Course MA2321: Michaelmas Term 2015.

Worked Solutions to Assignment 1.

1. *In answering this question, you should pay heed to the following definitions.*

   > *Let D be a subset of the set $\mathbb{R}$ of real numbers, and let $f\colon D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be* continuous *at s if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$.*

   > *Let D be a subset of the set $\mathbb{R}$ of real numbers, let $g\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. The real number l is said to be the* limit *of $g(x)$, as x tends to s in D, if and only if the following criterion is satisfied: given any strictly positive real number ε, there exists some strictly positive real number δ such that $|g(x) - l| < \varepsilon$ whenever $x \in D$ satisfies $0 < |x - s| < \delta$.*

   (a) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined such that*

   $$f(x) = \begin{cases} x^3 \cos \dfrac{1}{x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

   Using the formal definition of continuity (in terms of ε and δ etc.) prove that the function f is continuous at 0. What is the value of $\lim\limits_{x\to 0} f(x)$?

   Solution. Let some positive real number $\varepsilon > 0$ be given. Let $\delta = \sqrt[3]{\varepsilon}$. Now

   $$\left|\cos \frac{1}{x^2}\right| \leq 1$$

   for all non-zero real numbers x. It follows that if the real number x satisfies $0 < |x| < \delta$ then

   $$\left|x^3 \cos \frac{1}{x^2}\right| = |x^3| \left|\cos \frac{1}{x^2}\right| \leq |x^3| < \delta^3 = \varepsilon.$$

Also $f(0) = 0$. Thus $|f(x) - f(0)| < \varepsilon$ whenever $|x| < \delta$. It follows from the definition of continuity that the function f is continuous at 0. Moroever it follows from the continuity of f at 0 (or alternatively directly from the definition of limits and the inequalities proved above) that $\lim_{x \to 0} f(x) = f(0) = 0$.

Alternative. Instead of taking $\delta = \sqrt[3]{\varepsilon}$ one could choose $\delta = \min(1, \varepsilon)$, and indeed one could choose any value of δ small enough to ensure that $0 < \delta^3 \leq \varepsilon$.

(b) *Let* $g: \mathbb{R} \to \mathbb{R}$ *be defined such that*

$$g(y) = \begin{cases} 0 & \textit{if } y \neq 0; \\ 1 & \textit{if } y = 0. \end{cases}$$

Explain why the limit $\lim_{y \to 0} g(y)$ *exists. What is the value of* $\lim_{y \to 0} g(y)$*?*

Solution. We apply the definition of limit. Let $\varepsilon > 0$ be given. Whatever positive value of δ is chosen, the function g satisfies $|g(y)| < \epsilon$ whenever $0 < |y| < \delta$. It follows that $\lim_{y \to 0} g(y) = 0$.

(c) *Let* $f: \mathbb{R} \to \mathbb{R}$ *and* $g: \mathbb{R} \to \mathbb{R}$ *be the functions defined in parts (a) and (b) of this question. Determine whether or not it is the case that*

$$\lim_{x \to 0} g(f(x)) = l, \quad \textit{where } l = \lim_{y \to 0} g(y).$$

Solution. It is not the case that $\lim_{x \to 0} g(f(x)) = l$, where $l = \lim_{y \to 0} g(y)$. Indeed the function $g(f(x))$ does not tend to any limit whatsoever as $x \to 0$.

Note that $l = 0$, by part (b) of the question. Let

$$x_n = \frac{1}{\sqrt{(n + \frac{1}{2})\pi}}$$

for all positive integers n. Then $x_n \to 0$ as $n \to +\infty$. Now, for any positive integer n, $1/x_n^2 = (n + \frac{1}{2})\pi$, and $\cos(n + \frac{1}{2})\pi = 0$, and therefore $f(x_n) = 0$ for all positive integers n. It follows that $g(f(x_n)) = 1$ for all positive integers n.

Let ε be chosen satisfying $0 < \varepsilon \leq 1$ (e.g., $\varepsilon = \frac{1}{2}$). Given any positive real number δ, we can chosen the positive integer n large enough to ensure that $0 < |x_n| < \delta$. Then $g(f(x_n)) = 1$ and therefore $g(f(x_n)) \not< \varepsilon$. Therefore $\lim_{x \to 0} g(f(x)) \neq 0$.

Note. Given any positive real number δ, there exist values of x satisfying $0 < |x| < \delta$ for which $g(f(x)) = 0$ and other values for which $g(f(x)) = 1$. Therefore $g(f(x))$ cannot possibly tend to any limiting value as $x \to 0$.

2. *In this question, we employ partial derivatives, in the context of a real valued function* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *of two real variables. We also make use of the concept of the limit of such a function at a point of the plane* $\mathbb{R}^2$*. Here are the definitions of the partial derivatives of the function* f*:—*

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \\ \frac{\partial f(x,y)}{\partial y} &= \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}. \end{aligned}$$

Let D *be a subset of* $\mathbb{R}^2$*. A point* (u,v) *of* $\mathbb{R}^2$ *is said to be a* limit point *of* D *if, given any strictly positive real number* δ*, there exist points* (x,y) *of* D *that are distinct from* (u,v) *but lie within a distance* δ *of* (u,v)*. The definition of the limit of a function* f *of two variables may be formally stated as follows:*

> *Let* D *be a subset of* $\mathbb{R}^2$*, let* (u,v) *be a limit point of the set* D*, let* $f\colon D \to \mathbb{R}$ *be a real-valued function on* D*, and let* l *be a real number. Then* l *is said to be the* limit *of* $f(x,y)$*, as* (x,y) *tends to* (u,v) *in* D *if, given any strictly positive real number* ε*, there exists some strictly positive real number* δ *such that*
>
> $$|f(x,y) - f(u,v)| < \varepsilon$$
>
> *for all* $(x,y) \in D$ *satisfying*
>
> $$0 < \sqrt{(x-u)^2 + (y-v)^2} < \delta$$

Moreover a function f *of two real variables is continuous at a point* (u,v) *in the interior of its domain if and only if*

$$\lim_{(x,y) \to (u,v)} f(x,y) = f(u,v).$$

Throughout the remainder of this question, let

$$f(x,y) = \begin{cases} \dfrac{2x^2y^3}{x^4 + y^6} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Note that a certain amount of information about this function could be obtained using various software packages, or else by typing the following query

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z = (2 x^2 y^3) / (x^4 + y^6)
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into the search bar on the following website:

http://www.wolframalpha.com/

In particular, the above website will inform you that

$$\frac{\partial f(x,y)}{\partial x} = \frac{-4x^5y^3 + 4xy^9}{(x^4+y^6)^2}, \qquad \frac{\partial f(x,y)}{\partial y} = \frac{6x^2y^2(x^4-y^6)}{(x^4+y^6)^2}$$

when $(x,y) \neq (0,0)$.

Given any real numbers b *and* c, *we define* $g_{(b,c)}\colon \mathbb{R} \to \mathbb{R}$ *to be the function from the set* $\mathbb{R}$ *of real numbers to itself defined such that*

$$g_{(b,c)}(t) = f(tb, tc)$$

for all real numbers t, *where the function* f *is as defined above.*

(a) *Prove that* $-1 \leq f(x,y) \leq 1$ *for all real numbers* x *and* y. *Prove also that* $f(x,y) = 1$ *if and only if* $(x,y) \neq (0,0)$ *and* $x^2 = y^3$, *and also that* $f(x,y) = -1$ *if and only if* $(x,y) \neq (0,0)$ *and* $x^2 = -y^3$.

Let a and b be real numbers. Then $(a-b)^2 = a^2+b^2-2ab$, and therefore $2ab \leq a^2 + b^2$. Moreover $2ab = a^2 + b^2$ if and only if $a = b$. Applying this result with $a = x^2$ and $b = y^3$, we see that $2x^2y^3 \leq x^4 + y^6$, and moreover $2x^2y^3 = x^4 + y^6$ if and only if $x^2 = y^3$. Similarly if we apply the result with $a = x^2$ and $b = -y^3$ we find that $-2x^2y^3 \leq x^4 + y^6$, and moreover $-2x^2y^3 = x^4 + y^6$ if and only if $x^2 = -y^3$. It follows that $-1 \leq f(x,y) \leq 1$ for all $(x,y) \in \mathbb{R}^2$, and moreover $f(x,y) = 1$ if and only if $x^2 = y^3$ and $f(x,y) = -1$ if and only if $x^2 = -y^3$.

(b) *For each ordered pair* (b,c) *of real numbers, show that the associated function* $g_{(b,c)}$ *is differentiable, and determine the value of the derivative*

$$\frac{dg_{(b,c)}(t)}{dt}$$

for all values of the real variable t, *including* $t = 0$.

If $b = 0$ then $g_{(b,c)}(t) = 0$ for all real numbers t, and therefore the derivative $g'_{(b,c)}$ of $g_{(b,c)}$ satisfies $g'_{(b,c)}(t) = 0$ for all real numbers t.
If $b \neq 0$ and $t \neq 0$ then

$$g_{b,c}(t) = \frac{b^2 c^3 t^5}{b^4 t^4 + c^6 t^6} = \frac{b^2 c^3 t}{b^4 + c^6 t^2}.$$

It follows that if $b \neq 0$ then

$$g_{b,c}(t) = \frac{b^2 c^3 t}{b^4 + c^6 t^2},$$

for all real numbers t, and, applying the quotient rule, we find that

$$\begin{aligned} \frac{dg_{b,c}(t)}{dt} &= \frac{b^2 c^3 (b^4 + c^6 t^2) - 2b^2 c^9 t^2}{(b^4 + c^6 t^2)^2} \\ &= \frac{b^2 c^3 (b^4 - c^6 t^2)}{(b^4 + c^6 t^2)^2}. \end{aligned}$$

Note. Once $g_{(b,c)}$ has been expressed as a ratio of differentiable functions, where the denominator is non-zero for all real numbers t, the Quotient Rule can be applied. Indeed the use of the Quotient Rule in such situations has been shown to be valid. There is no need or reason in such a situation to differentiate the function from first principles, unless the questions specifically asks for this.

(c) *Show that the partial derivatives*

$$\frac{\partial f(x,y)}{\partial x}, \quad \frac{\partial f(x,y)}{\partial y}$$

are defined when $(x, y) = (0, 0)$, and determine the value of these partial derivatives at $(x, y) = (0, 0)$.

The function f satisfies $f(x, 0) = 0$ for all real numbers x, and therefore

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = \left.\frac{df(x,0)}{dx}\right|_{x=0} = 0.$$

Similarly the function f satisfies $f(0, y) = 0$ for all real numbers y, and therefore

$$\left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = \left.\frac{df(0,y)}{dy}\right|_{y=0} = 0.$$

(d) *Given any real number u, what are the supremum (i.e., the least upper bound) and infimum (i.e., the greatest lower bound) on the values of $f(x, y)$ on the line $x = u$?*

If $u = 0$ then $f(u, y) = 0$ for all real numbers y, and therefore the supremum and infimum on the values of f in this line are both equal to zero.
If $u \neq 0$ then $f(u, \sqrt[3]{u^2}) = 1$ and $f(u, \sqrt[3]{u^2}) = -1$. Moreover it follows from (a) that $-1 \leq f(u, y) \leq 1$ for all real numbers y. Therefore the supremum is 1 and the infimum is -1.

(e) *Given any real number v, what are the supremum and infimum values of the $f(x, y)$ on the line $y = v$?*

If $v = 0$ then $f(x, v) = 0$ for all real numbers x, and therefore the supremum and infimum on the values of f in this line are both equal to zero.
If $v > 0$ then $f(\sqrt{v^3}, v) = 1$. Moreover $f(x, v) \geq 0$ for all real numbers x, and it follows from (a) that $f(x, v) \leq 1$ for all real numbers x. Therefore the supremum is 1 and the infimum is 0.
If $v < 0$ then $f(\sqrt{-v^3}, v) = -1$. Moreover $f(x, v) \leq 0$ for all real numbers x, and it follows from (a) that $f(x, v) \geq -1$ for all real numbers x. Therefore the supremum is 0 and the infimum is -1.

(f) *Is the function f continuous at $(0, 0)$? [Justify your answer rigorously using an ε–δ definition of either limits of functions of two real variables or else of continuity for functions of two real variables.]*

The function f is not continuous at $(0, 0)$. Take, for example, $\varepsilon = \frac{1}{2}$. Then, given any $\delta < 1$ we can find a positive real number y small enough to ensure that $0 < y < 1$ and $y < \frac{1}{2}\delta$. Let $x = \sqrt{y^3}$. Then $0 < x < y < \frac{1}{2}\delta$, and therefore $0 < x^2 + y^2 < \delta^2$. But $f(x, y) = 1$, and therefore $f(x, y) \not< \varepsilon$. Thus the "epsilon-delta" definition of continuity is not satisfied by the function f at zero.

Remarks.

- Let

$$
\begin{aligned}
C_{++} &= \{(x, y) : x > 0,\ \ y > 0 \text{ and } x^2 = y^3\}, \\
C_{-+} &= \{(x, y) : x < 0,\ \ y > 0 \text{ and } x^2 = y^3\}, \\
C_{+-} &= \{(x, y) : x > 0,\ \ y < 0 \text{ and } x^2 = y^3\}, \\
C_{--} &= \{(x, y) : x < 0,\ \ y < 0 \text{ and } x^2 = y^3\},
\end{aligned}
$$

Then C_{++} and C_{-+} are curves in the plane on which the function achieves its maximum value of 1. Similarly C_{+-} and C_{--} are curves in the plane on which the function achieves its minimum value of -1. Each of these curves approaches the origin so as to come within any specified distance from the origin, no matter how small. It should be clear from this that the function f could not possibly be continuous at $(0, 0)$. Parts (a), (d) and (e) of the question were intended to drive home to those attempting the question these particular features of the function: they were *not* intended as exercises involving computation of partial derivatives and application of the complicated machinery of multivariate calculus! But on the other hand the function f is smooth along any line in $\mathbb{R}^2$ and the partial derivatives of this function with respect to the coordinate functions x and y are defined throughout $\mathbf{R}^2$. This is established through parts (b) and (c) of the question, taking it for granted that the function f is well-behaved away from the origin.

- In tackling an open-ended question like 2(f) (or 1(c)) one should analyse the behaviour of the function, decide whether one can show that it is continuous or discontinuous, and decide on a strategy that provides the required formal proof. It is not a good idea to embark on pages of calculations with epsilons and deltas without an idea of whether one is heading for the correct destination, and, if so, how to arrive at that destination. In such situations, one would probably eventually make a mistake in, for example, manipulating inequalities, which would make nonsense of the remainder of the attempt. In answering an open-ended question such as 2(f), one should ponder the question of whether or not the function is continuous as $(0, 0)$. If the function is thought to be continuous then, given any positive epsilon, how would one go about finding a valid value for the corresponding delta? If the function is thought not to be continuous, then how could one choose an epsilon small enough to ensure that no corresponding delta could be found? Are there "obvious" sequences tending to the limit point $(0, 0)$ where the values of the function along the sequence clearly do not converge to the value at $(0, 0)$? Such considerations can be worked through with mental thought and rough working, and a strategy for answering the question can be devised before setting out to write out the final solution.

- In a situation such as a homework assignment where one has computer access or web access, then the computer can prove to be a useful tool, and not only checking that calculus computations have been carried through correctly. For example, in the case of the function of question 2, entering the formula for the function (in the form "$z = \ldots$") into the Wolfram Alpha should result in a page of information about the function, including an attempt at a 3D plot of its graph and a contour diagram. Of course, in cases where the function is, or may be, discontinuous, such plots can only be approximations, or will not necessarily tell the full story. But nevertheless one can use the website to explore how the function behaves around the origin, and, ultimately, one can hopefully explain the features apparent in the presented 3D plot or contour diagram in terms of what you have ultimately learned about the function. And the website can provide both reality check and a tool for checking computations. After all, would it be advisable for anyone to submit a homework assignment asking them to find the solution of, for example, a simple ordinary differential equation without typing the differential equation into the search bar of a website like Wolfram Alpha, or using symbolical computation software, to check that they have arrived at the correct answer?