Course MA2321: Michaelmas Term 2015.

Assignment 2.

To be handed in by Friday 29th January, 2016.

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Please complete the attached cover sheet and attach it to your assignment, in particular signing the declaration with regard to plagiarism. Please make sure also that you include both name and student number on work handed in.

1. Let l, m and n be positive integers, let let D and E be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, let $\mathbf{p}$ and $\mathbf{q}$ be limit points of the sets D and E respectively, where in $\mathbf{p} \in \mathbb{R}^n$ and $\mathbf{q} \in \mathbb{R}^m$, let $\mathbf{r} \in \mathbb{R}^l$, let $\varphi\colon D \to \mathbb{R}^m$ be a function from D to $\mathbb{R}^m$ with the property that $\varphi(D) \subset E$, let $\psi\colon E \to \mathbb{R}^l$ be a function from E to $\mathbb{R}^l$. Suppose that the following three conditions are satisfied:

   (i) $\lim_{\mathbf{x}\to\mathbf{p}} \varphi(\mathbf{x}) = \mathbf{q}$;

   (ii) $\lim_{\mathbf{y}\to\mathbf{q}} \psi(\mathbf{y}) = \mathbf{r}$;

   (iii) there exists some positive real number δ_0 such that $\varphi(\mathbf{x}) \neq \mathbf{q}$ for all $\mathbf{x} \in D$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_0$.

   Prove that $\lim_{\mathbf{x}\to\mathbf{p}} \psi(\varphi(\mathbf{x})) = \mathbf{r}$.

   Comments.

   - Proofs that are concise but effective will be valued more highly than diffuse rambling attempts in assessing the quality of submitted work. It is possible to prove this result in a handful of well-drafted sentences that directly apply the appropriate definitions.
   - If in doubt as to how to proceed, you should review the definition of limits of functions of several real variables in subsection 6.4 of the module notes and the proofs of Lemma 6.5 and Lemma 6.6.

- Note that a valid proof will require all of conditions (i), (ii) and (iii) to be satisfied.
- After completing this question, it may be instructive to review question 1 on Assignment 1.

2. Let $\varphi\colon V \to \mathbb{R}^m$ be a function mapping some open subset V of $\mathbb{R}^n$ into $\mathbb{R}^m$, let $h\colon V \to \mathbb{R}$ be a real-valued function on V, and let $\mathbf{p}$ be a limit point of V. Suppose that $\lim_{\mathbf{x}\to\mathbf{p}} \varphi(\mathbf{x}) = 0$. Suppose also that there exist positive constants C and δ_0 such that $|h(\mathbf{x})| \leq C$ for all $\mathbf{x} \in V$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_0$. Using only the definition of limits (i.e., the "epsilon-delta" definition"), without citing to any other lemmas, propositions, theorems etc. concerning limits, prove that $\lim_{\mathbf{x}\to\mathbf{p}}(h(\mathbf{x})\varphi(\mathbf{x})) = 0$.

3. This question concerns real-valued functions $f\colon D \to \mathbb{R}$, where either $D = \mathbb{R}^n$ or else $D = \mathbb{R}^n \setminus \{\mathbf{0}\}$, and where the function satisfies an identity, determined by a real constant s, that requires that
$$f(t\mathbf{x}) = t^s f(\mathbf{x})$$
for all $\mathbf{x} \in D$ and for all positive real numbers t. Functions satisfying this condition as said to be *homogeneous of degree s*.

   We use the notation $\partial_j f$ to denote the partial derivative of f with respect to the jth coordinate on $\mathbb{R}^n$, so that $(\partial_j f)(\mathbf{p})$ denotes the value of partial derivative $\dfrac{\partial f}{\partial x_j}$ at the point $\mathbf{p}$ for $\mathbf{p} \in D$.

   Moreover, given $\mathbf{v} \in \mathbb{R}^n$, where $(v_1, v_2, \ldots v_n)$, the value $(Df)_\mathbf{x}\,\mathbf{v}$ resulting from the action of the (total) derivative $(Df)_\mathbf{x}$ on the vector $\mathbf{v}$ is expressed in terms of the partial derivatives of f according to the following formula:
$$(Df)_\mathbf{x}\,\mathbf{v} = \sum_{j=1}^{n} v_j (\partial_j f)(\mathbf{x}).$$

   Throughout this question, let $f\colon D \to \mathbb{R}$ be a differentiable real-valued function defined on D, where either $D = \mathbb{R}^n$ or $D = \mathbb{R}^n \setminus \{\mathbf{0}\}$ that satisfies the identity $f(t\mathbf{x}) = t^s f(\mathbf{x})$ for all $\mathbf{x} \in D$ and for all positive real numbers t.

   (a) Suppose that the function f is differentiable. For each positive real number t, let $\lambda_t\colon \mathbb{R}^n \to \mathbb{R}^n$ be the function defined such that $\lambda_t(\mathbf{x}) = t\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. The Chain Rule guarantees that
$$D(f \circ \lambda_t)_\mathbf{x} = (Df)_{\lambda_t(\mathbf{x})}\,(D\lambda_t)_\mathbf{x}.$$

The function $\lambda_t\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, and the derivative of any linear transformation is that linear transformation itself (see Lemma 7.4). Explain briefly why

$$(\partial_j f)(t\mathbf{x}) = t^{s-1}(\partial_j f)(\mathbf{x})$$

for $j = 1, 2, \ldots, n$, for all $\mathbf{x} \in D$ and for all positive real numbers t.

(b) Again suppose that the function f is differentiable. Let $P = \{t \in \mathbb{R} : t > 0\}$, and let $\mu_{\mathbf{x}}\colon P \to \mathbb{R}^n$ be defined such that $\mu_{\mathbf{x}}(t) = t\mathbf{x}$ for all $t \in P$ and $\mathbf{x} \in D$. The Chain Rule (Theorem 7.9) guarantees that

$$D(f \circ \mu_{\mathbf{x}})_t = (Df)_{\mu_{\mathbf{x}}(t)}\, (D\mu_{\mathbf{x}})_t,$$

where

$$D(f \circ \mu_{\mathbf{x}})_t = \frac{d}{dt}\left((f \circ \mu_{\mathbf{x}})(t)\right) = \frac{d}{dt}\left(f(t\mathbf{x})\right).$$

Use this result to prove that

$$\sum_{j=1}^{n} x_j (\partial_j f)(\mathbf{x}) = s f(\mathbf{x}),$$

or, in traditional notation,

$$\sum_{j=1}^{n} x_j \frac{\partial f}{\partial x_j} = s f.$$

Comments.

- The proofs for parts (a) and (b) of this question require little more than that you should unwrap to definitions of the functions, operators etc. involved in the statement of the Chain Rule, and do some obvious manipulations (e.g., cancel the quantity t^{s-1} from both sides of an equation).

4. Throughout this question, let s be a real number, and let $f\colon \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ be a continuous (but not necessarily differentiable) function that also satisfies the identity $f(t\mathbf{x}) = t^s f(\mathbf{x})$ for all $\mathbf{x} \in D$ and for all positive real numbers t, where s is a real constant. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined so that

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

The set S^{n-1} is a closed bounded subset of Euclidean space. The Extreme Value Theorem (Theorem 6.21) therefore guarantees the existence of points $\mathbf{u}$ and $\mathbf{v}$ on the unit sphere S^{n-1} such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in S^{n-1}$.

(a) Let $f\colon \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}$ satisfy the conditions stated above. In the case where $s > 0$ prove that $\lim_{\mathbf{x} \to \mathbf{0}} f(\mathbf{x}) = 0$.

Also, in the case where $s < 0$, prove that if the function f is non-zero at any point of $\mathbb{R}^n \setminus \{\mathbf{0}\}$ then a value $f(0,0)$ for f at the origin $(0,0)$ cannot be determined that will make the function continuous at $(0,0)$.

(b) The function f is C^k if all partial derivatives of the function of order less than or equal to k are defined and continuous throughout the domain. Prove that if a function f defined over the whole of $\mathbb{R}^n$ satisfies the conditions stated at the beginning of this question, and if it is C^k throughout $\mathbb{R}^n$ (and in particular at and around the origin) then either s is a non-negative integer or else $s \geq k$. (It follows from this that if f is smooth then s must necessarily be a non-negative integer.)

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function on $\mathbb{R}^n$ that is homogeneous of degree 1 (see Question 3). Suppose that f is differentiable at the origin $(0,0)$. Prove that f is a linear transformation.

Comments.

- The result can be obtained by a straightforward application of Lemma 7.7.
- In the special case where the function f is not only continuous but also differentiable, the result can also be deduced from a straightforward application of the results of Question 3.

6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a real-valued function on $\mathbb{R}^2$ that is homogeneous of degree two. Suppose that f is continuously differentiable on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, so that the first order partial derivatives exist at and are continuous around all points of $\mathbb{R}^2$ with the possible exception of the origin $(0,0)$. We denote the partial derivatives of f with respect to the Cartesian coordinates x and y by f_x and f_y respectively. We consider the possible existence and values of second order partial derivatives at the origin.

(a) Prove that $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(0,0)}$ exists if and only if $f(1,0) = f(-1,0)$ in which case

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 2f(1,0).$$

(b) Prove that $\left.\dfrac{\partial^2 f}{\partial x\,\partial y}\right|_{(0,0)}$ exists if and only if $f_y(1,0) = -f_y(-1,0)$ in which case

$$\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{(0,0)} = f_y(1,0).$$

Note that following results correspond to those of (a) and (b):—

- $\left.\dfrac{\partial^2 f}{\partial y^2}\right|_{(0,0)}$ exists if and only if $f(0,1) = f(0,-1)$ in which case

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2f(0,1).$$

- $\left.\dfrac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)}$ exists if and only if $f_x(0,1) = -f_y(0,-1)$ in which case

$$\left.\frac{\partial^2 f}{\partial y\,\partial x}\right|_{(0,0)} = f_x(0,1).$$

(c) Prove that if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a function that is twice continuously differentiable throughout $\mathbf{R}^2$ (including at $(0,0)$), and if the function f is homogeneous of degree two, then there exist constants E, F and G such that

$$f(x,y) = Ex^2 + 2Fxy + Gy^2.$$

Comments.

- The results of Question 3 are particularly relevant, and it is recommended that you make use of them in answering this question.

Module MA2321—Analysis in Several Real Variables. Assignment 2

Name (please print): ....

Student number: ....

Date submitted: ....

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`http://tcd-ie.libguides.com/plagiarism/ready-steady-write`

Signed:

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