non-zero real numbers t. We show that the function r is continuous. Let u be a non-zero real number, and let some strictly positive real number ε be given. Then

$$r(t) - r(u) = \frac{1}{t} - \frac{1}{u} = \frac{u - t}{tu}$$

for all non-zero real numbers t. Let δ be the minimum of $\frac{1}{2}|u|$ and $\frac{1}{2}|u|^2\varepsilon$. If t is a non-zero real number, and if $|t-u| < \delta$ then $|t| \geq |u| - |t-u| \geq \frac{1}{2}|u|$, and therefore

$$|r(t) - r(u)| \leq \frac{2}{|u|^2}|t-u| < \frac{2}{|u|^2}\delta \leq \varepsilon.$$

It follows that the function $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous at u.

Now $1/f(x) = r(f(x))$ for all $x \in X$. Moreover the function f is continuous at s, and the function f is continuous at $f(s)$. It follows from Lemma 3.4 that the composition function $r \circ f$ is continuous at s. Thus the function $1/f$ is continuous at s, as required. ∎

Proposition 3.6 *Let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ be functions defined over some subset D of $\mathbb{R}$. Suppose that f and g are continuous at some point s of D. Then the functions $f+g$, $f-g$ and $f \cdot g$ are also continuous at s. If moreover the function g is everywhere non-zero on D then the function f/g is continuous at s.*

Proof Now $f - g = f + (-g)$, and it follows from Lemma 3.2 that the function $-g$ is continuous on D. Lemma 3.1 therefore ensures that both $f+g$ and $f-g$ are continuous at s. Lemma 3.3 ensures that $f \cdot g$ is continuous at s. Moreover if the function g is everywhere non-zero on D then $f/g = f \cdot (1/g)$, and Lemma 3.5 ensures that the function $1/g$ is continuous on D. It then follows from Lemma 3.3 that the function f/g is continuous on D, as required. ∎

We shall show that if $f: D \to \mathbb{R}$ is a continuous real-valued function defined over some subset D of $\mathbb{R}$, then so is $|f|: D \to \mathbb{R}$, where $|f|(x) = |f(x)|$ for all $x \in D$.

Let u and v be real numbers. Then

$$\Big||u| - |v|\Big| \leq |u - v|.$$

To see this, note that $u = (u - v) + v$ and $v = (v - u) + u$ and therefore $|u| \leq |u - v| + |v|$ and $|v| \leq |v - u| + |u|$. But $|u - v| = |v - u|$. It follows that $|u| - |v| \leq |u - v|$ and $|v| - |u| \leq |u - v|$, and therefore

$$\Big||u| - |v|\Big| \leq |u - v|.$$

We apply this inequality in the proof of the next lemma.

Lemma 3.7 *Let D be a subset of $\mathbb{R}$, let $f: D \to \mathbb{R}$ be a real-valued function on D, and let $s \in D$. Suppose that the function f is continuous at s. Then so is the function $|f|$, where $|f|(x) = |f(x)|$ for all $x \in D$.*

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$\Big| |f(x)| - |f(s)| \Big| \leq |f(x) - f(s)| < \varepsilon$$

for all $x \in D$ satisfying $|x - s| < \delta$. It follows that $|f|: D \to \mathbb{R}$ is continuous at s, as required. ∎

Lemma 3.8 *Let $f: D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$, and let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers belonging to D. Suppose that $x_j \to s$ as $j \to +\infty$, where $s \in D$, and that f is continuous at s. Then $f(x_j) \to f(s)$ as $j \to +\infty$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then there exists some positive integer N such that $|x_j - s| < \delta$ for all j satisfying $j \geq N$. Thus $|f(x_j) - f(s)| < \varepsilon$ whenever $j \geq N$. Hence $f(x_j) \to f(s)$ as $j \to +\infty$. ∎

3.2 Limits of Functions of One Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $s \in \mathbb{R}$. The real number s is said to be a *limit point* of the set D if, given any strictly positive real number δ, there exists some real number x belonging to D such that $0 < |x - s| < \delta$.

It follows easily from the definition of convergence of sequences of real numbers that if D is a subset of the set $\mathbb{R}$ of real numbers, and if s is a point of $\mathbb{R}$ then the point s is a limit point of the set D if and only if there exists an infinite sequence $x_1, x_2, x_3, \ldots$ of points of D, all distinct from the point s, such that $\lim_{j \to +\infty} x_j = s$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f: D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. The real number l is said to be the *limit* of $f(x)$, as x tends to s in D, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - l| < \varepsilon$ whenever $x \in D$ satisfies $0 < |x - s| < \delta$.

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. If l is the limit of $f(x)$ as x tends to s in D then we can denote this fact by writing $\lim_{x \to s} f(x) = l$.

Proposition 3.9 *Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. Let $\tilde{D} = D \cup \{s\}$, and let $\tilde{f}\colon \tilde{D} \to \mathbb{R}$ be defined such that*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq s; \\ l & \text{if } x = s. \end{cases}$$

Then $\lim_{x \to s} f(x) = l$ if and only if the function $\tilde{f}$ is continuous at s.

Proof The result follows directly on comparing the relevant definitions. ∎

Corollary 3.10 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a point of the set D that is also a limit point of D. Then the function f is continuous at the point s if and only if $\lim_{x \to s} f(x) = f(s)$.*

Let D be a subset of $\mathbb{R}$, and let s be a real number belonging to the set D. Suppose that s is not a limit point of the set D. Then there exists some strictly positive real number δ_0 such that $|x - s| \geq \delta_0$ for all $x \in X$. The point s is then said to be an *isolated point* of D.

Let D be a subset of $\mathbb{R}$. The definition of continuity then ensures that any real-valued function $f\colon D \to \mathbb{R}$ on D is continuous at any isolated point of its domain D.

Corollary 3.11 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let s be a limit point of the set D. Suppose that $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ both exist. Then so do $\lim_{x \to s}(f(x) + g(x))$, $\lim_{x \to s}(f(x) - g(x))$ and $\lim_{x \to s}(f(x)g(x))$, and moreover*

$$\begin{aligned} \lim_{x \to s}(f(x) + g(x)) &= \lim_{x \to s} f(x) + \lim_{x \to s} g(x), \\ \lim_{x \to s}(f(x) - g(x)) &= \lim_{x \to s} f(x) - \lim_{x \to s} g(x), \\ \lim_{x \to s}(f(x)g(x)) &= \lim_{x \to s} f(x) \times \lim_{x \to s} g(x). \end{aligned}$$

If moreover $g(x) \neq 0$ for all $x \in X$ and $\lim_{x \to s} g(x) \neq 0$ then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \frac{\lim_{x \to s} f(x)}{\lim_{x \to s} g(x)}.$$

Proof Let $\tilde{D} = X \cup \{s\}$, and let $\tilde{f}\colon \tilde{D} \to \mathbb{R}$ and $\tilde{g}\colon \tilde{D} \to \mathbb{R}$ be defined such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq s; \\ l & \text{if } x = s. \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & \text{if } x \neq s; \\ m & \text{if } x = s. \end{cases},$$

where $l = \lim_{x \to s} f(x)$ and $m = \lim_{x \to s} g(x)$. Then the functions $\tilde{f}$ and $\tilde{g}$ are continuous at s. The result therefore follows on applying Proposition 3.6. ∎

3.3 The Intermediate Value Theorem

Proposition 3.12 *Let $f\colon [a,b] \to \mathbb{Z}$ continuous integer-valued function defined on a closed interval $[a,b]$. Then the function f is constant.*

Proof Let

$$S = \{x \in [a,b] : f \text{ is constant on the interval } [a,x]\},$$

and let $s = \sup S$. Now $s \in [a,b]$, and therefore the function f is continuous at s. Therefore there exists some strictly positive real number δ such that $|f(x) - f(s)| < \frac{1}{2}$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. But the function f is integer-valued. It follows that $f(x) = f(s)$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. Now $s - \delta$ is not an upper bound for the set S. Therefore there exists some element x_0 of S satisfying $s - \delta < x_0 \leq s$. But then $f(s) = f(x_0) = f(a)$, and therefore the function f is constant on the interval $[a,x]$ for all $x \in [a,b]$ satisfying $s \leq x < s+\delta$. Thus $x \in [a,b] \cap [s, s+\delta) \subset S$. In particular $s \in S$. Now S cannot contain any elements x of $[a,b]$ satisfying $x > s$. Therefore $[a,b] \cap [s, s+\delta) = \{s\}$, and therefore $s = b$. This shows that $b \in S$, and thus the function f is constant on the interval $[a,b]$, as required. ∎

Theorem 3.13 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on the interval $[a,b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a,b]$ for which $f(s) = c$.*

Proof The result is trivially true in the cases where $c = f(a)$ or $x = f(b)$. We may therefore suppose that either $f(a) < c < f(b)$ or else $f(a) > c > f(b)$. In either case, let $g_c\colon \mathbb{R} \setminus \{c\} \to \mathbb{Z}$ be the continuous integer-valued function on $\mathbb{R} \setminus \{c\}$ defined such that $g_c(x) = 0$ whenever $x < c$ and $g_c(x) = 1$ if $x > c$. Suppose that c were not in the range of the function f. Then the composition function $g_c \circ f\colon [a,b] \to \mathbb{R}$ would be a continuous integer-valued function defined throughout the interval $[a, b]$. This function would not be constant, since $g_c(f(a)) \neq g_c(f(b))$. But every continuous integer-valued function on the interval $[a, b]$ is constant (Proposition 3.12). It follows that every real number c lying between $f(a)$ and $f(b)$ must belong to the range of the function f, as required. ∎

Corollary 3.14 *Let $f\colon [a, b] \to [c, d]$ be a strictly increasing continuous function mapping an interval $[a, b]$ into an interval $[c, d]$, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. Suppose that $f(a) = c$ and $f(b) = d$. Then the function f has a continuous inverse $f^{-1}\colon [c, d] \to [a, b]$.*

Proof Let x_1 and x_2 be distinct real numbers belonging to the interval $[a, b]$ then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ or $x_1 > x_2$, in which case $f(x_1) > f(x_2)$. Thus $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. It follows that the function f is injective. The Intermediate Value Theorem (Theorem 3.13) ensures that f is surjective. It follows that the function f has a well-defined inverse $f^{-1}\colon [c, d] \to [a, b]$. It only remains to show that this inverse function is continuous.

Let y be a real number satisfying $c < y < d$, and let x be the unique real number such that $a < x < b$ and $f(x) = y$. Let some strictly positive real number ε be given. We can then choose $x_1, x_2 \in [a, b]$ such that $x - \varepsilon < x_1 < x < x_2 < x + \varepsilon$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $y_1 < y < y_2$. Choose a strictly positive real number δ for which $\delta < y - y_1$ and $\delta < y_2 - y$. If $v \in [c, d]$ satisfies $|v - y| < \delta$ then $y_1 < v < y_2$ and therefore $x_1 < f^{-1}(v) < x_2$. But then $|f^{-1}(v) - f^{-1}(y)| < \varepsilon$. We conclude that the function $f^{-1}\colon [c, d] \to [a, b]$ is continuous at all points in the interior of the interval $[a, b]$. A similar argument shows that it is continuous at the endpoints of this interval. Thus the function f has a continuous inverse, as required. ∎

3.4 The Extreme Value Theorem

Theorem 3.15 (The Extreme Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then there exist real numbers u and v belonging to the interval $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.*

Proof We prove the result for an arbitrary continuous real-valued function $f\colon [a, b] \to \mathbb{R}$ by showing that the result holds for a related continuous function $g\colon [a, b] \to \mathbb{R}$ that is known to be bounded above and below on $[a, b]$. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. If t_1 and t_2 are real numbers satisfying $0 \leq t_1 < t_2$ then

$$h(t_2) - h(t_1) = \frac{t_2}{1+t_2} - \frac{t_1}{1+t_1} = \frac{t_2 - t_1}{(1+t_1)(1+t_2)} > 0,$$

and thus $h(t_1) < h(t_2)$. Thus the function h is strictly increasing on the set of non-negative real numbers. Moreover $h(0) = 0$ and $h(-t) = -h(t)$ for all real numbers t. It follows easily from this that the continuous function $h\colon \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$.

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on the closed bounded interval $[a,b]$, and let $g\colon [a,b] \to \mathbb{R}$ be the continuous real-valued function defined on $[a,b]$ such that

$$g(x) = h(f(x)) = \frac{f(x)}{1 + |f(x)|}$$

for all $x \in [a,b]$. Then $-1 \leq g(x) \leq 1$ for all $x \in [a,b]$. The set of values of the function g is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(x) : a \leq x \leq b\}.$$

Then, for each positive integer j, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function g, and therefore there exists some real number x_j satisfying $a \leq x_j \leq b$ for which $M - j^{-1} < g(x_j) \leq M$. The sequence $x_1, x_2, x_3, \ldots$ is then a bounded sequence of real numbers. It follows from the Bolzano-Weierstrass Theorem that this sequence has a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ which converges to some real number v, where $a \leq v \leq b$. Now

$$M - \frac{1}{k_j} < g(x_{k_j}) \leq M$$

for all positive integers j, and therefore $g(x_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 3.8 that

$$g(v) = g\left(\lim_{j\to+\infty} x_{k_j}\right) = \lim_{j\to+\infty} g(x_{k_j}) = M.$$

But $g(x) \leq M$ for all $x \in [a,b]$. It follows that $h(f(x)) = g(x) \leq g(v) = h(f(v))$ for all $x \in [a,b]$. Moreover $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(x) \leq f(v)$ for all $x \in [a,b]$.

On applying this result with the continuous function f replaced by the function $-f$, we conclude also that there exists some real number u satisfying $a \leq u \leq b$ such that $f(u) \leq f(x)$ for all $x \in [a,b]$. The result follows. ∎

3.5 Uniform Continuity

Definition A function $f\colon D \to \mathbb{R}$ is said to be *uniformly continuous* over a subset D of $\mathbb{R}$ if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in [a, b]$ satisfying $|u - v| < \delta$. (where δ does not depend on u or v).

A continuous function defined over a subset D of $\mathbb{R}$ is not necessarily uniformly continuous on D. (One can verify for example that the function sending a non-zero real number x to $1/x$ is not uniformly continuous on the set of all non-zero real numbers.) However we show that continuity does imply uniform continuity when $D = [a, b]$ for some real numbers a and b satisfying $a < b$.

Theorem 3.16 *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function on a closed bounded interval $[a, b]$. Then the function f is uniformly continuuous on $[a, b]$.*

Proof Let some strictly positive real number ε be given. Suppose that there did not exist any strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$. Then, for each positive integer j, there would exist values u_j and v_j in the interval $[a, b]$ such that $|u_j - v_j| < 1/j$ and $|f(u_j) - f(v_j)| \geq \varepsilon$. But the sequence $u_1, u_2, u_3, \ldots$ would be bounded (since $a \leq u_j \leq b$ for all j) and thus would possess a convergent subsequence $u_{k_1}, u_{k_2}, u_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.5). Let $l = \lim_{j \to +\infty} u_{k_j}$. Then $l = \lim_{j \to +\infty} v_{k_j}$ also, since $\lim_{j \to +\infty} (v_{k_j} - u_{k_j}) = 0$. Moreover $a \leq l \leq b$. It follows from the continuity of f that $\lim_{j \to +\infty} f(u_{k_j}) = \lim_{j \to +\infty} f(v_{k_j}) = f(l)$ (see Lemma 3.8). Thus $\lim_{j \to +\infty} \bigl(f(u_{k_j}) - f(v_{k_j})\bigr) = 0$. But this is impossible, since u_j and v_j have been chosen so that $|f(u_j) - f(v_j)| \geq \varepsilon$ for all positive integers j. We conclude therefore that there must exist some strictly positive real number δ with the required property. ∎