Module MA2321: Analysis in Several Real Variables
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1 Ordered Fields and the Real Number System

1.1 Sets

A *set* is a collection of objects. These objects are referred to as the *elements* of the set. One can specify a set by enclosing a list of suitable objects within braces. Thus, for example, $\{1, 2, 3, 7\}$ denotes the set whose elements are the numbers 1, 2, 3 and 7. If x is an element of some set X then we denote this fact by writing $x \in X$. Conversely, if x is not an element of the set X then we write $x \notin X$. We denote by $\emptyset$ the *empty set*, which is defined to be the set with no elements.

We denote by $\mathbb{N}$ the set $\{1, 2, 3, 4, 5 \ldots\}$ of all *positive integers* (also known as *natural numbers*), and we denote by $\mathbb{Z}$ the set

$$\{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$$

of all *integers* (or 'whole numbers'). We denote by $\mathbb{Q}$ the set of *rational numbers* (i.e., numbers of the form p/q where p and q are integers and $q \neq 0$), and we denote be $\mathbb{R}$ and $\mathbb{C}$ the sets of real numbers and complex numbers respectively.

If X and Y are sets then the *union* $X \cup Y$ of X and Y is defined to be the set of all elements that belong either to X or to Y (or to both), the *intersection* $X \cap Y$ of X and Y is defined to be the set of all elements that belong to both X and Y, and the *difference* $X \setminus Y$ of X and Y is defined to be the set of all elements that belong to X but do not belong to Y. Thus, for example, if

$$X = \{2, 4, 6, 8\}, \qquad Y = \{3, 4, 5, 6, 7\}$$

then

$$X \cup Y = \{2, 3, 4, 5, 6, 7, 8\}, \qquad X \cap Y = \{4, 6\},$$
$$X \setminus Y = \{2, 8\}, \qquad Y \setminus X = \{3, 5, 7\}.$$

If X and Y are sets, and if every element of X is also an element of Y then we say that X is a *subset* of Y, and we write $X \subset Y$. We use the notation $\{y \in Y : P(y)\}$ to denote the subset of a given set Y consisting of all elements y of Y with some given property $P(y)$. Thus for example $\{n \in \mathbb{Z} : n > 0\}$ denotes the set of all integers n satisfying $n > 0$ (i.e., the set $\mathbb{N}$ of all positive integers).

1.2 Rational and Irrational Numbers

Rational numbers are numbers that can be expressed as fractions of the form p/q, where p and q are integers (i.e., 'whole numbers') and $q \neq 0$. The set of rational numbers is denoted by $\mathbb{Q}$. Operations of addition, subtraction, multiplication and division are defined on $\mathbb{Q}$ in the usual manner. In addition the set of rational numbers is ordered.

There are however certain familiar numbers which cannot be represented in the form p/q, where p and q are integers. These include $\sqrt{2}$, $\sqrt{3}$, π and e. Such numbers are referred to as *irrational numbers*. The irrationality of $\sqrt{2}$ is an immediate consequence of the following famous result, which was discovered by the Ancient Greeks.

Proposition 1.1 *There do not exist non-zero integers p and q with the property that $p^2 = 2q^2$.*

Proof Let us suppose that there exist non-zero integers p and q with the property that $p^2 = 2q^2$. We show that this leads to a contradiction. Without loss of generality we may assume that p and q are not both even (since if both p and q were even then we could replace p and q by $p/2^k$ and $q/2^k$ respectively, where k is the largest positive integer with the property that 2^k divides both p and q). Now $p^2 = 2q^2$, hence p^2 is even. It follows from this that p is even (since the square of an odd integer is odd). Therefore $p = 2r$ for some integer r. But then $2q^2 = 4r^2$, so that $q^2 = 2r^2$. Therefore q^2 is even, and hence q is even. We have thus shown that both p and q are even. But this contradicts our assumption that p and q are not both even. This contradiction shows that there cannot exist integers p and q with the property that $p^2 = 2q^2$, and thus proves that $\sqrt{2}$ is an irrational number. ∎

This result shows that the rational numbers are not sufficient for the purpose of representing lengths arising in familiar Euclidean geometry. Indeed consider the right-angled isosceles triangle whose short sides are q units long. Then the hypotenuse is $\sqrt{2}q$ units long, by Pythagoras' Theorem. Proposition 1.1 shows that it is not possible to find a unit of length for which the two short sides of this right-angled isosceles triangle are q units long and the hypotenuse is p units long, where both p and q are integers. We must therefore enlarge the system of rational numbers to obtain a number system which contains irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, π and e, and which is capable of representing the lengths of line segments and similar quantities arising in geometry and physics. The rational and irrational numbers belonging to this number system are known as *real numbers*.

1.3 Ordered Fields

An *ordered field* $\mathbb{F}$ consists of a set $\mathbb{F}$ on which are defined binary operations $+$ of addition and $\times$ of multiplication, together with an ordering relation $<$, where these binary operations and ordering relation satisfy the following axioms:—

1. if u and v are elements of $\mathbb{F}$ then their sum $u+v$ is also a element of $\mathbb{F}$;
2. (*the Commutative Law for addition*) $u + v = v + u$ for all elements u and v of $\mathbb{F}$;
3. (*the Associative Law for addition*) $(u + v) + w = u + (v + w)$ for all elements u, v and w of $\mathbb{F}$;
4. there exists an element of $\mathbb{F}$, denoted by 0, with the property that $u + 0 = x = 0 + u$ for all elements u of $\mathbb{F}$;
5. for each element u of $\mathbb{F}$ there exists some element $-u$ of $\mathbb{F}$ with the property that $u + (-u) = 0 = (-u) + u$;
6. if u and v are elements of $\mathbb{F}$ then their product $u \times v$ is also a element of $\mathbb{F}$;
7. (*the Commutative Law for multiplication*) $u \times v = v \times u$ for all elements u and v of $\mathbb{F}$;
8. (*the Associative Law for multiplication*) $(u \times v) \times w = u \times (v \times w)$ for all elements u, v and w of $\mathbb{F}$,
9. there exists an element of $\mathbb{F}$, denoted by 1, with the property that $u \times 1 = u = 1 \times u$ for all elements u of $\mathbb{F}$, and moreover $1 \neq 0$,
10. for each element u of $\mathbb{F}$ satisfying $u \neq 0$ there exists some element u^{-1} of $\mathbb{F}$ with the property that $u \times u^{-1} = 1 = u^{-1} \times u$,
11. (*the Distributive Law*) $u \times (v + w) = (u \times v) + (u \times w)$ for all elements u, v and w of $\mathbb{F}$,
12. (*the Trichotomy Law*) if u and v are elements of $\mathbb{F}$ then one and only one of the three statements $u < v$, $u = v$ and $u < v$ is true,
13. (*transitivity of the ordering*) if u, v and w are elements of $\mathbb{F}$ and if $u < v$ and $v < w$ then $u < w$,
14. if u, v and w are elements of $\mathbb{F}$ and if $u < v$ then $u + w < v + w$,

15. if u and v are elements of $\mathbb{F}$ which satisfy $0 < u$ and $0 < v$ then $0 < u \times v$,

The operations of subtraction and division are defined on an ordered field $\mathbb{F}$ in terms of the operations of addition and multiplication on that field in the obvious fashion: $u - v = u + (-v)$ for all elements u and v of $\mathbb{F}$, and moreover $u/v = uv^{-1}$ provided that $v \neq 0$.

Example The rational numbers, with the standard ordering, and the standard operations of addition, subtraction, multiplication, and division constitute an ordered field.

Example Let $\mathbb{Q}(\sqrt{2})$ denote the set of all numbers that can be represented in the form $b+c\sqrt{2}$, where b and c are rational numbers. The sum and difference of any two numbers belonging to $\mathbb{Q}(\sqrt{2})$ themselves belong to $\mathbb{Q}(\sqrt{2})$. Also the product of any two numbers $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$ because, for any rational numbers b, c, e and f,

$$(b + c\sqrt{2})(e + f\sqrt{2}) = (be + 2cf) + (bf + ce)\sqrt{2},$$

and both $be + 2cf$ and $bf + ce$ are rational numbers. The reciprocal of any non-zero element of $\mathbb{Q}(\sqrt{2})$ itself belongs to $\mathbb{Q}(\sqrt{2})$, because

$$\frac{1}{b + c\sqrt{2}} = \frac{b - c\sqrt{2}}{b^2 - 2c^2}.$$

for all rational numbers b and c. It is then a straightforward exercise to verify that $\mathbb{Q}(\sqrt{2})$ is an ordered field.

The *absolute value* $|x|$ of an element number x of an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

Note that $|x| \geq 0$ for all x and that $|x| = 0$ if and only if $x = 0$. Also $|x + y| \leq |x| + |y|$ and $|xy| = |x||y|$ for all elements x and y of the ordered field $\mathbb{F}$.

Let D be a subset of an ordered field $\mathbb{F}$. An element u of $\mathbb{F}$ is said to be an *upper bound* of the set D if $x \leq u$ for all $x \in D$. The set D is said to be *bounded above* if such an upper bound exists.

Definition Let $\mathbb{F}$ be an ordered field, and let D be some subset of $\mathbb{F}$ which is bounded above. An element s of $\mathbb{F}$ is said to be the *least upper bound* (or *supremum*) of D (denoted by $\sup D$) if s is an upper bound of D and $s \leq u$ for all upper bounds u of D.

Example The rational number 2 is the least upper bound, in the ordered field of rational numbers, of the sets $\{x \in \mathbb{Q} : x \leq 2\}$ and $\{x \in \mathbb{Q} : x < 2\}$. Note that the first of these sets contains its least upper bound, whereas the second set does not.

The axioms (1)–(15) listed above that characterize ordered fields are not in themselves sufficient to fully characterize the real number system. (Indeed any property of real numbers that could be derived solely from these axioms would be equally valid in any ordered field whatsoever, and in particular would be valid were the system of real numbers replaced by the system of rational numbers.) We require in addition the following axiom:—

> *the Least Upper Bound Axiom*: given any non-empty set D of real numbers that is bounded above, there exists a real number $\sup D$ that is the least upper bound for the set D.

A *lower bound* of a set D of real numbers is a real number l with the property that $l \leq x$ for all $x \in D$. A set D of real numbers is said to be *bounded below* if such a lower bound exists. If D is bounded below, then there exists a greatest lower bound (or *infimum*) $\inf D$ of the set D. Indeed $\inf D = -\sup\{x \in \mathbb{R} : -x \in D\}$.

Remark We have simply listed above a complete set of axioms for the real number system. We have not however proved the existence of a system of real numbers satisfying these axioms. There are in fact several constructions of the real number system: one of the most popular of these is the representation of real numbers as *Dedekind sections* of the set of rational numbers. For an account of the this construction, and for a proof that these axioms are sufficient to characterize the real number system, see chapters 27–29 of *Calculus*, by M. Spivak. The construction of the real number system using Dedekind cuts is also described in detail in the Appendix to Chapter 1 of *Principles of Real Analysis* by W. Rudin.

1.4 Remarks on the Existence of Least Upper Bounds

We present an argument here that is intended to show that if the system of real numbers has all the properties that one would expect it to possess, then it must satisfy the Least Upper Bound Axiom.

Let $\mathbb{F}$ be an ordered field that contains the field $\mathbb{Q}$ of rational numbers. The set $\mathbb{Z}$ is a subset of $\mathbb{Q}$. Thus $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Q} \subset \mathbb{F}$, and therefore $\mathbb{Z} \subset \mathbb{F}$.

Definition Let $\mathbb{F}$ be an ordered field that contains the field of rational numbers. The field $\mathbb{F}$ is said to satisfy the *Axiom of Archimedes* if, given any element x of $\mathbb{F}$, there exists some integer n satisfying $n \geq x$.

The Axiom of Archimedes excludes the possibility of “infinitely large” elements of the ordered field $\mathbb{F}$. Given that all real numbers should be representable in decimal arithmetic, any real number must be less than some positive integer. Thus we expect the system of real numbers to satisfy the Axiom of Archimedes.

Lemma 1.2 *Let $\mathbb{F}$ be an ordered field that satisfies the Axiom of Archimedes. Then, given any element x of $\mathbb{F}$ satisfying $x > 0$, there exists some positive integer n such that $x > \dfrac{1}{n} > 0$.*

Proof The Axiom of Archimedes ensures the existence of a positive integer n satisfying $n > \dfrac{1}{x}$. Then

$$n - \frac{1}{x} > 0 \quad \text{and} \quad \frac{x}{n} = x \times \frac{1}{n} > 0,$$

and therefore

$$x - \frac{1}{n} = \left(n - \frac{1}{x}\right) \times \frac{x}{n} > 0,$$

and thus $x > \dfrac{1}{n}$, as required. ∎

Now let $\mathbb{F}$ be an ordered field containing as a subfield the field $\mathbb{Q}$ of rational numbers. We suppose also that $\mathbb{F}$ satisfies the Axiom of Archimedes. Let D be a subset of $\mathbb{F}$ which is bounded above. The Axiom of Archimedes then ensures that there exists some integer that is an upper bound for the set D. It follows from this that there exists some integer m that is the largest integer that is *not* an upper bound for the set D. Then m is not an upper bound for D, but $m + 1$ is. Let

$$E = \{x \in \mathbb{F} : x \geq 0 \quad \text{and} \quad m + x \in D\}.$$

Then E is non-empty and $x \leq 1$ for all $x \in E$. Suppose that there exists a least upper bound $\sup E$ in $\mathbb{F}$ for the set E. Then $m + \sup E$ is a least upper bound for the set D, and thus $\sup D$ exists, and $\sup D = m + \sup E$. Thus, in order to show that every non-empty subset of D that is bounded above has a least upper bound, it suffices to show this for subsets D of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$.

Now let $\mathbb{F}$ be an ordered field containing the field $\mathbb{Q}$ of rational numbers that satisfies the Axiom of Archimedes, and let D be a subset of $\mathbb{F}$ with the property that $0 \leq x \leq 1$ for all $x \in D$. Then, for each positive integer m, let

u_m denote the largest non-negative integer for which $u_m \times (10)^{-m}$ is *not* an upper bound for the set D. Then $0 \leq u_m < (10)^m$ and $(u_m + 1)(10)^{-m}$ is an upper bound for the set D. Thus if there were to exist a least upper bound s for the set D, then s would have to satisfy

$$\frac{u_m}{(10)^m} < s \leq \frac{u_m}{(10)^m} + \frac{1}{(10)^m}$$

for $m = 1, 2, 3, \ldots$. Now if $m > 1$ then definitions of u_m and u_{m-1} ensure that $(10u_{m-1}) \times (10)^{-m}$ is not an upper bound for the set D but $(10u_{m-1} + 10) \times (10)^{-m}$ is an upper bound for the set D. It follows that

$$10u_{m-1} \leq u_m < 10u_{m-1} + 10.$$

Let $d_1 = u_1$, and let $d_m = u_m - 10u_{m-1}$ for all integers m satisfying $m > 1$. Then d_m is an integer satisfying $0 \leq d_m < 10$ for $m = 1, 2, 3, \ldots$, and

$$\frac{u_m}{(10)^m} = \frac{d_m}{(10)^m} + \frac{u_{m-1}}{(10)^{m-1}}.$$

It follows that

$$\frac{u_m}{(10)^m} = \sum_{k=1}^{m} \frac{d_k}{(10)^k}.$$

Any least upper bound t for the set D would therefore have to satisfy the inequalities

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} < t \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}$$

for all positive integers m.

Now suppose that every well-formed decimal expansion determines a corresponding element of the ordered field $\mathbb{F}$. Assuming this, we conclude that there must exist some element s of the ordered field $\mathbb{F}$ whose decimal expansion takes the form

$$0.d_1\, d_2\, d_3\, d_4\, d_5, \ldots$$

The basic properties of decimal expansions then ensure that

$$\sum_{k=1}^{m} \frac{d_k}{(10)^k} \leq s \leq \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m}.$$

Let ε be an element of $\mathbb{F}$ satisfying $\varepsilon > 0$. Then, because the ordered field $\mathbb{F}$ is required to satisfy the Axiom of Archimedes, a positive integer m can be chosen large enough to ensure that $0 < (10)^{-m} < \varepsilon$. Then

$$s - \varepsilon < \sum_{k=1}^{m} \frac{d_k}{(10)^k} = \frac{u_m}{(10)^m},$$

and therefore $s - \varepsilon$ cannot be an upper bound for the set D. Also

$$s + \varepsilon > \sum_{k=1}^{m} \frac{d_k}{(10)^k} + \frac{1}{(10)^m} = \frac{u_m}{(10)^m} + \frac{1}{(10)^m},$$

and therefore $s + \varepsilon$ is an upper bound for the set D. We see therefore if s is an element of $\mathbb{F}$ satisfying $0 \leq s \leq$, and if s is determined by the decimal expansion whose successive decimal digits are $d_1, d_2, d_3, \ldots$, where these digits are determined by D as described above, then $s-\varepsilon$ cannot be an upper bound for the set D for any $\varepsilon > 0$, but $s + \varepsilon$ must be an upper bound for the set D for all $\varepsilon > 0$.

Now if there were to exist any element x of D satisfying $x > s$, then we could obtain a contradiction on choosing $\varepsilon \in \mathbb{F}$ such that $0 < \varepsilon < x - s$. It follows that $x \leq s$ for all $x \in D$, and thus s is an upper bound for the set D. But if $\varepsilon > 0$ then $s - \varepsilon$ is not an upper bound for the set D. Therefore s must be the least upper bound for the set D.

This analysis shows that if $\mathbb{F}$ is an ordered field, containing the field of rational numbers, that satisfies the Axiom of Archimedes, and if every decimal expansion determines a corresponding element of $\mathbb{F}$ then every non-empty subset of $\mathbb{F}$ that is bounded above must have a least upper bound. The ordered field $\mathbb{F}$ must therefore satisfy the Least Upper Bound Axiom.

This justifies the characterization of the field $\mathbb{R}$ of real numbers as an ordered field that satisfies the Least Upper Bound Axiom.

1.5 Intervals

Given real numbers a and b satisfying $a \leq b$, we define

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

If $a < b$ then we define

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}, \qquad [a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}.$$

For each real number c, we also define

$$[c, +\infty) = \{x \in \mathbb{R} : c \leq x\}, \qquad (c, +\infty) = \{x \in \mathbb{R} : c < x\},$$

$$(-\infty, c] = \{x \in \mathbb{R} : x \leq c\}, \qquad (-\infty, c) = \{x \in \mathbb{R} : x < c\}.$$

All these subsets of $\mathbb{R}$ are referred to as *intervals*. An *interval* I may be defined as a non-empty set of real numbers with the following property: if s,

t and u are real numbers satisfying $s < t < u$ and if s and u both belong to the interval I then t also belongs to the interval I. Using the Least Upper Bound Axiom, one can prove that every interval in $\mathbb{R}$ is either one of the intervals defined above, or else is the whole of $\mathbb{R}$.

1.6 The Real Number System

From the time of the ancient Greeks to the present day, mathematicians have recognized the necessity of establishing rigorous foundations for the discipline. This led mathematicians such as Bolzano, Cauchy and Weierstrass to establish in the nineteenth century the definitions of continuity, limits and convergence that are required in order to establish a secure foundation upon which to build theories of real and complex analysis that underpin the application of standard techiques of the differential calculus in one or more variables.

But mathematicians in the nineteenth century realised that, in order to obtain satisfactory proofs of basic theorems underlying the applications of calculus, they needed a deeper understanding of the nature of the real number system. Accordingly Dedekind developed a theory in which real numbers were represented by *Dedekind sections*, in which each real number was characterized by means of a partition of the set of rational numbers into two subsets, where every rational number belonging to the first subset is less than every rational number belonging to the second. Dedekind published his construction of the real number system in 1872, in the work *Stetigkeit und irrationale Zahlen*. In the same year, Georg Cantor published a construction of the real number system in which real numbers are represented by sequences of rational numbers satisfying an appropriate convegence criterion.

It has since been shown that the system of real numbers is completely characterized by the statement that the real numbers constitute an ordered field which satisfies the Least Upper Bound Axiom.

2 Infinite Sequences of Real Numbers

2.1 Convergence

An *infinite sequence* of real numbers is a sequence of the form $x_1, x_2, x_3, \ldots$, where x_j is a real number for each positive integer j. (More formally, one can view an infinite sequence of real numbers as a function from $\mathbb{N}$ to $\mathbb{R}$ which sends each positive integer j to some real number x_j.)

Definition An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to *converge* to some real number l if and only if the following criterion is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - l| < \varepsilon$ for all positive integers j satisfying $j \geq N$.

If the sequence $x_1, x_2, x_3, \ldots$ converges to the *limit* l then we denote this fact by writing '$x_j \to l$ as $j \to +\infty$', or by writing '$\lim\limits_{j\to+\infty} x_j = l$'.

Let x and l be real numbers, and let ε be a strictly positive real number. Then $|x - l| < \varepsilon$ if and only if both $x - l < \varepsilon$ and $l - x < \varepsilon$. It follows that $|x - l| < \varepsilon$ if and only if $l - \varepsilon < x < l + \varepsilon$. The condition $|x - l| < \varepsilon$ essentially requires that the value of the real number x should agree with l to within an error of at most ε. An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers converges to some real number l if and only if, given any positive real number ε, there exists some positive integer N such that $l-\varepsilon < x_j < l+\varepsilon$ for all positive integers j satisfying $j \geq N$.

Example A straightforward application of the definition of convergence shows that $1/j \to 0$ as $j \to +\infty$. Indeed suppose that we are given any strictly positive real number ε. If we pick some positive integer N large enough to satisfy $N > 1/\varepsilon$ then $|1/j| < \varepsilon$ for all positive integers j satisfying $j \geq N$, as required.

Example We show that $(-1)^j/j^2 \to 0$ as $j \to +\infty$. Indeed, given any strictly positive real number ε, we can find some positive integer N satisfying $N^2 > 1/\varepsilon$. If $j \geq N$ then $|(-1)^j/j^2| < \varepsilon$, as required.

Example The infinite sequence $x_1, x_2, x_3, \ldots$ defined by $x_j = j$ is not convergent. To prove this formally, we suppose that it were the case that $\lim\limits_{j\to+\infty} x_j = l$ for some real number l, and derive from this a contradiction. On setting $\varepsilon = 1$ (say) in the formal definition of convergence, we would deduce that there would exist some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $x_j < l + 1$ for all $j \geq N$, which is impossible. Thus the sequence cannot converge.

Example The infinite sequence $u_1, u_2, u_3, \ldots$ defined by $u_j = (-1)^j$ is not convergent. To prove this formally, we suppose that it were the case that $\lim_{j \to +\infty} u_j = l$ for some real number l. On setting $\varepsilon = \frac{1}{2}$ in the criterion for convergence, we would deduce the existence of some positive integer N such that $|u_j - l| < \frac{1}{2}$ for all $j \geq N$. But then

$$|u_j - u_{j+1}| \leq |u_j - l| + |l - u_{j+1}| < \tfrac{1}{2} + \tfrac{1}{2} = 1$$

for all $j \geq N$, contradicting the fact that $u_j - u_{j+1} = \pm 2$ for all j. Thus the sequence cannot converge.

Definition We say that an infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is *bounded above* if there exists some real number B such that $x_j \leq B$ for all positive integers j. Similarly we say that this sequence is *bounded below* if there exists some real number A such that $x_j \geq A$ for all positive integers j. A sequence is said to be *bounded* if it is bounded above and bounded below. Thus a sequence is bounded if and only if there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j.

Lemma 2.1 *Every convergent sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers converging to some real number l. On applying the formal definition of convergence (with $\varepsilon = 1$), we deduce the existence of some positive integer N such that $|x_j - l| < 1$ for all $j \geq N$. But then $A \leq x_j \leq B$ for all positive integers j, where A is the minimum of $x_1, x_2, \ldots, x_{N-1}$ and $l - 1$, and B is the maximum of $x_1, x_2, \ldots, x_{N-1}$ and $l + 1$. ∎

Proposition 2.2 *Let $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3,$ be convergent infinite sequences of real numbers. Then the sum, difference and product of these sequences are convergent, and*

$$\begin{aligned}
\lim_{j \to +\infty} (x_j + y_j) &= \lim_{j \to +\infty} x_j + \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j - y_j) &= \lim_{j \to +\infty} x_j - \lim_{j \to +\infty} y_j, \\
\lim_{j \to +\infty} (x_j y_j) &= \left(\lim_{j \to +\infty} x_j\right) \left(\lim_{j \to +\infty} y_j\right).
\end{aligned}$$

If in addition $y_j \neq 0$ for all positive integers j and $\lim_{j\to+\infty} y_j \neq 0$, then the quotient of the sequences (x_j) and (y_j) is convergent, and

$$\lim_{j\to+\infty} \frac{x_j}{y_j} = \frac{\lim_{j\to+\infty} x_j}{\lim_{j\to+\infty} y_j}.$$

Proof Throughout this proof let $l = \lim_{j\to+\infty} x_j$ and $m = \lim_{j\to+\infty} y_j$.

First we prove that $x_j + y_j \to l + m$ as $j \to +\infty$. Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j + y_j - (l + m)| < \varepsilon$ whenever $j \geq N$. Now $x_j \to l$ as $j \to +\infty$, and therefore, given any strictly positive real number ε_1, there exists some positive integer N_1 with the property that $|x_j - l| < \varepsilon_1$ whenever $j \geq N_1$. In particular, there exists a positive integer N_1 with the property that $|x_j - l| < \frac{1}{2}\varepsilon$ whenever $j \geq N_1$. (To see this, let $\varepsilon_1 = \frac{1}{2}\varepsilon$.) Similarly there exists some positive integer N_2 such that $|y_j - m| < \frac{1}{2}\varepsilon$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then

$$\begin{aligned} |x_j + y_j - (l + m)| &= |(x_j - l) + (y_j - m)| \leq |x_j - l| + |y_j - m| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Thus $x_j + y_j \to l + m$ as $j \to +\infty$.

Let c be some real number. We show that $cy_j \to cm$ as $j \to +\infty$. The case when $c = 0$ is trivial. Suppose that $c \neq 0$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|y_j - m| < \varepsilon/|c|$ whenever $j \geq N$. But then $|cy_j - cm| = |c||y_j - m| < \varepsilon$ whenever $j \geq N$. Thus $cy_j \to cm$ as $j \to +\infty$.

If we combine this result, for $c = -1$, with the previous result, we see that $-y_j \to -m$ as $j \to +\infty$, and therefore $x_j - y_j \to l - m$ as $j \to +\infty$.

Next we show that if $u_1, u_2, u_3, \ldots$ and $v_1, v_2, v_3, \ldots$ are infinite sequences, and if $u_j \to 0$ and $v_j \to 0$ as $j \to +\infty$, then $u_j v_j \to 0$ as $j \to +\infty$. Let some strictly positive real number ε be given. Then there exist positive integers N_1 and N_2 such that $|u_j| < \sqrt{\varepsilon}$ whenever $j \geq N_1$ and $|v_j| < \sqrt{\varepsilon}$ whenever $j \geq N_2$. Let N be the maximum of N_1 and N_2. If $j \geq N$ then $|u_j v_j| < \varepsilon$. We deduce that $u_j v_j \to 0$ as $j \to +\infty$.

We can apply this result with $u_j = x_j - l$ and $v_j = y_j - m$ for all positive integers j. Using the results we have already obtained, we see that

$$\begin{aligned} 0 &= \lim_{j\to+\infty} (u_j v_j) = \lim_{j\to+\infty} (x_j y_j - x_j m - l y_j + lm) \\ &= \lim_{j\to+\infty} (x_j y_j) - m \lim_{j\to+\infty} x_j - l \lim_{j\to+\infty} y_j + lm = \lim_{j\to+\infty} (x_j y_j) - lm. \end{aligned}$$

Thus $x_j y_j \to lm$ as $j \to +\infty$.

Next we show that if $w_1, w_2, w_3, \ldots$ is an infinite sequence of non-zero real numbers, and if $w_j \to 1$ as $j \to +\infty$ then $1/w_j \to 1$ as $j \to +\infty$. Let some strictly positive real number ε be given. Let ε_0 be the minimum of $\frac{1}{2}\varepsilon$ and $\frac{1}{2}$. Then there exists some positive integer N such that $|w_j - 1| < \varepsilon_0$ whenever $j \geq N$. Thus if $j \geq N$ then $|w_j - 1| < \frac{1}{2}\varepsilon$ and $\frac{1}{2} < w_j < \frac{3}{2}$. But then

$$\left|\frac{1}{w_j} - 1\right| = \left|\frac{1 - w_j}{w_j}\right| = \frac{|w_j - 1|}{|w_j|} < 2|w_j - 1| < \varepsilon.$$

We deduce that $1/w_j \to 1$ as $j \to +\infty$.

Finally suppose that $\lim\limits_{j \to +\infty} x_j = l$ and $\lim\limits_{j \to +\infty} y_j = m$, where $m \neq 0$. Let $w_j = y_j/m$. Then $w_j \to 1$ as $j \to +\infty$, and hence $1/w_j \to 1$ as $j \to +\infty$. We see therefore that $m/y_j \to 1$, and thus $1/y_j \to 1/m$, as $j \to +\infty$. The result we have already obtained for products of sequences then enables us to deduce that $x_j/y_j \to l/m$ as $j \to +\infty$. ∎

Example We shall show that if $s_j \to 2$ as $j \to +\infty$, where $s_j = \dfrac{6j^2 - 4j}{3j^2 + 7}$ for all positive integers j. Now neither $6j^2 - 4j$ nor $3j^2 + 7$ converges to any (finite) limit as $j \to +\infty$; and therefore we cannot directly apply the result in Proposition 2.2 concerning the convergence of the quotient of two convergent sequences. However on dividing both the numerator and the denominator of the fraction defining s_j by j^2, we see that

$$s_j = \frac{6j^2 - 4j}{3j^2 + 7} = \frac{6 - \dfrac{4}{j}}{3 + \dfrac{7}{j^2}}.$$

Moreover $6 - \dfrac{4}{j} \to 6$ and $3 + \dfrac{7}{j^2} \to 3$ as $j \to +\infty$, and therefore, on applying Proposition 2.2, we see that

$$\lim_{j \to +\infty} \frac{6j^2 - 4j}{3j^2 + 7} = \lim_{j \to +\infty} \frac{6 - \dfrac{4}{j}}{3 + \dfrac{7}{j^2}} = \frac{\lim\limits_{j \to +\infty} \left(6 - \dfrac{4}{j}\right)}{\lim\limits_{j \to +\infty} \left(3 + \dfrac{7}{j^2}\right)} = \frac{6}{3} = 2.$$

2.2 Monotonic Sequences

An infinite sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be *strictly increasing* if $x_{j+1} > x_j$ for all positive integers j, *strictly decreasing* if $x_{j+1} < x_j$ for all positive integers j, *non-decreasing* if $x_{j+1} \geq x_j$ for all positive integers j, *non-increasing* if $x_{j+1} \leq x_j$ for all positive integers j. A sequence satisfying any one of these conditions is said to be *monotonic*; thus a monotonic sequence is either non-decreasing or non-increasing.

Theorem 2.3 *Any non-decreasing sequence of real numbers that is bounded above is convergent. Similarly any non-increasing sequence of real numbers that is bounded below is convergent.*

Proof Let $x_1, x_2, x_3, \ldots$ be a non-decreasing sequence of real numbers that is bounded above. It follows from the Least Upper Bound Axiom that there exists a least upper bound l for the set $\{x_j : j \in \mathbb{N}\}$. We claim that the sequence converges to l.

Let some strictly positive real number ε be given. We must show that there exists some positive integer N such that $|x_j - l| < \varepsilon$ whenever $j \geq N$. Now $l - \varepsilon$ is not an upper bound for the set $\{x_j : j \in \mathbb{N}\}$ (since l is the least upper bound), and therefore there must exist some positive integer N such that $x_N > l - \varepsilon$. But then $l - \varepsilon < x_j \leq l$ whenever $j \geq N$, since the sequence is non-decreasing and bounded above by l. Thus $|x_j - l| < \varepsilon$ whenever $j \geq N$. Therefore $x_j \to l$ as $j \to +\infty$, as required.

If the sequence $x_1, x_2, x_3, \ldots$ is non-increasing and bounded below then the sequence $-x_1, -x_2, -x_3, \ldots$ is non-decreasing and bounded above, and is therefore convergent. It follows that the sequence $x_1, x_2, x_3, \ldots$ is also convergent. ∎

Example Let $x_1 = 2$ and

$$x_{j+1} = x_j - \frac{x_j^2 - 2}{2x_j}$$

for all positive integers j. Now

$$x_{j+1} = \frac{x_j^2 + 2}{2x_j} \text{ and } x_{j+1}^2 = x_j^2 - (x_j^2 - 2) + \left(\frac{x_j^2 - 2}{2x_j}\right)^2 = 2 + \left(\frac{x_j^2 - 2}{2x_j}\right)^2.$$

It therefore follows by induction on j that $x_j > 0$ and $x_j^2 > 2$ for all positive integers j. But then $x_{j+1} < x_j$ for all j, and thus the sequence $x_1, x_2, x_3, \ldots$ is decreasing and bounded below. It follows from Theorem 2.3 that this sequence converges to some real number α. Also $x_j > 1$ for all positive integers j (since $x_j > 0$ and $x_j^2 > 2$), and therefore $\alpha \geq 1$. But then, on applying Proposition 2.2, we see that

$$\alpha = \lim_{j \to +\infty} x_{j+1} = \lim_{j \to +\infty} \left(x_j - \frac{x_j^2 - 2}{2x_j}\right) = \alpha - \frac{\alpha^2 - 2}{2\alpha}.$$

Thus $\alpha^2 = 2$, and so $\alpha = \sqrt{2}$.

2.3 Subsequences of Sequences of Real Numbers

Definition Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. A *subsequence* of this infinite sequence is a sequence of the form $x_{j_1}, x_{j_2}, x_{j_3}, \ldots$ where $j_1, j_2, j_3, \ldots$ is an infinite sequence of positive integers with

$$j_1 < j_2 < j_3 < \cdots.$$

Let $x_1, x_2, x_3, \ldots$ be an infinite sequence of real numbers. The following sequences are examples of subsequences of the above sequence:—

$$x_1, x_3, x_5, x_7, \ldots$$
$$x_1, x_4, x_9, x_{16}, \ldots$$

2.4 The Bolzano-Weierstrass Theorem

Proposition 2.4 *Let $x_1, x_2, x_3, \ldots$ be a bounded infinite sequence of real numbers. Then there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$.*

First Proof The infinite sequence $(x_j : j \in \mathbb{N})$ is bounded, and therefore there exist real numbers A and B such that $A \leq x_j \leq B$ for all positive integers j. For each real number s let

$$Q_s = \{j \in \mathbb{N} : x_j > s\}.$$

Then $Q_s = \emptyset$ whenever $s \geq B$, and $Q_s = \mathbb{N}$ whenever $s < A$.

Let S be the set consisting of all real numbers s for which the corresponding set Q_s is infinite. Then $s \notin S$ whenever $s \geq B$, and $s \in S$ whenever $s < A$. It follows that the set S is a non-empty subset of $\mathbb{R}$ that is bounded above by B. The Least Upper Bound Principle therefore ensures that the set S has a well-defined least upper bound. Let c be the least upper bound bound of the set S.

Let some strictly positive real number ε be given. Let v satisfy $c < v < c + \varepsilon$. Then $v \notin S$, because c is an upper bound for the set S, and therefore the set Q_v is a finite subset of $\mathbb{N}$. Also $c - \varepsilon$ is not an upper bound for the set S, because c is the least upper bound for this set, and therefore there exists some element u of S satisfying $c - \varepsilon < u \leq c$. Then Q_u is an infinite subset of $\mathbb{N}$. It follows that the complement $Q_u \setminus Q_v$ of Q_v in Q_u is a subset of $\mathbb{N}$ with infinitely many elements.

Now

$$Q_u \setminus Q_v = \{j \in \mathbb{N} : x_j > u\} \setminus \{j \in \mathbb{N} : x_j > v\} = \{j \in \mathbb{N} : u < x_j \leq v\}.$$

Thus $c - \varepsilon < u < x_j \leq v < c + \varepsilon$ for all $j \in Q_u \setminus Q_v$. Therefore the number of positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$ must be infinite, as required. ∎

Theorem 2.5 (Bolzano-Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

First Proof Let $x_1, x_2, x_3, \ldots$ be an bounded infinite sequence of real numbers. It follows from Proposition 2.4 that there exists a real number c with the property that, given any strictly positive real number ε, there are infinitely many positive integers j for which $c - \varepsilon < x_j < c + \varepsilon$. There then exists some positive integer k_1 such that $c - 1 < x_{k_1} < c + 1$.

Now suppose that positive integers $k_1, k_2, \ldots, k_m$ have been determined such that $k_1 < k_2 < \cdots < k_m$ and

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for $j = 1, 2, \ldots, m$. The interval

$$\left\{ x \in \mathbb{R} : c - \frac{1}{m+1} < x < c + \frac{1}{m+1} \right\}$$

must then contain infinitely many members of the original sequence, and therefore there exists some positive integer k_{m+1} for which $k_m < k_{m+1}$ and

$$c - \frac{1}{m+1} < x_{k_{m+1}} < c + \frac{1}{m+1}.$$

Thus we can construct in this fashion a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ of the original sequence with the property that

$$c - \frac{1}{j} < x_{k_j} < c + \frac{1}{j}$$

for all positive integers j. This subsequence then converges to c. The given sequence therefore has a convergent subsequence, as required. ∎

Second Proof Let $a_1, a_2, a_3, \ldots$ be a bounded sequence of real numbers, and let

$$S = \{ j \in \mathbb{N} : a_j \geq a_k \text{ for all } k \geq j \}$$

(i.e., S is the set of all positive integers j with the property that a_j is greater than or equal to all the succeeding members of the sequence).

First let us suppose that the set S is infinite. Arrange the elements of S in increasing order so that $S = \{j_1, j_2, j_3, j_4, \ldots\}$, where $j_1 < j_2 < j_3 < j_4 < \cdots$. It follows from the manner in which the set S was defined that $a_{j_1} \geq a_{j_2} \geq a_{j_3} \geq a_{j_4} \geq \cdots$. Thus $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a non-increasing subsequence of the original sequence $a_1, a_2, a_3, \ldots$. This subsequence is bounded below (since the original sequence is bounded). It follows from Theorem 2.3 that $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ is a convergent subsequence of the original sequence.

Now suppose that the set S is finite. Choose a positive integer j_1 which is greater than every positive integer belonging to S. Then j_1 does not belong to S. Therefore there must exist some positive integer j_2 satisfying $j_2 > j_1$ such that $a_{j_2} > a_{j_1}$. Moreover j_2 does not belong to S (since j_2 is greater than j_1 and j_1 is greater than every positive integer belonging to S). Therefore there must exist some positive integer j_3 satisfying $j_3 > j_2$ such that $a_{j_3} > a_{j_2}$. We can continue in this way to construct (by induction on j) a strictly increasing subsequence $a_{j_1}, a_{j_2}, a_{j_3}, \ldots$ of our original sequence. This increasing subsequence is bounded above (since the original sequence is bounded) and thus is convergent, by Theorem 2.3. This completes the proof of the Bolzano-Weierstrass Theorem. ∎

2.5 Cauchy's Criterion for Convergence

Definition A sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to be a *Cauchy sequence* if the following condition is satisfied:

> given any strictly positive real number ε, there exists some positive integer N such that $|x_j - x_k| < \varepsilon$ for all positive integers j and k satisfying $j \geq N$ and $k \geq N$.

Lemma 2.6 *Every Cauchy sequence of real numbers is bounded.*

Proof Let $x_1, x_2, x_3, \ldots$ be a Cauchy sequence. Then there exists some positive integer N such that $|x_j - x_k| < 1$ whenever $j \geq N$ and $k \geq N$. In particular, $|x_j| \leq |x_N| + 1$ whenever $j \geq N$. Therefore $|x_j| \leq R$ for all positive integers j, where R is the maximum of the real numbers $|x_1|, |x_2|, \ldots, |x_{N-1}|$ and $|x_N| + 1$. Thus the sequence is bounded, as required. ∎

The following important result is known as *Cauchy's Criterion for convergence*, or as the *General Principle of Convergence*.

Theorem 2.7 (Cauchy's Criterion for Convergence) *An infinite sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof First we show that convergent sequences are Cauchy sequences. Let $x_1, x_2, x_3, \ldots$ be a convergent sequence of real numbers, and let $l = \lim_{j \to +\infty} x_j$. Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - l| < \frac{1}{2}\varepsilon$ for all $j \geq N$. Thus if $j \geq N$ and $k \geq N$ then $|x_j - l| < \frac{1}{2}\varepsilon$ and $|x_k - l| < \frac{1}{2}\varepsilon$, and hence

$$|x_j - x_k| = |(x_j - l) - (x_k - l)| \leq |x_j - l| + |x_k - l| < \varepsilon.$$

Thus the sequence $x_1, x_2, x_3, \ldots$ is a Cauchy sequence.

Conversely we must show that any Cauchy sequence $x_1, x_2, x_3, \ldots$ is convergent. Now Cauchy sequences are bounded, by Lemma 2.6. The sequence $x_1, x_2, x_3, \ldots$ therefore has a convergent subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.5). Let $l = \lim_{j \to +\infty} x_{k_j}$. We claim that the sequence $x_1, x_2, x_3, \ldots$ itself converges to l.

Let some strictly positive real number ε be given. Then there exists some positive integer N such that $|x_j - x_k| < \frac{1}{2}\varepsilon$ whenever $j \geq N$ and $k \geq N$ (since the sequence is a Cauchy sequence). Let m be chosen large enough to ensure that $k_m \geq N$ and $|x_{k_m} - l| < \frac{1}{2}\varepsilon$. Then

$$|x_j - l| \leq |x_j - x_{k_m}| + |x_{k_m} - l| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

whenever $j \geq N$. It follows that $x_j \to l$ as $j \to +\infty$, as required. ∎

3 Continuity for Functions of a Real Variable

3.1 The Definition of Continuity for Functions of a Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $f: D \to \mathbb{R}$ be a real-valued function on D. Let s be a point of D. The function f is said to be *continuous* at s if, given any positive real number ε, there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. If f is continuous at every point of D then we say that f is continuous on D.

Example Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

The function f is not continuous at 0. To prove this formally we note that when $0 < \varepsilon \leq 1$ there does not exist any strictly positive real number δ with the property that $|f(x) - f(0)| < \varepsilon$ for all x satisfying $|x| < \delta$ (since $|f(x) - f(0)| = 1$ for all $x > 0$).

Example Let $g: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We show that this function is not continuous at 0. Suppose that ε is chosen to satisfy $0 < \varepsilon < 1$. No matter how small we choose the strictly positive real number δ, we can always find $x \in \mathbb{R}$ for which $|x| < \delta$ and $|g(x) - g(0)| \geq \varepsilon$. Indeed, given any strictly positive real number δ, we can choose some integer j large enough to ensure that $0 < x_j < \delta$, where x_j satisfies $1/x_j = (4j+1)\pi/2$. Moreover $g(x_j) = 1$. This shows that the criterion defining the concept of continuity is not satisfied at $x = 0$.

Example Let $h: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$h(x) = \begin{cases} 3x \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

We claim that the function h is continuous at 0. To prove this, we must apply the definition of continuity directly. Let some strictly positive real number ε be given. If $\delta = \frac{1}{3}\varepsilon$ then $|h(x)| \leq 3|x| < \varepsilon$ for all real numbers x satisfying $|x| < \delta$, as required.

Lemma 3.1 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let $s \in D$. Suppose that the functions f and g are continuous at s. Then so is the function $f+g$, where $(f+g)(x) = f(x)+g(x)$ for all $x \in D$.*

Proof Suppose that $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ are continuous at s, where $s \in D$. We show that $f + g$ is continuous at s. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(x) - f(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta_1$, and $|g(x)-g(s)| < \frac{1}{2}\varepsilon$ for all $x \in D$ satisfying $|x-s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. If $|x - s| < \delta$ then

$$|f(x) + g(x) - (f(s) + g(s))| \leq |f(x) - f(s)| + |g(x) - g(s)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

showing that $f + g$ is continuous at s, as required. ∎

Lemma 3.2 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let c be a real number, and let $s \in D$. Suppose that the function f is continuous at s. Then so is the function cf, where $(cf)(x) = cf(x)$ for all $x \in D$.*

Proof If $c = 0$ then the function cf is the zero function, and is therefore continuous. We may therefore restrict attention to the case where $c \neq 0$.

Let some strictly positive real number ε be given, and let $\varepsilon_0 = \varepsilon/|c|$. Then $\varepsilon_0 > 0$, and the continuity of f at s then ensures the existence of some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon_0$ whenever $x \in D$ satisfies $|x - s| < \delta$. But then

$$|cf(x) - cf(s)| = |c|\,|f(x) - f(s)| < |c|\varepsilon_0 = \varepsilon$$

whenever $x \in D$ satisfies $|x - s| < \delta$. This shows that the function cf is continuous as s, as required. ∎

Lemma 3.3 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let $s \in D$. Suppose that the functions f and g are continuous at s. Then so is the function $f \cdot g$, where $(f \cdot g)(x) = f(x)g(x)$ for all $x \in D$.*

Proof Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $|f(x)-f(s)| < \sqrt{\varepsilon}$ whenever $x \in D$ satisfies $|x - s| < \delta_1$ and $|g(x) - g(s)| < \sqrt{\varepsilon}$ whenever $x \in D$ satisfies

$|x-s| < \delta_2$. Let δ be the minimum of δ_1 and δ_2. Then $\delta > 0$, and if $|x-s| < \delta$ then

$$|(f(x) - f(s)(g(x) - g(s))| = |(f(x) - f(s)|\,|(g(x) - g(s))| < \sqrt{\varepsilon} \times \sqrt{\varepsilon} = \varepsilon.$$

But

$$\begin{aligned}(f(x) - f(s)(g(x) - g(s)) &= f(x)g(x) - f(s)g(x) - g(s)f(x) + f(s)g(s) \\ &= h(x) - h(s),\end{aligned}$$

where $h: D \to \mathbb{R}$ is the real-valued function on D defined such that

$$h(x) = f(x)g(x) - f(s)g(x) - g(s)f(x)$$

for all $x \in D$. It follows that $|h(x) - h(s)| < \varepsilon$ whenever $x \in D$ satisfies $|x - s| < \delta$. We conclude from this that the function $h: D \to \mathbb{R}$ is continuous at s. Now

$$f(x)g(x) = h(x) + f(s)g(x) + g(s)f(x).$$

It therefore follows from Lemma 3.1 and Lemma 3.2 that the function $f \cdot g$ is continuous as s, as required. ∎

Proposition 3.4 *Let $f: D \to \mathbb{R}$ and $g: E \to \mathbb{R}$ be functions defined on D and E respectively, where D and E are subsets of $\mathbb{R}$ satisfying $f(D) \subset E$. Let s be an element of D. Suppose that the function f is continuous at s and that the function g is continuous at $f(s)$. Then the composition $g \circ f$ of f and g is continuous at s.*

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number η such that $|g(u) - g(f(s))| < \varepsilon$ for all $u \in E$ satisfying $|u - f(s)| < \eta$. But then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \eta$ for all $x \in D$ satisfying $|x - s| < \delta$. Thus if $|x - s| < \delta$ then $|g(f(x)) - g(f(s))| < \varepsilon$. Hence $g \circ f$ is continuous at s. ∎

Lemma 3.5 *Let $f: D \to \mathbb{R}$ be a function defined on a subset D of $\mathbb{R}$, and let s be an element of D. Suppose that $f(x) \neq 0$ for all $x \in D$ and that the function f is continuous at s for some $s \in D$. Then the function $1/f$ is also continuous at s, where $(1/f)(x) = 1/f(x)$ for all $x \in D$.*

Proof Let $r: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined such that $r(t) = 1/t$ for all non-zero real numbers t. We show that the function r is continuous. Let u be a non-zero real number, and let some strictly positive real number ε be given. Then

$$r(t) - r(u) = \frac{1}{t} - \frac{1}{u} = \frac{u - t}{tu}$$

for all non-zero real numbers t. Let δ be the minimum of $\frac{1}{2}|u|$ and $\frac{1}{2}|u|^2\varepsilon$. If t is a non-zero real number, and if $|t-u| < \delta$ then $|t| \geq |u| - |t-u| \geq \frac{1}{2}|u|$, and therefore

$$|r(t) - r(u)| \leq \frac{2}{|u|^2}|t-u| < \frac{2}{|u|^2}\delta \leq \varepsilon.$$

It follows that the function $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is continuous at u.

Now $1/f(x) = r(f(x))$ for all $x \in X$. Moreover the function f is continuous at s, and the function f is continuous at $f(s)$. It follows from Lemma 3.4 that the composition function $r \circ f$ is continuous at s. Thus the function $1/f$ is continuous at s, as required. ∎

Proposition 3.6 *Let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be functions defined over some subset D of $\mathbb{R}$. Suppose that f and g are continuous at some point s of D. Then the functions $f+g$, $f-g$ and $f \cdot g$ are also continuous at s. If moreover the function g is everywhere non-zero on D then the function f/g is continuous at s.*

Proof Now $f - g = f + (-g)$, and it follows from Lemma 3.2 that the function $-g$ is continuous on D. Lemma 3.1 therefore ensures that both $f+g$ and $f-g$ are continuous at s. Lemma 3.3 ensures that $f \cdot g$ is continuous at s. Moreover if the function g is everywhere non-zero on D then $f/g = f \cdot (1/g)$, and Lemma 3.5 ensures that the function $1/g$ is continuous on D. It then follows from Lemma 3.3 that the function f/g is continuous on D, as required. ∎

We shall show that if $f\colon D \to \mathbb{R}$ is a continuous real-valued function defined over some subset D of $\mathbb{R}$, then so is $|f|\colon D \to \mathbb{R}$, where $|f|(x) = |f(x)|$ for all $x \in D$.

Let u and v be real numbers. Then

$$\Bigl||u| - |v|\Bigr| \leq |u - v|.$$

To see this, note that $u = (u-v) + v$ and $v = (v-u) + u$ and therefore $|u| \leq |u-v| + |v|$ and $|v| \leq |v-u| + |u|$. But $|u-v| = |v-u|$. It follows that $|u| - |v| \leq |u-v|$ and $|v| - |u| \leq |u-v|$, and therefore

$$\Bigl||u| - |v|\Bigr| \leq |u - v|.$$

We apply this inequality in the proof of the next lemma.

Lemma 3.7 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let $s \in D$. Suppose that the function f is continuous at s. Then so is the function $|f|$, where $|f|(x) = |f(x)|$ for all $x \in D$.*

Proof Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$\Big||f(x)| - |f(s)|\Big| \leq |f(x) - f(s)| < \varepsilon$$

for all $x \in D$ satisfying $|x - s| < \delta$. It follows that $|f|: D \to \mathbb{R}$ is continuous at s, as required. ∎

Lemma 3.8 *Let $f: D \to \mathbb{R}$ be a function defined on some subset D of $\mathbb{R}$, and let $x_1, x_2, x_3, \ldots$ be a sequence of real numbers belonging to D. Suppose that $x_j \to s$ as $j \to +\infty$, where $s \in D$, and that f is continuous at s. Then $f(x_j) \to f(s)$ as $j \to +\infty$.*

Proof Let some positive real number ε be given. Then there exists some positive real number δ such that $|f(x) - f(s)| < \varepsilon$ for all $x \in D$ satisfying $|x-s| < \delta$. But then there exists some positive integer N such that $|x_j - s| < \delta$ for all j satisfying $j \geq N$. Thus $|f(x_j) - f(s)| < \varepsilon$ whenever $j \geq N$. Hence $f(x_j) \to f(s)$ as $j \to +\infty$. ∎

3.2 Limits of Functions of One Real Variable

Definition Let D be a subset of $\mathbb{R}$, and let $s \in \mathbb{R}$. The real number s is said to be a *limit point* of the set D if, given any strictly positive real number δ, there exists some real number x belonging to D such that $0 < |x - s| < \delta$.

It follows easily from the definition of convergence of sequences of real numbers that if D is a subset of the set $\mathbb{R}$ of real numbers, and if s is a point of $\mathbb{R}$ then the point s is a limit point of the set D if and only if there exists an infinite sequence $x_1, x_2, x_3, \ldots$ of points of D, all distinct from the point s, such that $\lim\limits_{j \to +\infty} x_j = s$.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f: D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. The real number l is said to be the *limit* of $f(x)$, as x tends to s in D, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(x) - l| < \varepsilon$ whenever $x \in D$ satisfies $0 < |x - s| < \delta$.

Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. If l is the limit of $f(x)$ as x tends to s in D then we can denote this fact by writing $\lim\limits_{x\to s} f(x) = l$.

Proposition 3.9 *Let D be a subset of the set $\mathbb{R}$ of real numbers, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, let s be a limit point of the set D, and let l be a real number. Let $\tilde{D} = D \cup \{s\}$, and let $\tilde{f}\colon \tilde{D} \to \mathbb{R}$ be defined such that*

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq s; \\ l & \text{if } x = s. \end{cases}$$

Then $\lim\limits_{x\to s} f(x) = l$ if and only if the function $\tilde{f}$ is continuous at s.

Proof The result follows directly on comparing the relevant definitions. ∎

Corollary 3.10 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ be a real-valued function on D, and let s be a point of the set D that is also a limit point of D. Then the function f is continuous at the point s if and only if $\lim\limits_{x\to s} f(x) = f(s)$.*

Let D be a subset of $\mathbb{R}$, and let s be a real number belonging to the set D. Suppose that s is not a limit point of the set D. Then there exists some strictly positive real number δ_0 such that $|x - s| \geq \delta_0$ for all $x \in X$. The point s is then said to be an *isolated point* of D.

Let D be a subset of $\mathbb{R}$. The definition of continuity then ensures that any real-valued function $f\colon D \to \mathbb{R}$ on D is continuous at any isolated point of its domain D.

Corollary 3.11 *Let D be a subset of $\mathbb{R}$, let $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ be real-valued functions on D, and let s be a limit point of the set D. Suppose that $\lim\limits_{x\to s} f(x)$ and $\lim\limits_{x\to s} g(x)$ both exist. Then so do $\lim\limits_{x\to s}(f(x) + g(x))$, $\lim\limits_{x\to s}(f(x) - g(x))$ and $\lim\limits_{x\to s}(f(x)g(x))$, and moreover*

$$\begin{aligned} \lim_{x\to s}(f(x) + g(x)) &= \lim_{x\to s} f(x) + \lim_{x\to s} g(x), \\ \lim_{x\to s}(f(x) - g(x)) &= \lim_{x\to s} f(x) - \lim_{x\to s} g(x), \\ \lim_{x\to s}(f(x)g(x)) &= \lim_{x\to s} f(x) \times \lim_{x\to s} g(x). \end{aligned}$$

If moreover $g(x) \neq 0$ for all $x \in X$ and $\lim_{x\to s} g(x) \neq 0$ then

$$\lim_{x\to s} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to s} f(x)}{\lim\limits_{x\to s} g(x)}.$$

Proof Let $\tilde{D} = X \cup \{s\}$, and let $\tilde{f}\colon \tilde{D} \to \mathbb{R}$ and $\tilde{g}\colon \tilde{D} \to \mathbb{R}$ be defined such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \neq s; \\ l & \text{if } x = s. \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & \text{if } x \neq s; \\ m & \text{if } x = s. \end{cases},$$

where $l = \lim_{x \to s} f(x)$ and $m = \lim_{x \to s} g(x)$. Then the functions $\tilde{f}$ and $\tilde{g}$ are continuous at s. The result therefore follows on applying Proposition 3.6. ∎

3.3 The Intermediate Value Theorem

Proposition 3.12 *Let $f\colon [a,b] \to \mathbb{Z}$ continuous integer-valued function defined on a closed interval $[a,b]$. Then the function f is constant.*

Proof Let

$$S = \{x \in [a,b] : f \text{ is constant on the interval } [a,x]\},$$

and let $s = \sup S$. Now $s \in [a,b]$, and therefore the function f is continuous at s. Therefore there exists some strictly positive real number δ such that $|f(x) - f(s)| < \frac{1}{2}$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. But the function f is integer-valued. It follows that $f(x) = f(s)$ for all $x \in [a,b]$ satisfying $|x - s| < \delta$. Now $s - \delta$ is not an upper bound for the set S. Therefore there exists some element x_0 of S satisfying $s - \delta < x_0 \leq s$. But then $f(s) = f(x_0) = f(a)$, and therefore the function f is constant on the interval $[a,x]$ for all $x \in [a,b]$ satisfying $s \leq x < s+\delta$. Thus $x \in [a,b] \cap [s, s+\delta) \subset S$. In particular $s \in S$. Now S cannot contain any elements x of $[a,b]$ satisfying $x > s$. Therefore $[a,b] \cap [s, s+\delta) = \{s\}$, and therefore $s = b$. This shows that $b \in S$, and thus the function f is constant on the interval $[a,b]$, as required. ∎

Theorem 3.13 (The Intermediate Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a continuous function defined on the interval $[a,b]$. Let c be a real number which lies between $f(a)$ and $f(b)$ (so that either $f(a) \leq c \leq f(b)$ or else $f(a) \geq c \geq f(b)$.) Then there exists some $s \in [a,b]$ for which $f(s) = c$.*

Proof The result is trivially true in the cases where $c = f(a)$ or $x = f(b)$. We may therefore suppose that either $f(a) < c < f(b)$ or else $f(a) > c > f(b)$. In either case, let $g_c\colon \mathbb{R} \setminus \{c\} \to \mathbb{Z}$ be the continuous integer-valued function on $\mathbb{R} \setminus \{c\}$ defined such that $g_c(x) = 0$ whenever $x < c$ and $g_c(x) = 1$ if $x > c$. Suppose that c were not in the range of the function f. Then the composition function $g_c \circ f\colon [a,b] \to \mathbb{R}$ would be a continuous integer-valued function defined throughout the interval $[a, b]$. This function would not be constant, since $g_c(f(a)) \neq g_c(f(b))$. But every continuous integer-valued function on the interval $[a, b]$ is constant (Proposition 3.12). It follows that every real number c lying between $f(a)$ and $f(b)$ must belong to the range of the function f, as required. ∎

Corollary 3.14 *Let $f: [a, b] \to [c, d]$ be a strictly increasing continuous function mapping an interval $[a, b]$ into an interval $[c, d]$, where a, b, c and d are real numbers satisfying $a < b$ and $c < d$. Suppose that $f(a) = c$ and $f(b) = d$. Then the function f has a continuous inverse $f^{-1}: [c, d] \to [a, b]$.*

Proof Let x_1 and x_2 be distinct real numbers belonging to the interval $[a, b]$ then either $x_1 < x_2$, in which case $f(x_1) < f(x_2)$ or $x_1 > x_2$, in which case $f(x_1) > f(x_2)$. Thus $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. It follows that the function f is injective. The Intermediate Value Theorem (Theorem 3.13) ensures that f is surjective. It follows that the function f has a well-defined inverse $f^{-1}: [c, d] \to [a, b]$. It only remains to show that this inverse function is continuous.

Let y be a real number satisfying $c < y < d$, and let x be the unique real number such that $a < x < b$ and $f(x) = y$. Let some strictly positive real number ε be given. We can then choose $x_1, x_2 \in [a, b]$ such that $x - \varepsilon < x_1 < x < x_2 < x + \varepsilon$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $y_1 < y < y_2$. Choose a strictly positive real number δ for which $\delta < y - y_1$ and $\delta < y_2 - y$. If $v \in [c, d]$ satisfies $|v - y| < \delta$ then $y_1 < v < y_2$ and therefore $x_1 < f^{-1}(v) < x_2$. But then $|f^{-1}(v) - f^{-1}(y)| < \varepsilon$. We conclude that the function $f^{-1}: [c, d] \to [a, b]$ is continuous at all points in the interior of the interval $[a, b]$. A similar argument shows that it is continuous at the endpoints of this interval. Thus the function f has a continuous inverse, as required. ∎

3.4 The Extreme Value Theorem

Theorem 3.15 (The Extreme Value Theorem) *Let a and b be real numbers satisfying $a < b$, and let $f: [a, b] \to \mathbb{R}$ be a continuous real-valued function defined on the closed interval $[a, b]$. Then there exist real numbers u and v belonging to the interval $[a, b]$ such that $f(u) \leq f(x) \leq f(v)$ for all $x \in [a, b]$.*

Proof We prove the result for an arbitrary continuous real-valued function $f: [a, b] \to \mathbb{R}$ by showing that the result holds for a related continuous function $g: [a, b] \to \mathbb{R}$ that is known to be bounded above and below on $[a, b]$. Let $h: \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. If t_1 and t_2 are real numbers satisfying $0 \leq t_1 < t_2$ then

$$h(t_2) - h(t_1) = \frac{t_2}{1+t_2} - \frac{t_1}{1+t_1} = \frac{t_2 - t_1}{(1+t_1)(1+t_2)} > 0,$$

and thus $h(t_1) < h(t_2)$. Thus the function h is strictly increasing on the set of non-negative real numbers. Moreover $h(0) = 0$ and $h(-t) = -h(t)$ for all real numbers t. It follows easily from this that the continuous function $h\colon \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$.

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous real-valued function on the closed bounded interval $[a,b]$, and let $g\colon [a,b] \to \mathbb{R}$ be the continuous real-valued function defined on $[a,b]$ such that

$$g(x) = h(f(x)) = \frac{f(x)}{1 + |f(x)|}$$

for all $x \in [a,b]$. Then $-1 \leq g(x) \leq 1$ for all $x \in [a,b]$. The set of values of the function g is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(x) : a \leq x \leq b\}.$$

Then, for each positive integer j, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function g, and therefore there exists some real number x_j satisfying $a \leq x_j \leq b$ for which $M - j^{-1} < g(x_j) \leq M$. The sequence $x_1, x_2, x_3, \ldots$ is then a bounded sequence of real numbers. It follows from the Bolzano-Weierstrass Theorem that this sequence has a subsequence $x_{k_1}, x_{k_2}, x_{k_3}, \ldots$ which converges to some real number v, where $a \leq v \leq b$. Now

$$M - \frac{1}{k_j} < g(x_{k_j}) \leq M$$

for all positive integers j, and therefore $g(x_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 3.8 that

$$g(v) = g\left(\lim_{j \to +\infty} x_{k_j}\right) = \lim_{j \to +\infty} g(x_{k_j}) = M.$$

But $g(x) \leq M$ for all $x \in [a,b]$. It follows that $h(f(x)) = g(x) \leq g(v) = h(f(v))$ for all $x \in [a,b]$. Moreover $h\colon \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(x) \leq f(v)$ for all $x \in [a,b]$.

On applying this result with the continuous function f replaced by the function $-f$, we conclude also that there exists some real number u satisfying $a \leq u \leq b$ such that $f(u) \leq f(x)$ for all $x \in [a,b]$. The result follows. ∎

3.5 Uniform Continuity

Definition A function $f\colon D \to \mathbb{R}$ is said to be *uniformly continuous* over a subset D of $\mathbb{R}$ if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ for all $u, v \in [a, b]$ satisfying $|u - v| < \delta$. (where δ does not depend on u or v).

A continuous function defined over a subset D of $\mathbb{R}$ is not necessarily uniformly continuous on D. (One can verify for example that the function sending a non-zero real number x to $1/x$ is not uniformly continuous on the set of all non-zero real numbers.) However we show that continuity does imply uniform continuity when $D = [a, b]$ for some real numbers a and b satisfying $a < b$.

Theorem 3.16 *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous real-valued function on a closed bounded interval $[a, b]$. Then the function f is uniformly continuuous on $[a, b]$.*

Proof Let some strictly positive real number ε be given. Suppose that there did not exist any strictly positive real number δ such that $|f(u) - f(v)| < \varepsilon$ whenever $|u - v| < \delta$. Then, for each positive integer j, there would exist values u_j and v_j in the interval $[a, b]$ such that $|u_j - v_j| < 1/j$ and $|f(u_j) - f(v_j)| \geq \varepsilon$. But the sequence $u_1, u_2, u_3, \ldots$ would be bounded (since $a \leq u_j \leq b$ for all j) and thus would possess a convergent subsequence $u_{k_1}, u_{k_2}, u_{k_3}, \ldots$, by the Bolzano-Weierstrass Theorem (Theorem 2.5). Let $l = \lim_{j \to +\infty} u_{k_j}$. Then $l = \lim_{j \to +\infty} v_{k_j}$ also, since $\lim_{j \to +\infty} (v_{k_j} - u_{k_j}) = 0$. Moreover $a \leq l \leq b$. It follows from the continuity of f that $\lim_{j \to +\infty} f(u_{k_j}) = \lim_{j \to +\infty} f(v_{k_j}) = f(l)$ (see Lemma 3.8). Thus $\lim_{j \to +\infty} \bigl(f(u_{k_j}) - f(v_{k_j})\bigr) = 0$. But this is impossible, since u_j and v_j have been chosen so that $|f(u_j) - f(v_j)| \geq \varepsilon$ for all positive integers j. We conclude therefore that there must exist some strictly positive real number δ with the required property. ∎

4 Differentiation

4.1 Interior Points and Open Sets in the Real Line

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers, and let s be a real number belonging to D. We say that s is an *interior point* of D if there exists some strictly positive number δ such that $x \in D$ for all real numbers x satisfying $s - \delta < x < s + \delta$. The *interior* of D is then the subset of D consisting of all real numbers belonging to D that are interior points of D.

Definition Let D be a subset of the set $\mathbb{R}$ of real numbers. We say that D is an *open set* in $\mathbb{R}$ if every point of D is an interior point of D.

Let s be a real number. We say that a function $f\colon D \to \mathbb{R}$ is defined *around* s if the real number s is an interior point of the domain D of the function f. It follows that the function f is defined around s if and only if there exists some strictly positive real number δ such that $f(x)$ is defined for all real numbers x satisfying $s - \delta < x < s + \delta$.

4.2 Differentiable Functions

Definition Let s be some real number, and let f be a real-valued function defined around s. The function f is said to be *differentiable* at s, with *derivative* $f'(s)$, if and only if the limit

$$f'(s) = \lim_{h \to 0} \frac{f(s+h) - f(s)}{h}$$

is well-defined. We denote by f', or by $\dfrac{df}{dx}$ the function whose value at s is the derivative $f'(s)$ of f at s.

Suppose that the real-valued function f is defined around some real number s and is differentiable at s. Then

$$f(s+h) = f(s) + h\,\frac{f(s+h) - f(s)}{h}$$

for all real numbers h sufficiently close to zero. It follows that

$$\begin{aligned} \lim_{x \to s} f(x) &= \lim_{h \to 0} f(s+h) = \lim_{h \to 0} f(s) + \left(\lim_{h \to 0} h\right)\left(\lim_{h \to 0} \frac{f(s+h) - f(s)}{h}\right) \\ &= f(s) + 0.f'(s) = f(s), \end{aligned}$$

and therefore f is continuous at s (see Lemma 3.10). Thus differentiability implies continuity.

Example Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2$. Let s be a real number. If $h \neq 0$ then

$$\frac{f(s+h) - f(s)}{h} = \frac{(s+h)^2 - s^2}{h} = 2s + h.$$

Therefore the function f is differentiable at s, and $f'(s) = \lim_{h\to 0}(2s + h) = 2s$.

Example Let $g\colon [0, +\infty) \to \mathbb{R}$ be the function defined by $g(x) = \sqrt{x}$, and let $s \in (0, +\infty)$. If h is any real number satisfying $h > -s$ and $h \neq 0$ then

$$\begin{aligned} \frac{g(s+h) - g(s)}{h} &= \frac{\sqrt{s+h} - \sqrt{s}}{h} = \frac{(\sqrt{s+h} - \sqrt{s})(\sqrt{s+h} + \sqrt{s})}{h(\sqrt{s+h} + \sqrt{s})} \\ &= \frac{(s+h) - s}{h(\sqrt{s+h} + \sqrt{s})} = \frac{1}{\sqrt{s+h} + \sqrt{s}}. \end{aligned}$$

Now $\lim_{h\to 0} \sqrt{s+h} = \sqrt{s}$ (since the function $x \mapsto \sqrt{x}$ is continuous at s). It follows that the function g is differentiable at s, and

$$g'(s) = \lim_{h\to 0} \frac{g(s+h) - g(s)}{h} = \frac{1}{\lim_{h\to 0}(\sqrt{s+h} + \sqrt{s})} = \frac{1}{2\sqrt{s}}.$$

Proposition 4.1 *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f + g$ and $f - g$ are differentiable at s, and*

$$(f+g)'(s) = f'(s) + g'(s), \qquad (f-g)'(s) = f'(s) - g'(s).$$

Proof It follows from Proposition 3.11 that

$$\begin{aligned} &\lim_{h\to 0} \frac{(f+g)(s+h) - (f+g)(s)}{h} \\ &\qquad = \lim_{h\to 0} \frac{f(s+h) - f(s)}{h} + \lim_{h\to 0} \frac{g(s+h) - g(s)}{h} \\ &\qquad = f'(s) + g'(s). \end{aligned}$$

Thus the function $f + g$ is differentiable at s, and $(f+g)'(s) = f'(s) + g'(s)$. An analogous proof shows that the function $f - g$ is also differentiable at s and $(f-g)'(s) = f'(s) - g'(s)$. ∎

Proposition 4.2 (Product Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s. Then $f \cdot g$ is also differentiable at s, and $(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$.*

Proof Note that

$$\begin{aligned}&\frac{f(s+h)g(s+h) - f(s)g(s)}{h}\\ &\qquad = \frac{f(s+h) - f(s)}{h} g(s+h) + f(s)\frac{g(s+h) - g(s)}{h}.\end{aligned}$$

Moreover $\lim_{h\to 0} g(s+h) = g(s)$ since g is differentiable, and hence continuous, at s. It follows that

$$\begin{aligned}&\lim_{h\to 0}\frac{f(s+h)g(s+h) - f(s)g(s)}{h}\\ &\qquad = \lim_{h\to 0}\frac{f(s+h) - f(s)}{h}\lim_{h\to 0} g(s+h) + f(s)\lim_{h\to 0}\frac{g(s+h) - g(s)}{h}\\ &\qquad = f'(s)g(s) + f(s)g'(s).\end{aligned}$$

Thus the function $f \cdot g$ is differentiable at s, and $(f \cdot g)'(s) = f'(s)g(s) + f(s)g'(s)$, as required. ∎

Proposition 4.3 (Quotient Rule) *Let s be some real number, and let f and g be real-valued functions defined around s. Suppose that the functions f and g are differentiable at s and that the function g is non-zero around s. Then f/g is differentiable at s, and*

$$(f/g)'(s) = \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2}.$$

Proof Note that

$$\begin{aligned}\frac{f(s+h)}{g(s+h)} - \frac{f(s)}{g(s)} &= \frac{f(s+h)g(s) - f(s)g(s+h)}{g(s+h)g(s)}\\ &= \frac{(f(s+h) - f(s))\,g(s) - f(s)\,(g(s+h) - g(s))}{g(s)g(s+h)}.\end{aligned}$$

Therefore

$$\begin{aligned}(f/g)'(s) &= \lim_{h\to 0}\frac{1}{h}\left(\frac{f(s+h)}{g(s+h)} - \frac{f(s)}{g(s)}\right)\\ &= \frac{1}{g(s)^2}\left(\lim_{h\to 0}\frac{f(s+h) - f(s)}{h} g(s) - f(s)\lim_{h\to 0}\frac{g(s+h) - g(s)}{h}\right)\\ &= \frac{f'(s)g(s) - f(s)g'(s)}{g(s)^2},\end{aligned}$$

since $\lim_{h\to 0} g(s)g(s+h) = g(s)^2 > 0$. ∎

Proposition 4.4 (Chain Rule) *Let a be some real number, let f be a real-valued function defined around a, and let g be a real-valued function defined around $f(a)$. Suppose that the function f is differentiable at a, and the function g is differentiable at $f(a)$. Then the composition function $g \circ f$ is differentiable at a, and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof Let $b = f(a)$, and let

$$R(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & \text{if } y \neq b; \\ g'(b) & \text{if } y = b. \end{cases}$$

for values of y around b. By considering separately the cases when $f(a+h) \neq f(a)$ and $f(a+h) = f(a)$, we see that

$$g(f(a+h)) - g(f(a)) = R(f(a+h))(f(a+h) - f(a)).$$

Now the function f is continuous at a, because it is differentiable at a. Also the function R is continuous at b, where $b = f(a)$, since

$$\lim_{y \to b} R(y) = \lim_{y \to b} \frac{g(y) - g(b)}{y - b} = \lim_{k \to 0} \frac{g(b+k) - g(b)}{k} = g'(b) = R(b).$$

It follows from Proposition 3.4 that the composition function $R \circ f$ is continuous at a, and therefore

$$\lim_{h \to 0} R(f(a+h)) = R(f(a)) = g'(f(a))$$

by Lemma 3.10. It follows that $g \circ f$ is differentiable at a, and

$$\begin{aligned} (g \circ f)'(a) &= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \\ &= \lim_{h \to 0} R(f(a+h)) \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = g'(f(a))f'(a), \end{aligned}$$

as required. ∎

4.3 Rolle's Theorem and the Mean Value Theorem

Theorem 4.5 (Rolle's Theorem) *Let $f\colon [a,b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a,b]$. Suppose that f is continuous on $[a,b]$ and is differentiable on (a,b). Suppose also that $f(a) = f(b)$. Then there exists some real number s satisfying $a < s < b$ which has the property that $f'(s) = 0$.*

Proof First we show that if the function f attains its minimum value at u, and if $a < u < b$, then $f'(u) = 0$. Now the difference quotient

$$\frac{f(u+h) - f(u)}{h}$$

is non-negative for all sufficiently small positive values of h; therefore $f'(u) \geq 0$. On the other hand, this difference quotient is non-positive for all sufficiently small negative values of h; therefore $f'(u) \leq 0$. We deduce therefore that $f'(u) = 0$.

Similarly if the function f attains its maximum value at v, and if $a < v < b$, then $f'(v) = 0$. (Indeed the result for local maxima can be deduced from the corresponding result for local minima simply by replacing the function f by $-f$.)

Now the function f is continuous on the closed bounded interval $[a, b]$. It therefore follows from the Extreme Value Theorem (Theorem 3.15) that there must exist real numbers u and v in the interval $[a, b]$ with the property that $f(u) \leq f(x) \leq f(v)$ for all real numbers x satisfying $a \leq x \leq b$. If $a < u < b$ then $f'(u) = 0$ and we can take $s = u$. Similarly if $a < v < b$ then $f'(v) = 0$ and we can take $s = v$. The only remaining case to consider is when both u and v are endpoints of the interval $[a, b]$. In that case the function f is constant on $[a, b]$, since $f(a) = f(b)$, and we can choose s to be any real number satisfying $a < s < b$. ∎

4.4 The Mean Value Theorem

Rolle's Theorem can be generalized to yield the following important theorem.

Theorem 4.6 (The Mean Value Theorem) *Let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function defined on some interval $[a, b]$. Suppose that f is continuous on $[a, b]$ and is differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$f(b) - f(a) = f'(s)(b - a).$$

Proof Let $g\colon [a, b] \to \mathbb{R}$ be the real-valued function on the closed interval $[a, b]$ defined by

$$g(x) = f(x) - \frac{b - x}{b - a} f(a) - \frac{x - a}{b - a} f(b).$$

Then the function g is continuous on $[a, b]$ and differentiable on (a, b). Moreover $g(a) = 0$ and $g(b) = 0$. It follows from Rolle's Theorem (Theorem 4.5) that $g'(s) = 0$ for some real number s satisfying $a < s < b$. But

$$g'(s) = f'(s) - \frac{f(b) - f(a)}{b - a}.$$

Therefore $f(b) - f(a) = f'(s)(b - a)$, as required. ∎

4.5 Cauchy's Mean Value Theorem

We now prove a generalization of the standard Mean Value Theorem, known as *Cauchy's Mean Value Theorem.*

Theorem 4.7 (Cauchy's Mean Value Theorem) *Let f and g be real-valued functions defined on some interval $[a, b]$. Suppose that f and g are continuous on $[a, b]$ and are differentiable on (a, b). Then there exists some real number s satisfying $a < s < b$ which has the property that*

$$(f(b) - f(a))\, g'(s) = (g(b) - g(a))\, f'(s).$$

In particular, if $g(b) \neq g(a)$ and the function g' is non-zero throughout (a, b), then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(s)}{g'(s)}.$$

Proof Consider the function $h\colon [a, b] \to \mathbb{R}$ defined by

$$h(x) = f(x)\,(g(b) - g(a)) - g(x)\,(f(b) - f(a))\,.$$

Then $h(a) = f(a)g(b) - g(a)f(b) = h(b)$, and the function h satisfies the hypotheses of Rolle's Theorem on the interval $[a, b]$. We deduce from Rolle's Theorem (Theorem 4.5) that $h'(s) = 0$ for some s satisfying $a < s < b$. The required result then follows immediately. ∎

4.6 One-Sided Limits and Limits at Infinity

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s and l be real numbers. We say that l is the *limit* $\lim\limits_{x \to s^+} f(x)$ *of $f(x)$ as x tends to s from above* if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $s < x < s + \delta$.

If f is a real-valued function, if $f(x)$ is defined for all real numbers x greater than but sufficiently close to some real number s, if l is a real number, and if l is the limit of $f(x)$ as x tends to s from above, then we may denote this fact by writing

$$l = \lim_{x \to s^+} f(x).$$

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined over some subset D of $\mathbb{R}$, and let s and l be real numbers. We say that l is the *limit* $\lim\limits_{x \to s^-} f(x)$ *of* $f(x)$ *as* x *tends to* s *from below* if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $s - \delta < x < s$.

Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on some subset D of $\mathbb{R}$, and let s and l be real numbers. Suppose that there exists some positive real number δ_0 with the property that $x \in D$ for all real numbers x satisfying $s < x < s + \delta_0$. Then $\lim\limits_{x \to s^+} f(x) = l$ if and only if the real number l is the limit of $f(x)$ as x tends to s in the subset $D \cap (s, +\infty)$ of D. Thus the properties of "one-sided limits" taken as a variable x tends to some given value s from above, or from below, are consequences of properties of limits in general, and thus there is no need to develop a separate theory of "one-sided limits".

Lemma 4.8 *Let* $f\colon D \to \mathbb{R}$ *be a real-valued function defined on a subset* D *of* $\mathbb{R}$, *let* s *be a real number that is an interior point of* $D \cup \{s\}$, *and let* l *be a real number. Then* $\lim\limits_{x \to s} f(x) = l$ *if and only if* $\lim\limits_{x \to s^+} f(x) = l$ *and* $\lim\limits_{x \to s^-} f(x) = l$.

Proof It follows directly from the definition of limits that if $\lim\limits_{x \to s} f(x) = l$ then $\lim\limits_{x \to s^+} f(x) = l$ and $\lim\limits_{x \to s^-} f(x) = l$. To prove the converse, suppose that $\lim\limits_{x \to s^+} f(x) = l$ and $\lim\limits_{x \to s^-} f(x) = l$. Let some strictly positive real number ε be given. Then there exist strictly positive real numbers δ_1 and δ_2 such that $l - \varepsilon < f(x) < l + \varepsilon$ both for all real numbers x satisfying $s < x < s + \delta_1$ and also for all real numbers x satisfying $s - \delta_2 < x < s$. Let δ be the minimum of δ_1 and δ_2. Then $l - \varepsilon < f(x) < l + \varepsilon$ for all real numbers x satisfying $0 < |x - s| < \delta$. It follows that $\lim\limits_{x \to s} f(x) = l$, as required.

Definition Let $f\colon D \to \mathbb{R}$ be a real-valued function defined on some subset D of $\mathbb{R}$, and let l be some real number. We say that l is the *limit* $\lim\limits_{x \to +\infty} f(x)$ of $f(x)$ as $x \to +\infty$ if, given any strictly positive real number ε, there exists some real number K such that $x \in D$ and $l - \varepsilon < f(x) < l + \varepsilon$ whenever $x > K$.

On comparing definitions, it we see that if $f\colon D \to \mathbb{R}$ is a real-valued function defined on a subset D of $\mathbb{R}$, where D contains all real numbers greater than some given real number, if l is a real number, then $\lim\limits_{x\to+\infty} f(x) = l$ if and

$$\lim_{t\to 0^+} f\left(\frac{1}{t}\right) = l.$$

It follows that properties of limits taken "at infinity" can be deduced from corresponding properties of "one-sided limits" and thus follow from the general theory of limits. In particular, if f and g are real valued functions, if $f(x)$ and $g(x)$ are defined for all sufficiently large values of x, and if the limits $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to+\infty} g(x)$ both exist, then so do the corresponding limits of the functions $f+g$, $f-g$, $f.g$ and $|f|$, and moreover

$$\begin{aligned}
\lim_{x\to+\infty}(f(x)+g(x)) &= \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x),\\
\lim_{x\to+\infty}(f(x)-g(x)) &= \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x),\\
\lim_{x\to+\infty}(f(x)g(x)) &= \lim_{x\to+\infty} f(x) \times \lim_{x\to+\infty} g(x),\\
\lim_{x\to+\infty}|f(x)| &= \left|\lim_{x\to+\infty} f(x)\right|.
\end{aligned}$$

Moreover if in addition $\lim\limits_{x\to+\infty} g(x) \neq 0$ then

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to+\infty} f(x)}{\lim\limits_{x\to+\infty} g(x)}.$$

4.7 L'Hôpital's Rule

An important corollary of Cauchy's Mean Value Theorem is *l'Hôpital's Rule* for evaluating the limit of a quotient of two functions at a point where both functions vanish.

Proposition 4.9 (L'Hôpital's Rule for Limits from above) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ_0 such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $s < x < s + \delta_0$, and that $\lim\limits_{x\to s^+}\dfrac{f'(x)}{g'(x)}$ exists (and is finite). Then $\lim\limits_{x\to s^+}\dfrac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x\to s+}\frac{f(x)}{g(x)} = \lim_{x\to s+}\frac{f'(x)}{g'(x)}.$$

Proof Let $l = \lim_{x \to s^+} \dfrac{f'(x)}{g'(x)}$, and let some strictly positive real number ε be given. By choosing a sufficiently small strictly positive real number δ we can ensure that $f(x)/g(x)$ and $f'(x)/g'(x)$ are well-defined and

$$l - \varepsilon < \frac{f'(x)}{g'(x)} < l + \varepsilon$$

for all real numbers x satisfying $s < x < s + \delta$. Now $f(s) = g(s) = 0$. An application of Cauchy's Mean Value Theorem to the functions f and g on the interval $[s, x]$ therefore ensures that there exists some real number t satisfying $s < t < x$ for which

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(s)}{g(x) - g(s)} = \frac{f'(t)}{g'(t)}.$$

But then $s < t < s + \delta$. It follows that

$$l - \varepsilon < \frac{f'(t)}{g'(t)} < l + \varepsilon,$$

and therefore

$$l - \varepsilon < \frac{f(x)}{g(x)} < l + \varepsilon.$$

This shows that $\lim_{x \to s^+} f(x)/g(x) = l$, as required. ∎

Corollary 4.10 (L'Hôpital's Rule for Limits from below) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ_0 such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $s - \delta_0 < x < s$, and that $\lim_{x \to s^-} \dfrac{f'(x)}{g'(x)}$ exists (and is finite). Then $\lim_{x \to s^-} \dfrac{f(x)}{g(x)}$ also exists, and*

$$\lim_{x \to s-} \frac{f(x)}{g(x)} = \lim_{x \to s-} \frac{f'(x)}{g'(x)}.$$

Proof It follows from Proposition 4.9 and the definitions of limits from above and from below that

$$\lim_{x \to s^-} \frac{f(x)}{g(x)} = \lim_{h \to 0^+} \frac{f(s-h)}{g(s-h)} = \lim_{h \to 0^+} \frac{f'(s-h)}{g'(s-h)} = \lim_{x \to s^-} \frac{f'(x)}{g'(x)},$$

as required. ∎

Proposition 4.11 (L'Hôpital's Rule) *Let f and g be differentiable real-valued functions defined around some real number s for which $f(s) = g(s) = 0$. Suppose that there exists some strictly positive real number δ such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $0 < |x - s| < \delta$, and that the limit of $f'(x)/g'(x)$ exists (and is finite) as $x \to s$. Then the limit of $f(x)/g(x)$ exists as $x \to s$, and*

$$\lim_{x \to s} \frac{f(x)}{g(x)} = \lim_{x \to s} \frac{f'(x)}{g'(x)}.$$

Proof Let $l = \lim\limits_{x \to s} \dfrac{f'(x)}{g'(x)}$. It follows from Proposition 4.9 and Corollary 4.10 that

$$\lim_{x \to s^+} \frac{f(x)}{g(x)} = \lim_{x \to s+} \frac{f'(x)}{g'(x)} = l$$

and

$$\lim_{x \to s^-} \frac{f(x)}{g(x)} = \lim_{x \to s-} \frac{f'(x)}{g'(x)} = l.$$

It then follows from Lemma 4.8 that

$$\lim_{x \to s} \frac{f(x)}{g(x)} = l,$$

as required. ∎

Example Using l'Hôpital's Rule twice, we see that

$$\lim_{x \to 2} \frac{x^3 + x^2 - 16x + 20}{x^3 - 3x^2 + 4} = \lim_{x \to 2} \frac{3x^2 + 2x - 16}{3x^2 - 6x} = \lim_{x \to 2} \frac{6x + 2}{6x - 6} = \frac{7}{3}.$$

Proposition 4.12 (L'Hôpital's Rule for Limits at Infinity) *Let f and g be differentiable real-valued functions defined for all real numbers that are greater than some given real number. Suppose that $\lim\limits_{x \to +\infty} f(x) = 0$ and $\lim\limits_{g \to +\infty} g(x) = 0$. Suppose also that there exists some real number K such that $g(x)$ and $g'(x)$ are non-zero for all real numbers x satisfying $x > K$, and that the limit of $f'(x)/g'(x)$ exists (and is finite) as $x \to +\infty$. Then the limit of $f(x)/g(x)$ exists as $x \to +\infty$, and*

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{f'(x)}{g'(x)}.$$

Proof Suppose that

$$\lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = l.$$

Let $p\colon [0, 1/K) \to \mathbb{R}$ and $q\colon [0, 1/K) \to \mathbb{R}$ be defined such that $p(0) = q(0) = 0$, $p(t) = f(1/t)$ and $q(t) = g(1/t)$ for all real numbers t satisfying $0 < t < 1/K$. The requirements that $\lim\limits_{x\to+\infty} f(x) = 0$ and $\lim\limits_{g\to+\infty} g(x) = 0$ ensure that the functions p and q defined on the interval $[0, 1/K)$ are continuous at 0. Moreover

$$p'(t) = -\frac{1}{t^2} f'\left(\frac{1}{t}\right) \quad \text{and} \quad q'(t) = -\frac{1}{t^2} g'\left(\frac{1}{t}\right)$$

for all real numbers t satisfying $0 < t < 1/K$, and thus

$$\lim_{t\to 0^+} \frac{p'(t)}{q'(t)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} = l.$$

It follows that there exists some positive real number δ such that $l - \varepsilon < p'(t)/q'(t) < l + \varepsilon$ for all real numbers t satisfying $0 < t < \delta$. Let s be a real number satisfying $0 < s < \delta$. application of Cauchy's Mean Value Theorem shows that there exists some real number t satisfying $0 < t < s < \delta$ for which

$$\frac{p(s)}{q(s)} = \frac{p(s) - p(0)}{q(s) - q(0)} = \frac{p'(t)}{q'(t)}.$$

But then $l - \varepsilon < p(s)/q(s) < l + \varepsilon$. It follows that $\lim\limits_{s\to 0^+} p(s)/q(s) = l$, and thus $\lim\limits_{x\to+\infty} f(x)/g(x) = l$, as required. ∎

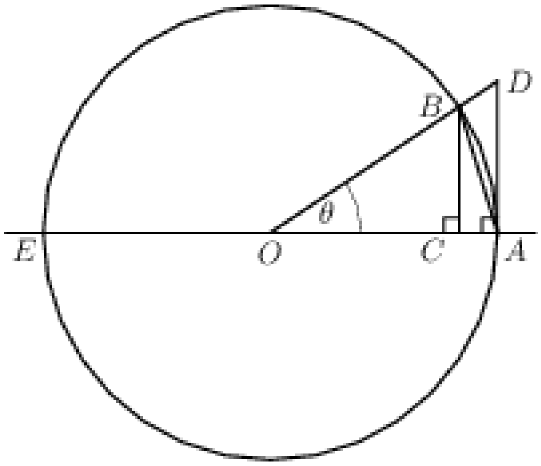
4.8 Derivatives of Trigonometrical Functions

Proposition 4.13 *Let* $\sin\colon \mathbb{R} \to \mathbb{R}$ *be the sine function whose value* $\sin\theta$*, for a given real number* θ *is the sine of an angle of* θ *radians. Then*

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

Proof Let E and A be the endpoints of a diameter of a circle of unit radius, let O be the centre of the circle, and let B be a point on the circle for which the line OB makes an angle of θ radians with the line OA, where $0 < \theta < \frac{\pi}{2}$. Let C be the point on the line segment OA for which the angle OCB is a right angle, and let the line OB be produced to the point D determined so that the angle OAD is a right angle.

The sector OAB of the unit circle is by definition the region bounded by the arc AB of the circle and the radii OA and OB. Now the area of a sector



of a circle subtending at the centre an angle of θ radians is equal to the area of the circle multiplied by $\dfrac{\theta}{2\pi}$. But the area of a circle of unit radius is π. It follows that a sector of the unit circle subtending at the centre an angle of θ radians has area $\frac{1}{2}\theta$. Also the triangles OAB and OAD have heights equal to lengths of the line segments BC and AD respectively, and the definitions of the sine, cosine and tangent functions ensure that the lengths of BC and AD are $\sin\theta$ and $\tan\theta$ respectively. Also the common base OA of the triangles OAB and OAD has length one unit, because the circle has unit radius. Now, in Euclidean geometry, the area of any triangle is half the base of the triangle multiplied by the height of the triangle. Therefore

$$\begin{aligned}\text{area of triangle OAB} &= \tfrac{1}{2}\sin\theta,\\ \text{area of sector OAB} &= \tfrac{1}{2}\theta,\\ \text{area of triangle OAD} &= \tfrac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}.\end{aligned}$$

Moreover the triangle OAB is strictly contained in the sector OAB, which in turn is strictly contained in the triangle OAD. It follows that

$$\sin\theta < \theta < \frac{\sin\theta}{\cos\theta},$$

for all real numbers θ satisfying $0 < \theta < \frac{\pi}{2}$, and therefore

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

for all real numbers θ satisfying $0 < \theta < \frac{\pi}{2}$. Now, given any positive real number ε, there exists some real numbeer δ satisfying $0 < \delta < \frac{\pi}{2}$ such that

$1-\varepsilon < \cos\theta < 1$ whenever $0 < \theta < \delta$. (In geometrical terms, we are choosing δ so that the length of the line segment BA in the figure associated with this proof is less than ε whenever $0 < \theta < \delta$.) But then

$$1 - \varepsilon < \frac{\sin\theta}{\theta} < 1$$

whenever $0 < \theta < \delta$. These inequalities also hold when $-\delta < \theta < 0$, because the value of $\dfrac{\sin\theta}{\theta}$ is unchanged on replacing θ by $-\theta$. It follows that $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$, as required. ∎

Corollary 4.14 *Let* $\cos\colon \mathbb{R} \to \mathbb{R}$ *be the cosine function whose value* $\cos\theta$*, for a given real number* θ *is the cosine of an angle of* θ *radians. Then*

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = 0.$$

Proof Basic trigonometrical identities ensure that

$$1-\cos\theta = 2\sin^2\tfrac{1}{2}\theta \quad\text{and}\quad \sin\theta = 2\sin\tfrac{1}{2}\theta\,\cos\tfrac{1}{2}\theta$$

for all real numbers θ. Therefore

$$\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta$$

for all real numbers θ. It follows that

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} = 0,$$

and therefore

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta} \times \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 0\times 1 = 0,$$

as required. ∎

Corollary 4.15 *The derivatives of the sine and cosine functions satisfy*

$$\frac{d}{dx}(\sin x) = \cos x, \quad \textit{and} \quad \frac{d}{dx}(\cos x) = -\sin x.$$

Proof Using standard principles of differential calculus we see that

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x\, \cos h - \cos x\, \sin h - \sin x}{h} \\
&= \cos x\, \lim_{h\to 0} \frac{\sin h}{h} - \sin x\, \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= \cos x, \\
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x\, \cos h - \sin x\, \sin h - \sin x}{h} \\
&= -\sin x\, \lim_{h\to 0} \frac{\sin h}{h} - \cos x\, \lim_{h\to 0} \frac{1-\cos h}{h} \\
&= -\sin x,
\end{aligned}$$

as required. ∎

4.9 Derivatives of Logarithmic and Exponential Functions

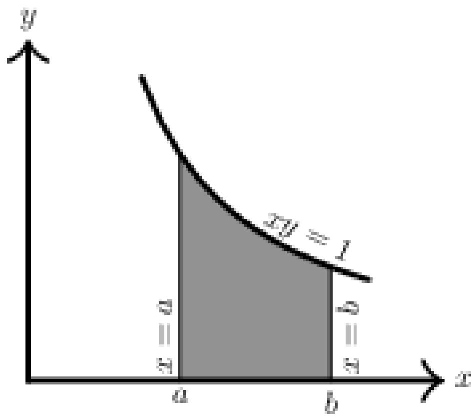
Given any real numbers a and b satisfying $a < b$, let $L(a, b)$ denote the area of the region

$$\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ \ y \geq 0 \ \text{ and } \ xy \leq 1\}$$

of the Euclidean plane bounded by the x-axis (i.e., the line $y = 0$), the line $x = a$, the line $y = b$ and the hyperbola $xy = 1$. (The quantity $L(a, b)$ thus denotes the area under the graph of the function sending x to $1/x$ (i.e., between the graph of that function and the x-axis) in the interval from $x = a$ and $x = b$.

Let r be a positive real number, and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ denote the transformation of the Euclidean plane defined such that $T(x, y) = (rx, r^{-1}x)$ for all real numbers x and y. Given any rectangle in $\mathbb{R}^2$ with sides parallel to the coordinate axes, the image of that rectangle under the transformation T has the same area as the rectangle itself. It follows from this that the $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ preserves the area of any geometrical figure whose boundary can be approximated sufficiently closely by a polygonal curve with sides parallel to the coordinate axes. Now the transformation T maps the hyperbola $xy = 1$ onto itself. It therefore maps the region

$$\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b,\ \ y \geq 0 \ \text{ and } \ xy \leq 1\}$$



onto the region

$$\{(x, y) \in \mathbb{R}^2 : ra \leq x \leq rb, \;\; y \geq 0 \;\text{ and }\; xy \leq 1\}.$$

It follows that $L(ra, rb) = L(a, b)$ for all strictly positive real numbers a, b and r satisfying $a < b$.

Let us define $L(a, a) = 0$ and $L(b, a) = -L(a, b)$ for all positive real numbers a and b satisfying $a < b$. Then $L(a, b) = L(ra, rb)$ for all positive real numbers a, b and r, irrespective of whether $a < b$, $a = b$ or $a > b$. Moreover $L(a, c) = L(a, b) + L(b, c)$ for all positive real numbers a, b and c.

We define $\log x = L(1, x)$ for all positive real numbers x. The real-valued function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ defined on the set $\mathbb{R}^+$ of positive real numbers is the *natural logarithm function*.

If u and v are real numbers satisfying $u < v$ then $\log v - \log u = L(u, v) > 0$, and thus $\log u < \log v$. Thus the logarithm function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ is a monotonically increasing function.

Lemma 4.16 *The natural logarithm function* $\log\colon \mathbb{R}^+ \to \mathbb{R}$ *satisfies*

$$\log(xy) = \log x + \log y.$$

for all real numbers x and y.

Proof Given real numbers a and b, let $L(a, b)$ denote the area of the region $X_{a,b}$ of the plane defined such that

$$X_{a,b} = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \;\; y \geq 0 \;\text{ and }\; xy \leq 1\}.$$

Then $L(a, c) = L(a, b) + L(b, c)$ for all positive real numbers a, b and c. It follows that

$$\log xy = L(1, xy) = L(1, x) + L(x, xy) = L(1, x) + L(1, y) = \log x + \log y,$$

as required. ∎

Lemma 4.17 *The natural logarithm function* $\log\colon \mathbb{R}^+ \to \mathbb{R}$ *satisfies*

$$\frac{d}{dx}(\log x) = \frac{1}{x}.$$

for all real numbers x.

Proof Given real numbers a and b, let $L(a,b)$ denote the area of the region $X_{a,b}$ of the plane defined such that

$$X_{a,b} = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ \ y \geq 0 \ \text{ and } \ xy \leq 1\}.$$

Let s be a positive real number. Then

$$\frac{\log(s+h) - \log s}{h} = \frac{1}{h} L(s, s+h)$$

for all real numbers h satisfying $h > -s$. Suppose that $h > 0$. Then

$$X_{s,h} \supset \{(x,y) \in \mathbb{R}^2 : s \leq x \leq s+h \ \text{ and } \ 0 \leq y \leq 1/(s+h)\}$$

and

$$X_{s,s+h} \subset \{(x,y) \in \mathbb{R}^2 : s \leq x \leq s+h \ \text{ and } \ 0 \leq y \leq 1/s\},$$

and therefore

$$\frac{1}{s+h} < \frac{1}{h}\, L(s, s+h) < \frac{1}{s}.$$

Taking the limit as h tends to zero from above, we find that

$$\lim_{h \to 0^+} \frac{\log(s+h) - \log s}{h} = \lim_{h \to 0^+} \frac{1}{h} L(s, s+h) = \frac{1}{s}.$$

Similarly

$$\frac{1}{s} < \frac{1}{k}\, L(s-k, s) < \frac{1}{s-k}$$

for all real numbers k satisfying $0 < k < s$, and therefore

$$\lim_{h \to 0^-} \frac{\log(s+h) - \log s}{h} = \lim_{k \to 0^+} \frac{\log s - \log(s-k)}{k} = \lim_{k \to 0^+} \frac{1}{k} L(s-k, s) = \frac{1}{s}.$$

It follows that

$$\lim_{h \to 0} \frac{\log(x+h) - \log x}{h} = \frac{1}{s}.$$

We deduce that the natural logarithm function is differentiable, and

$$\frac{d}{dx}(\log x) = \frac{1}{x}$$

for all positive real numbers x, as required. ∎

Let s be a real number satisfying $s > 1$, and let n be a positive integer. Then $\log s > 0$, $\log s^n = n \log s$ and $\log s^{-n} = -n \log s$. The Intermediate Value Theorem (Theorem 3.13) then ensures that all real numbers between $-n \log s$ and $n \log s$ belong to the range of the natural logarithm function. Now, given any real number y, we can choose n large enough to ensure that $|y| < n \log s$. It follows that there exists some positive real number x satisfying $\log x = y$. This shows that the range of the logarithm function is the set $\mathbb{R}$ of real numbers. Also $\log u < \log v$ for all real numbers u and v satisfying $u < v$. It follows that the function $\log\colon \mathbb{R}^+ \to \mathbb{R}$ provides a one-to-one correspondence between the set $\mathbb{R}^+$ of positive real numbers and the set $\mathbb{R}$ of real numbers, and therefore there exists a well-defined function $\exp\colon \mathbb{R} \to \mathbb{R}$ whose value $\exp(t)$ at any real number t is equal to the unique positive real number s satisfying $\log s = t$. This function $\exp\colon \mathbb{R} \to \mathbb{R}$ is the *exponential function*. The range of the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is the set $\mathbb{R}^+$ of positive real numbers. It follows from the definition of the exponential function that $\exp(\log x) = x$ for all positive real numbers x.

Lemma 4.18 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *is differentiable, and*

$$\frac{d}{dx}\,(\exp(x)) = \exp(x)$$

for all real numbers x.

Proof Let t be a real number. Then there exists some positive real number s satisfying $\log s = t$. Now the logarithm function is differentiable at s, and its derivative at s is equal to $1/s$. It follows that

$$s = \lim_{k \to 0} \frac{k}{\log(s+k) - \log s} = \lim_{u \to s} \frac{u - s}{\log u - \log s}.$$

Let some strictly positive number ε be given. Then there exists some strictly positive number η such that

$$s - \varepsilon < \frac{u - s}{\log u - \log s} < s + \varepsilon$$

for all real numbers u satisfying $s - \eta < u < s + \eta$ that are not equal to s. Now $t = \log s$, and therefore $\log(s - \eta) < t < \log(s + \eta)$. Let δ be the minimum of $\log(s + \eta) - t$ and $t - \log(s - \eta)$. Then $\delta > 0$, and, given any real number x that differs from t but satisfies the inequalities $t - \delta < x < t + \delta$, there exists some positive real number u satisfying $s - \eta < u < s + \eta$ for which $x = \log u$. Moreover $u \neq s$, because $x = \log u$, $t = \log s$ and $x \neq t$. $x \neq s$. But then $u = \exp(x)$ and $s = \exp(t)$, and therefore

$$s - \varepsilon < \frac{\exp(x) - \exp(t)}{x - t} < s + \varepsilon.$$

Thus, given any positive real number ε, there exists some positive real number δ such that

$$\exp(t) - \varepsilon < \frac{\exp(t+h) - \exp(t)}{h} < \exp(t) + \varepsilon.$$

for all real numbers h satisfying $0 < |h| < \delta$. It follows that

$$\lim_{h \to 0} \frac{\exp(t+h) - \exp(t)}{h} = \exp(t),$$

as required. ∎

4.10 Continuous Differentiability and Smoothness

Definition An *open set* in $\mathbb{R}$ is a subset D of $\mathbb{R}$ with the property that, given any element s of D, there exists some strictly positive real number δ such that every real number x satisfying $|x - s| < \delta$ belongs to the set D.

Definition Let $f\colon D \to \mathbb{R}$ be a real valued function defined on an open set D in $\mathbb{R}$. The function f is said to be *k-times continuously differentiable* (or C^k) on D if the function f itself and its first k derivatives $f', f'', \ldots, f^{(k)}$ are well-defined and continuous on D.

Definition Let $f\colon D \to \mathbb{R}$ be a real valued function defined on an open set D in $\mathbb{R}$. The function f is said to be *smooth* (or C^∞) on D if the function f itself and its derivatives $f', f'', f''', \ldots$ of all orders are well-defined and continuous on D.

Sums, differences and products of smooth functions are smooth. Also a quotient of a smooth function by another smooth function that is everywhere non-zero is itself smooth.

In particular polynomial functions are smooth, and the sine, cosine, tangent, logarithm and exponential functions are smooth where they are defined.

Lemma 4.19 *Let $f\colon D \to \mathbb{R}$ and $g\colon E \to \mathbb{R}$ be smooth functions defined over open subsets D and E of $\mathbb{R}$, where $f(D) \subset E$. Then the composition function $g \circ f\colon D \to \mathbb{R}$ is smooth.*

Proof Let $f^{(0)} = f$, $g^{(0)} = g$, $f^{(1)} = f'$, $g^{(1)} = g'$ etc., and let $\mathcal{C}$ denote the the collection of functions that either are of the form $g^{(k)} \circ f$ for some non-negative integer k or else are of the form

$$(g^{(k)} \circ f) \cdot f^{(j_1)} \cdot f^{(j_2)} \cdot \cdots \cdot f^{(j_m)}$$

for some non-negative integer k and positive integers $j_1, j_2, \ldots, j_m$. Now it follows from the Chain Rule (Proposition 4.4) and the Product Rule (Proposition 4.2) that any function belonging to this collection $\mathcal{C}$ is differentiable, and moreover the derivative of a function belonging to $\mathcal{C}$ either belongs itself to $\mathcal{C}$ or else is expressible as a sum of functions belonging to the collection $\mathcal{C}$. Thus any function expressible as a sum of functions belonging to $\mathcal{C}$ is differentiable, and its derivative is expressible as a sum of functions belonging to the collection $\mathcal{C}$. It follows that any function belonging to the collection $\mathcal{C}$ is smooth. In particular, the composition function $g \circ f$ is smooth, as required. ∎

4.11 Taylor's Theorem for Functions of One Real Variable

A subset I of $\mathbb{R}$ is an *interval* if and only if $(s, u) \subset I$ for all $s, u \in I$, where

$$(s, u) = \{x \in \mathbb{R} : s \leq x \leq u\}.$$

Thus a subset I of $\mathbb{R}$ is an interval if and only if, given real numbers s, x and u satisfying $s < x < u$ for which $s \in I$ and $u \in I$, the real number x also satisfies $x \in I$. An *open interval* is an interval that is also an open set in $\mathbb{R}$. Given real numbers c and d satisfying $c < d$, the intervals (c, d), $(c, +\infty)$ and $(-\infty, d)$ are open intervals, as is the whole real line $\mathbb{R}$. It is a straightforward exercise to verify, using the Least Upper Bound Principle, that all open intervals in $\mathbb{R}$ conform to one of the types just described.

Lemma 4.20 *Let s and h be real numbers, let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$, let $c_0, c_1, \ldots, c_{k-1}$ be real numbers, and let*

$$p(t) = f(s + th) - \sum_{n=0}^{k-1} c_n t^n.$$

for all real numbers t belonging to some open interval I for which $0 \in I$ and $1 \in I$. Then $p^{(n)}(0) = 0$ for all integers n satisfying $0 \leq n < k$ if and only if

$$c_n = \frac{h^n f^{(n)}(s)}{n!}$$

for all integers n satisfying $0 \leq n < k$.

Proof On setting $t = 0$, we find that $p(0) = f(s) - c_0$, and thus $p(0) = 0$ if and only if $c_0 = f(s)$.

Let the integer n satisfy $0 < n < k$. On differentiating the function p n times (using in particular the Chain Rule to differentiate $f(s + th)$ and its derivatives as functions of t), we find that

$$p^{(n)}(t) = h^n f^{(n)}(s + th) - \sum_{j=n}^{k-1} \frac{j!}{(j-n)!} c_j t^{j-n}.$$

Then, on setting $t = 0$, we find that only the term with $j = n$ contributes to the value of the sum on the right hand side of the above identity, and therefore

$$p^{(n)}(0) = h^n f^{(n)}(s) - n! c_n.$$

The result follows. ∎

Theorem 4.21 (Taylor's Theorem) *Let s and h be real numbers, and let f be a k times differentiable real-valued function defined on some open interval containing s and $s + h$. Then*

$$f(s + h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h)$$

for some real number θ satisfying $0 < \theta < 1$.

Proof Let I be an open interval, containing the real numbers 0 and 1, chosen to ensure that $f(s + th)$ is defined for all $t \in I$, and let $p: I \to \mathbb{R}$ be defined so that

$$p(t) = f(s + th) - f(s) - \sum_{n=1}^{k-1} \frac{t^n h^n}{n!} f^{(n)}(s)$$

for all $t \in I$. A straightforward calculation shows that $p^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$ (see Lemma 4.20). Thus if $q(t) = p(t) - p(1)t^k$ for all $s \in [0, 1]$ then $q^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k - 1$, and $q(1) = 0$. We can therefore apply Rolle's Theorem (Theorem 4.5) to the function q on the interval $[0, 1]$ to deduce the existence of some real number t_1 satisfying $0 < t_1 < 1$ for which $q'(t_1) = 0$. We can then apply Rolle's Theorem to the function q' on the interval $[0, t_1]$ to deduce the existence of some real number t_2 satisfying $0 < t_2 < t_1$ for which $q''(t_2) = 0$. By continuing in this fashion, applying Rolle's Theorem in turn to the functions $q'', q''', \ldots, q^{(k-1)}$, we deduce the existence of real numbers $t_1, t_2, \ldots, t_k$ satisfying $0 < t_k < t_{k-1} < \cdots < t_1 < 1$ with the property that $q^{(n)}(t_n) = 0$ for $n = 1, 2, \ldots, k$. Let $\theta = t_k$. Then $0 < \theta < 1$ and

$$0 = \frac{1}{k!} q^{(k)}(\theta) = \frac{1}{k!} p^{(k)}(\theta) - p(1) = \frac{h^k}{k!} f^{(k)}(s + \theta h) - p(1),$$

hence

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + p(1) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s+\theta h),$$

as required. ∎

Corollary 4.22 *Let $f\colon D \to \mathbb{R}$ be a k-times continuously differentiable function defined over an open subset D of $\mathbb{R}$ and let $s \in \mathbb{R}$. Then given any strictly positive real number ε, there exists some strictly positive real number δ such that*

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| < \varepsilon |h|^k$$

whenever $|h| < \delta$.

Proof The function f is k-times continuously differentiable, and therefore its kth derivative $f^{(k)}$ is continuous. Let some strictly positive real number ε be given. Then there exists some strictly positive real number δ that is small enough to ensure that $s + h \in D$ and $|f^{(k)}(s+h) - f^{(k)}(s)| < k!\varepsilon$ whenever $|h| < \delta$. If h is an real number satisfying $|h| < \delta$, and if θ is a real number satisfying $0 < \theta < 1$, then $s + \theta h \in D$ and $|f^{(k)}(s + \theta h) - f^(k)(s)| < k!\varepsilon$. Now it follows from Taylor's Theorem (Theorem 4.21) that, given any real number h satisfying $|h| < \delta$ there exists some real number θ satisfying $0 < \theta < 1$ for which

$$f(s+h) = f(s) + \sum_{n=1}^{k-1} \frac{h^n}{n!} f^{(n)}(s) + \frac{h^k}{k!} f^{(k)}(s + \theta h).$$

Then

$$\left| f(s+h) - f(s) - \sum_{n=1}^{k} \frac{h^n}{n!} f^{(n)}(s) \right| = \frac{|h|^k}{k!} |f^{(k)}(s+\theta h) - f^(k)(s)| < \varepsilon |h|^k,$$

as required. ∎

Corollary 4.23 *The exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ *satisfies*

$$\exp(x) = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

for all real numbers x.

Proof The derivative of the exponential function is the exponential function itself (Lemma 4.18). It follows from Taylor's Theorem (Theorem 4.21) that

$$\exp x = \sum_{n=0}^{m} \frac{x^n}{n!} + \frac{x^{m+1}}{(m+1)!} \exp(\theta x)$$

for some real number θ satisfying $0 < \theta < 1$. It follows that

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \leq b_{m+1}(x) \exp(|x|),$$

where

$$b_n(x) = \frac{|x|^n}{n!}$$

for all real numbers x and non-negative integers n. Note that $b_n(x) \geq 0$ for all real numbers x and non-negative integers n.

Let N be some positive integer satisfying $N \geq 2|x|$. If n is a positive integer satisfying $n \geq N$ then $n + 1 > 2|x|$, and therefore

$$b_{n+1}(x) = \frac{|x|}{n+1} \times b_n(x) < \tfrac{1}{2} b_n(x).$$

It follows that $0 \leq b_n(x) < \dfrac{1}{2^{n-N}} b_N(x)$ whenever $n \geq N$, and therefore $\lim\limits_{n \to +\infty} b_n(x) = 0$. Thus

$$\left| \exp x - \sum_{n=0}^{m} \frac{x^n}{n!} \right| \to 0$$

as $m \to +\infty$, and thus

$$\exp x = \lim_{m \to +\infty} \sum_{n=0}^{m} \frac{x^n}{n!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!},$$

as required. ∎

Corollary 4.24 *The sine function* $\sin\colon \mathbb{R} \to \mathbb{R}$ *and cosine function* $\cos\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$\sin x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \textit{and} \quad \cos x = \sum_{k=0}^{+\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

for all real numbers x.

Proof The derivatives of the sine function are given by

$$\sin^{(2k)}(x) = (-1)^k \sin(x) \quad \text{and} \quad \sin^{(2k+1)}(x) = (-1)^k \cos(x)$$

for all positive integers k. It follows from Taylor's Theorem that, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\sin x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \frac{(-1)^{m+1} x^{2m+3}}{(2m+3)!} \cos(\theta x)$$

(The value of θ will depend on x and m.) It follows that

$$\left| \sin x - \sum_{k=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \right| \leq b_{2m+3}(x),$$

for all non-negative integers m, where $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But it was shown in the proof of Corollary 4.23 that $\lim\limits_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\sin x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{n=0}^{+\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}.$$

Similarly the derivatives of the cosine function are given by

$$\cos^{(2k)}(x) = (-1)^k \cos(x) \quad \text{and} \quad \cos^{(2k-1)}(x) = (-1)^k \sin(x)$$

for all positive integers k. Therefore, given any real number x, and given any non-negative integer m, there exists some θ satisfying $0 < \theta < 1$ such that

$$\cos x = \sum_{k=0}^{m} \frac{(-1)^k x^{2k}}{(2k)!} + \frac{(-1)^{k+1} x^{2m+2}}{(2k+2)!} \cos(\theta x)$$

But then

$$\left| \cos x - \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} \right| \leq b_{2m+2}(x),$$

where, as before, $b_n(x) = |x|^n/n!$ for all real numbers x and non-negative integers n. But $\lim_{n\to+\infty} b_n(x) = 0$ for all real numbers x. It follows that

$$\cos x = \lim_{m\to+\infty} \sum_{n=0}^{m} \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

as required. ▮

4.12 Real-Analytic Functions

Definition A real-valued function $f\colon D \to \mathbb{R}$ defined over an open subset D of the set $\mathbb{R}$ of real numbers is said to be *real-analytic* if, given any real number s belonging to the domain D of the function, there exists some strictly positive real number δ such that

$$f(s+h) = f(s) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(s)$$

for all real numbers h satisfying $|h| < \delta$.

It can be shown that sums, differences, products, quotients and compositions of real-analytic functions are themselves real-analytic over their domains of definition. In particular, polynomial functions and quotients of polynomial functions are real-analytic. The natural logarithm function is real-analytic over the set of positive real numbers because its derivative is real-analytic. It follows from Corollary 4.23 that the exponential function is real-analytic. and it follows from Corollary 4.24 that the sine and cosine functions are real-analytic. Inverses of real-analytic functions are real-analytic.

All real-analytic functions are smooth. However not all smooth functions are real-analytic.

4.13 Smooth Functions that are not the Sum of their Taylor Series

Let f be an infinitely differentiable real-valued function defined around some real number a. The infinite series

$$f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a)$$

is referred to as the *Taylor expansion* of the function f about a. For many functions, typically including those constructed from polynomial functions, logarithm functions, exponential functions, trigonometrical functions and their inverses, identities of the form

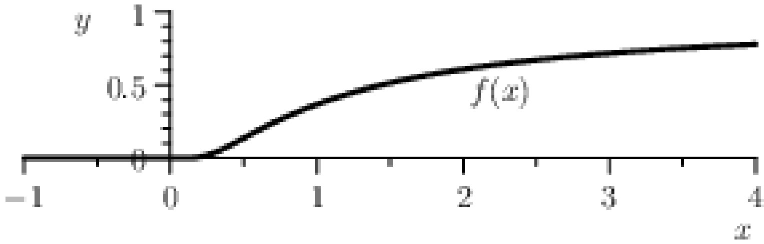
$$f(a+h) = f(a) + \sum_{n=1}^{+\infty} \frac{h^n}{n!} f^{(n)}(a) = f(a) + \lim_{m\to+\infty} \left(\sum_{n=1}^{m} \frac{h^n}{n!} f^{(n)}(a) \right)$$

for all sufficiently small values of h. Such functions are said to be *real-analytic*. However there exist functions whose Taylor expansion about some real number a does not converge to the given function for any non-zero value of h. Such a function is the subject of the following lemma.

Proposition 4.25 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that*

$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Then the function $f\colon \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. In particular $f^{(k)}(0) = 0$ for all positive integers k.



Proof We show by induction on k that the function f is k times differentiable on $\mathbb{R}$ and $f^{(k)}(0) = 0$ for all positive integers k. Now it follows from standard rules for differentiating functions that

$$f^{(k)}(x) = \frac{p_k(x)}{x^{2k}} \exp\left(-\frac{1}{x}\right)$$

for all strictly positive real numbers x, where $p_1(x) = 1$ and

$$p_{k+1}(x) = x^2 p_k'(x) + (1 - 2kx)p_k(x)$$

for all k. A straightforward proof by induction shows that $p_k(x)$ is a polynomial in x of degree $k-1$ for all positive integers k with leading term $(-1)^{k-1}k!x^{k-1}$.

Now

$$\frac{d}{dt}\left(t^n e^{-t}\right) = t^{n-1}(n-t)e^{-t}$$

for all positive real numbers t. It follows that function sending each positive real number t to $t^n e^{-t}$ is increasing when $0 \le t < n$ and decreasing when $t > n$, and therefore $t^n e^{-t} \le M_n$ for all positive real numbers t, where $M_n = n^n e^{-n}$. It follows that

$$0 \le \frac{1}{x^{2k+1}} \exp\left(-\frac{1}{x}\right) \le M_{2k+2}x$$

for all positive real numbers x, and therefore

$$\lim_{h\to 0^+} \frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right) = 0.$$

It then follows that

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f^{(k)}(h)}{h} &= \lim_{h\to 0^+} \left(\frac{p_k(h)}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\
&= \lim_{h\to 0^+} p_k(h) \times \lim_{h\to 0^+} \left(\frac{1}{h^{2k+1}} \exp\left(-\frac{1}{h}\right)\right) \\
&= p_k(0) \times 0 = 0
\end{aligned}$$

for all positive integers k.

Now

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{f(h)}{h} = 0 = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}.$$

It follows that the function f is differentiable at zero, and $f'(0) = 0$.

Suppose that the function $f(x)$ is k-times differentiable at zero for some positive integer k, and that $f^{(k)}(0) = 0$. Then

$$\lim_{h\to 0^+} \frac{f^{(k)}(h)-f^{(k)}(0)}{h} = \lim_{h\to 0^+} \frac{f^{(k)}(h)}{h} = 0 = \lim_{h\to 0^-} \frac{f^{(k)}(h)-f^{(k)}(0)}{h}.$$

It then follows that the function $f^{(k)}$ is differentiable at zero, and moreover the derivative $f^{(k+1)}(0)$ of this function at zero is equal to zero. The function f is thus $(k+1)$-times differentiable at zero.

It now follows by induction on k that $f^{(k)}(x)$ exists for all positive integers k and real numbers x, and moreover

$$f^{(k)}(x) = \begin{cases} \dfrac{p_k(x)}{x^{2k}} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

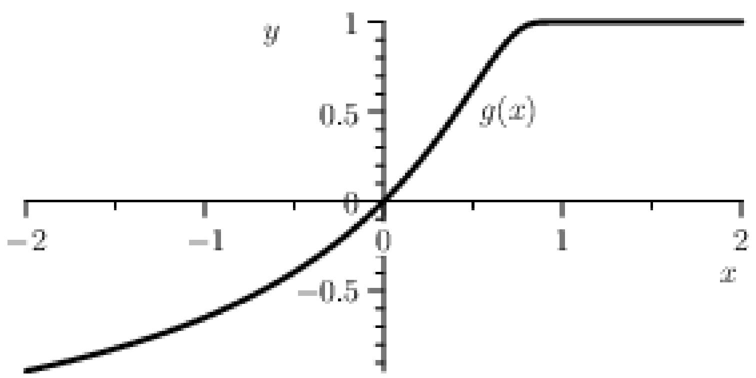
The function $f: \mathbb{R} \to \mathbb{R}$ is thus a smooth function, as required. ∎

Remark Note that the function $f: \mathbb{R} \to \mathbf{R}$ defined in the statement of Lemma 4.25 has a well-defined Taylor expansion about $x = 0$. Moreover all the terms of this Taylor expansion are zero, and therefore the Taylor expansion of f converges to the zero function. This function therefore provides an example of a function where the Taylor expansion is well-defined but does not converge to the given function.

Corollary 4.26 *Let $g: \mathbb{R} \to \mathbb{R}$ be the function mapping the set $\mathbb{R}$ of real numbers to itself defined such that*

$$g(x) = \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \textit{if } x < 0; \\ 1 & \textit{if } x \geq 1. \end{cases}$$

Then the function $g: \mathbb{R} \to \mathbb{R}$ is smooth on $\mathbb{R}$. Moreover the function g is a strictly increasing function on $\{x \in \mathbb{R} : x < 1\}$, and $g(0) = 0$.



Proof Let $f: \mathbb{R} \to \mathbb{R}$ be the real-valued function defined on the set $\mathbb{R}$ of real numbers so that

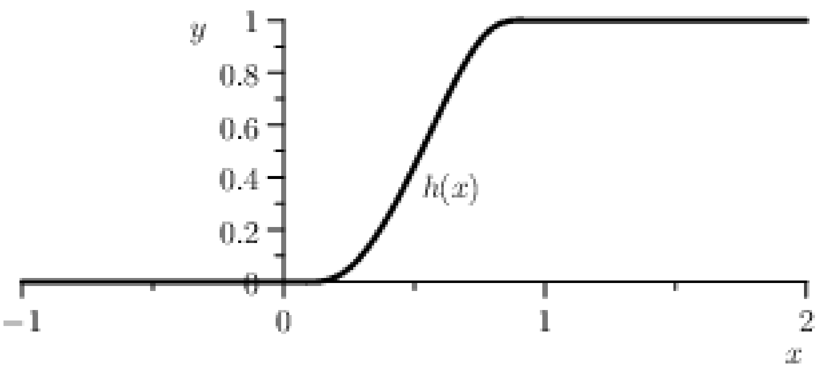
$$f(x) = \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0; \\ 0 & \text{if } x \leq 0. \end{cases}$$

Now
$$-\frac{x}{1-x} = 1 - \frac{1}{1-x}$$
for all real numbers x. It follows from the definition of the functions f and g that $g(x) = 1 - ef(1-x)$ for all real numbers x, where $e = \exp(1)$. Now Proposition 4.25 ensures that the function f is smooth on $\mathbb{R}$. It follows that the function g is also smooth on $\mathbb{R}$. Also $g(0) = 0$. Now $f(1-x)$ is a strictly decreasing function of x on $\{x \in \mathbb{R} : x < 1\}$. It follows that the function g is strictly increasing on that set, as required. ∎

Corollary 4.27 *Let $h\colon \mathbb{R} \to \mathbb{R}$ be defined such that $h(x) = g(f(x)/f(1))$ for all real numbers x, where*

$$\begin{aligned} f(x) &= \begin{cases} \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \\ g(x) &= \begin{cases} 1 - \exp\left(-\dfrac{x}{1-x}\right) & \text{if } x < 0; \\ 1 & \text{if } x \geq 1. \end{cases} \end{aligned}$$

Then the function $h\colon \mathbb{R} \to \mathbb{R}$ is smooth, $h(x) = 0$ whenever $x \leq 0$, $h(1) = 1$ whenever $x \geq 1$, and $h(x)$ is a strictly increasing function of x when restricted to the interval $\{x \in \mathbb{R} : 0 < x < 1\}$.



Proof The function h is a composition of smooth functions, and is therefore smooth (see Lemma 4.19). If $x \leq 0$ then $h(x) = g(f(0)) = g(0) = 0$. If $x \geq 1$ then $f(x)/f(1) \geq 1$ and therefore $h(x) = 1$. The function sending a real number x satisfying $0 < x < 1$ to $f(x)/f(1)$ is strictly increasing on the interval $(0, 1)$ and maps that interval into itself. Also the function g is strictly increasing on the interval $(0, 1)$. Thus the function h restricted to the interval $(0, 1)$ is a composition of two strictly increasing functions, and is thus itself strictly increasing, as required. ∎

4.14 Historical Note

Representation of functions as sums of infinite series have been known to mathematicians for centuries. The standard representation of the sine and cosine and arctangent functions was known to, and presumably discovered by, Madhava of Sangramagrama (c. 1340–c. 1425), whose work gave impetus to the flourishing of the study of astronomy and mathematics in Kerala, in southern India. The theory of infinite series was extensively developed in Western Europe in the 17th century, with Isaac Newton (1642–1726/7) being particularly active in the field. Isaac Newton's manuscript on the *Method of fluxions and infinite series* was completed in 1671, and was posthumously published in 1736.

In 1797, Joseph-Louis Lagrange published his *Théorie des fonctions analytiques*. One of the primary aims of this book was to develop an approach to the principles of differential and integral calculus taking as its starting point the principle that functions of a real variable studied by mathematicians could be represented around a particular value through an infinite series expansion, so that, in particular, an analytic function $f(x)$ defined for values of x close to some given value s could be represented through an infinite series expansion of the form

$$f(s+h) = \sum_{n=0}^{+\infty} a_n h^n$$

for all sufficiently small values of the increment h. Lagrange defined the derivative of such a function f to be the function $f'(x)$ whose infinite series expansion takes the form

$$f'(s+h) = \sum_{n=1}^{+\infty} n a_n h^{n-1}.$$

Lagrange intended that his theory of analytic functions would supply an approach to the foundations of calculus that required neither "infinitesimal quantities" nor the use of limits.

In 1830, William Rowan Hamilton published a paper in the *Transactions of the Royal Irish Academy* entitled *On the Error of a received Principle of Analysis, respecting Functions which vanish with their Variables.* In this paper, Hamilton pointed that the function whose value at x, for non-zero real numbers x, is $e^{-x^{-2}}$ cannot be expressed around zero as the sum of a power series. The following year Hamilton published a note in the *Transactions of the Royal Irish Academy* to put on record the fact that, prior to Hamilton's earlier paper, Cauchy had published a paper citing this same function as an example of a function whose derivatives at zero of all orders are all equal to zero though the function itself takes non-zero values at non-zero values of its argument.

These examples demonstrated that the theory of calculus could not be founded on the assumption that all functions relevant to mathematical analysis could be represented as sums of power series in the neighbourhood of any value at which they are defined. Accordingly mathematicians in the nineteenth century returned to the approach of justifying the basic principles of differential and integral calculus on the theory of limits and quadratures. A theory of limits had already been employed by Isaac Newton, using the terminology of *prime and ultimate ratios*. However the concept of limit employed by Newton was only applicable to variable geometrical quantities that approached their limiting values monotonically. The Newton version of the limit concept was not applicable to functions such as $x \sin\left(\frac{1}{x}\right)$ which oscillates round zero as the value of x approaches zero from above, but nevertheless can be made to approximate to zero to within any given margin of error, provided that the value of x is sufficiently close to zero. The theory of limits was accordingly generalized and further developed in the nineteenth century by mathematicians such as Bolzano (1781–1848) and Cauchy (1789–1857) to cover such situations. The generalized concept of limit developed by Bolzano and Cauchy proved to be more appropriate to serve as the basis for defining the basic concepts and proving the basic theorems that justify the principles of calculus. The definitive treatment of mathematical analysis was provided by Karl Weierstrass (1815–1897), whose lectures at Berlin established the standard approach to the foundations of real and complex analysis through the use of "epsilon-delta" definitions and proofs, together with the systematic use of standard theorems such as the Bolzano-Weierstrass Theorem.

5 The Riemann Integral

The approach to the theory of integration discussed below was developed by Jean-Gaston Darboux (1842–1917). The integral defined using lower and upper sums in the manner described below is sometimes referred to as the *Darboux integral* of a function on a given interval. However the class of functions that are integrable according to the definitions introduced by Darboux is the class of *Riemann-integrable* functions. Thus the approach using Darboux sums provides a convenient approach to define and establish the basic properties of the *Riemann integral*.

A *partition* P of an interval $[a, b]$ is a set $\{x_0, x_1, x_2, \ldots, x_n\}$ of real numbers satisfying $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

Given any bounded real-valued function f on $[a, b]$, the *lower sum* (or *lower Darboux sum*) $L(P, f)$ and the *upper sum* (or *upper Darboux sum*) $U(P, f)$ of f for the partition P of $[a, b]$ are defined by

$$L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \qquad U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$. Clearly $L(P, f) \leq U(P, f)$. Moreover $\sum\limits_{i=1}^{n}(x_i - x_{i-1}) = b - a$, and therefore

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a),$$

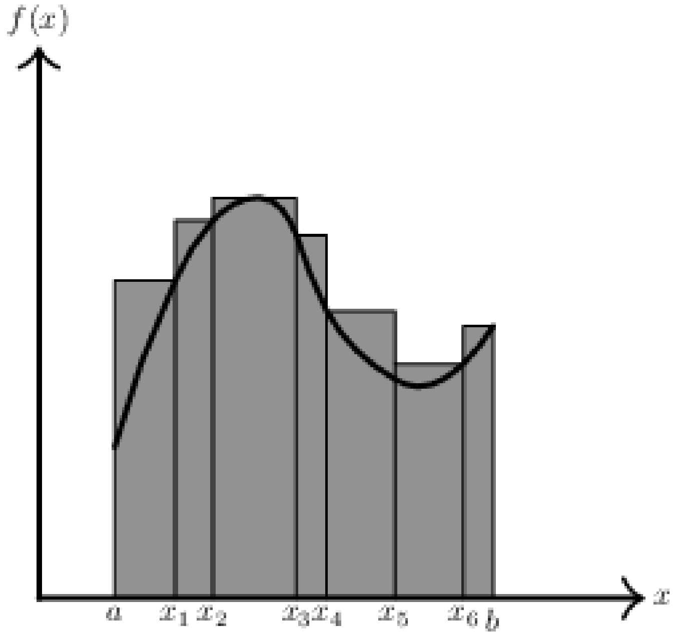
for any real numbers m and M satisfying $m \leq f(x) \leq M$ for all $x \in [a, b]$.

Definition Let f be a bounded real-valued function on the interval $[a, b]$, where $a < b$. The *upper Riemann integral* $\mathcal{U}\int_a^b f(x)\,dx$ (or *upper Darboux integral*) and the *lower Riemann integral* $\mathcal{L}\int_a^b f(x)\,dx$ (or *lower Darboux integral*) of the function f on $[a, b]$ are defined by

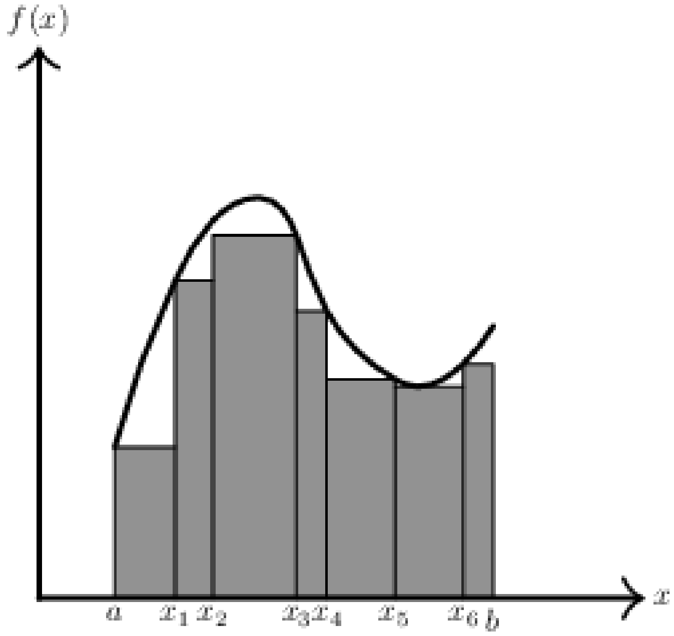
$$\begin{aligned} \mathcal{U}\int_a^b f(x)\,dx &\equiv \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}, \\ \mathcal{L}\int_a^b f(x)\,dx &\equiv \sup\{L(P, f) : P \text{ is a partition of } [a, b]\} \end{aligned}$$

(i.e., $\mathcal{U}\int_a^b f(x)\,dx$ is the infimum of the values of $U(P, f)$ and $\mathcal{L}\int_a^b f(x)\,dx$ is the supremum of the values of $L(P, f)$ as P ranges over all possible partitions of the interval $[a, b]$). If

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx$$



The upper sum $U(P, f)$



The lower sum $L(P, f)$

then the function f is said to be *Riemann-integrable* (or *Darboux-integrable*) on $[a, b]$, and the *Riemann integral* $\int_a^b f(x)\,dx$ (or *Darboux integral*) of f on $[a, b]$ is defined to be the common value of $\mathcal{U}\int_a^b f(x)\,dx$ and $\mathcal{L}\int_a^b f(x)\,dx$.

When $a > b$ we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

for all Riemann-integrable functions f on $[b, a]$. We set $\int_a^b f(x) = 0$ when $b = a$.

If f and g are bounded Riemann-integrable functions on the interval $[a, b]$, and if $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$, since $L(P, f) \leq L(P, g)$ and $U(P, f) \leq U(P, g)$ for all partitions P of $[a, b]$.

Definition Let P and R be partitions of $[a, b]$, given by $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$. We say that the partition R is a *refinement* of P if $P \subset R$, so that, for each x_i in P, there is some u_j in R with $x_i = u_j$.

Lemma 5.1 *Let R be a refinement of some partition P of $[a, b]$. Then*

$$L(R, f) \geq L(P, f) \quad \textit{and} \quad U(R, f) \leq U(P, f)$$

for any bounded function $f \colon [a, b] \to \mathbb{R}$.

Proof Let $P = \{x_0, x_1, \ldots, x_n\}$ and $R = \{u_0, u_1, \ldots, u_m\}$, where $a = x_0 < x_1 < \cdots < x_n = b$ and $a = u_0 < u_1 < \cdots < u_m = b$. Now for each integer i between 0 and n there exists some integer $j(i)$ between 0 and m such that $x_i = u_{j(i)}$ for each i, since R is a refinement of P. Moreover $0 = j(0) < j(1) < \cdots < j(n) = n$. For each i, let R_i be the partition of $[x_{i-1}, x_i]$ given by $R_i = \{u_j : j(i-1) \leq j \leq j(i)\}$. Then $L(R, f) = \sum_{i=1}^n L(R_i, f)$ and $U(R, f) = \sum_{i=1}^n U(R_i, f)$. Moreover

$$m_i(x_i - x_{i-1}) \leq L(R_i, f) \leq U(R_i, f) \leq M_i(x_i - x_{i-1}),$$

since $m_i \leq f(x) \leq M_i$ for all $x \in [x_{i-1}, x_i]$. On summing these inequalities over i, we deduce that $L(P, f) \leq L(R, f) \leq U(R, f) \leq U(P, f)$, as required. ∎

Given any two partitions P and Q of $[a, b]$ there exists a partition R of $[a, b]$ which is a refinement of both P and Q. For example, we can take $R = P \cup Q$. Such a partition is said to be a *common refinement* of the partitions P and Q.

Lemma 5.2 *Let f be a bounded real-valued function on the interval $[a, b]$. Then*

$$\mathcal{L}\int_a^b f(x)\,dx \le \mathcal{U}\int_a^b f(x)\,dx.$$

Proof Let P and Q be partitions of $[a, b]$, and let R be a common refinement of P and Q. It follows from Lemma 5.1 that $L(P, f) \le L(R, f) \le U(R, f) \le U(Q, f)$. Thus, on taking the supremum of the left hand side of the inequality $L(P, f) \le U(Q, f)$ as P ranges over all possible partitions of the interval $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \le U(Q, f)$ for all partitions Q of $[a, b]$. But then, taking the infimum of the right hand side of this inequality as Q ranges over all possible partitions of $[a, b]$, we see that $\mathcal{L}\int_a^b f(x)\,dx \le \mathcal{U}\int_a^b f(x)\,dx$, as required. ∎

Example Let $f(x) = cx+d$, where $c \ge 0$. We shall show that f is Riemann-integrable on $[0, 1]$ and evaluate $\int_0^1 f(x)\,dx$ from first principles.

For each positive integer n, let P_n denote the partition of $[0, 1]$ into n subintervals of equal length. Thus $P_n = \{x_0, x_1, \ldots, x_n\}$, where $x_i = i/n$. Now the function f takes values between $(i-1)c/n + d$ and $ic/n + d$ on the interval $[x_{i-1}, x_i]$, and therefore

$$m_i = \frac{(i-1)c}{n} + d, \qquad M_i = \frac{ic}{n} + d$$

where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$. Thus

$$\begin{aligned}
L(P_n, f) &= \sum_{i=1}^n m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d - \frac{c}{n}\right) \\
&= \frac{c(n+1)}{2n} + d - \frac{c}{n} = \frac{c}{2} + d - \frac{c}{2n}, \\
U(P_n, f) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^n \left(\frac{ci}{n} + d\right) \\
&= \frac{c(n+1)}{2n} + d = \frac{c}{2} + d + \frac{c}{2n}.
\end{aligned}$$

But $L(P_n, f) \le \mathcal{L}\int_a^b f(x)\,dx \le \mathcal{U}\int_a^b f(x)\,dx \le U(P_n, f)$ for all n. It follows that $\mathcal{L}\int_a^b f(x)\,dx = \frac{1}{2}c + d = \mathcal{U}\int_a^b f(x)\,dx$. Thus f is Riemann-integrable on the interval $[0, 1]$, and $\int_0^1 f(x)\,dx = \frac{1}{2}c + d$.

Example Let $f\colon [0,1] \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational;} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Let P be a partition of the interval $[0,1]$ given by $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$. Then

$$\inf\{f(x) : x_{i-1} \leq x \leq x_i\} = 0, \qquad \sup\{f(x) : x_{i-1} \leq x \leq x_i\} = 1,$$

for $i = 1, 2, \ldots, n$, and thus $L(P, f) = 0$ and $U(P, f) = 1$ for all partitions P of the interval $[0,1]$. It follows that $\mathcal{L}\int_0^1 f(x)\,dx = 0$ and $\mathcal{U}\int_0^1 f(x)\,dx = 1$, and therefore the function f is not Riemann-integrable on the interval $[0,1]$.

It can be shown that sums and products of Riemann-integrable functions are themselves Riemann-integrable.

Proposition 5.3 *Let f be a bounded real-valued function on the interval $[a,c]$. Suppose that f is Riemann-integrable on the intervals $[a,b]$ and $[b,c]$, where $a < b < c$. Then f is Riemann-integrable on $[a,c]$, and*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Proof Let Q and R be any partitions of the intervals $[a,b]$ and $[b,c]$ respectively. These partitions combine to give a partition $Q \cup R$ of the interval $[a,c]$: thus if $Q = \{a, x_1, \ldots, x_{n-1}, b\}$ and $R = \{b, u_1, \ldots, u_{m-1}, c\}$, where

$$a < x_1 < x_2 < \cdots < x_{n-1} < b < u_1 < u_2 < \cdots < u_{m-1} < c,$$

then $Q \cup R = \{a, x_1, \ldots, x_{n-1}, b, u_1, \ldots, u_{m-1}, c\}$. Clearly the lower and upper sums of f satisfy $L(Q, f) + L(R, f) = L(Q \cup R, f)$ and $U(Q, f) + U(R, f) = U(Q \cup R, f)$. It follows that

$$L(Q, f) + L(R, f) \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Taking the supremum of the left hand side of this inequality over all partitions Q of $[a,b]$ and all partitions R of $[b,c]$, we deduce that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \leq \mathcal{L}\int_a^c f(x)\,dx.$$

Similarly $U(Q,f) + U(R,f) \geq \mathcal{U}\int_a^c f(x)\,dx$, and hence

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \geq \mathcal{U}\int_a^c f(x)\,dx.$$

But $\mathcal{L}\int_a^c f(x)\,dx \leq \mathcal{U}\int_a^c f(x)\,dx$ by Lemma 5.2. It follows that

$$\mathcal{L}\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \mathcal{U}\int_a^c f(x)\,dx,$$

as required. ∎

5.1 Integrability of Monotonic functions

Let a and b be real numbers satisfying $a < b$. A real-valued function $f\colon [a,b] \to \mathbb{R}$ defined on the closed bounded interval $[a,b]$ is said to be *non-decreasing* if $f(u) \leq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. Similarly $f\colon [a,b] \to \mathbb{R}$ is said to be *non-increasing* if $f(u) \geq f(v)$ for all real numbers u and v satisfying $a \leq u \leq v \leq b$. The function $f\colon [a,b] \to \mathbb{R}$ is said to be *monotonic* on $[a,b]$ if either it is non-decreasing on $[a,b]$ or else it is non-increasing on $[a,b]$.

Proposition 5.4 *Let a and b be real numbers satisfying $a < b$. Then every monotonic function on the interval $[a,b]$ is Riemann-integrable on $[a,b]$.*

Proof Let $f\colon [a,b] \to \mathbb{R}$ be a non-decreasing function on the closed bounded interval $[a,b]$. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in [a,b]$, and therefore the function f is bounded on $[a,b]$. Let some positive real number ε be given. Let δ be some strictly positive real number for which $(f(b) - f(a))\delta < \varepsilon$, and let P be a partition of $[a,b]$ of the form $P = \{x_0, x_1, x_2, \ldots, x_n\}$, where

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

and $x_i - x_{i-1} < \delta$ for $i = 1, 2, \ldots, n$. Then the maximum and minimum values of $f(x)$ on the interval $[x_{i-1}, x_i]$ are attained at x_i and x_{i-1} respectively, and therefore the upper sum $U(P,f)$ and $L(P,f)$ of f for the partition P satisfy

$$U(P,f) = \sum_{i=1}^n f(x_i)(x_i - x_{i-1}) \quad \text{and} \quad L(P,f) = \sum_{i=1}^n f(x_{i-1})(x_i - x_{i-1}).$$

Moreover $f(x_i) - f(x_{i-1}) \geq 0$ for $i = 1, 2, \ldots, n$. It follows that

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^n (f(x_i) - f(x_{i-1})(x_i - x_{i-1}) \\ &< \delta \sum_{i=1}^n (f(x_i) - f(x_{i-1}) = \delta(f(b) - f(a)) < \varepsilon. \end{aligned}$$

We have thus shown that

$$\mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx < \varepsilon$$

for all strictly positive numbers ε. But $\mathcal{U}\displaystyle\int_a^b f(x)\,dx \geq \mathcal{L}\int_a^b f(x)\,dx$. It follows that

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$.

Now let $f\colon [a, b] \to \mathbb{R}$ be a non-increasing function on $[a, b]$. Then $-f$ is a non-decreasing function on $[a, b]$ and it follows from what we have just shown that $-f$ is Riemann-integrable on $[a, b]$. It follows that the function f itself must be Riemann-integrable on $[a, b]$, as required. ∎

Corollary 5.5 *Let a and b be real numbers satisfing $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a real-valued function on the interval $[a, b]$. Suppose that there exist real numbers $x_0, x_1, \ldots, x_n$, where*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

such that the function f restricted to the interval $[x_{i-1}, x_i]$ is monotonic on $[x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n$. Then f is Riemann-integrable on $[a, b]$.

Proof The result follows immediately on applying the results of Proposition 5.3 and Proposition 5.4. ∎

Remark The result and proof of Proposition 5.4 are to be found in their essentials, though expressed in different language, in Isaac Newton, *Philosophiae naturalis principia mathematica* (1686), Book 1, Section 1, Lemmas 2 and 3.

5.2 Integrability of Continuous functions

Theorem 5.6 *Let a and b be real numbers satisfying $a < b$. Then any continuous real-valued function on the interval $[a, b]$ is Riemann-integrable.*

Proof Let f be a continuous real-valued function on $[a, b]$. It follows from the Extreme Value Theorem (Theorem 3.15) that f is bounded above and below on the interval $[a, b]$.

Let some strictly positive real number ε be given. It follows from Proposition 3.16 that the function f is uniformly continuous on the interval $[a, b]$, and therefore there exists some strictly positive real number δ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in [a, b]$ satisfy $|x - y| < \delta$. Choose a partition P of the interval $[a, b]$ such that each subinterval in the partition has length less than δ. Write $P = \{x_0, x_1, \ldots, x_n\}$, where $a = x_0 < x_1 < \cdots < x_n = b$. Now if $x_{i-1} \le x \le x_i$ then $|x - x_i| < \delta$, and hence $f(x_i) - \varepsilon < f(x) < f(x_i) + \varepsilon$. It follows that

$$f(x_i) - \varepsilon \le m_i \le M_i \le f(x_i) + \varepsilon \qquad (i = 1, 2, \ldots, n),$$

where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$. Therefore

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) - \varepsilon(b - a) &\le L(P, f) \le U(P, f) \\ &\le \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) + \varepsilon(b - a),\end{aligned}$$

where $L(P, f)$ and $U(P, f)$ denote the lower and upper sums of the function f for the partition P, and hence

$$0 \le \mathcal{U}\int_a^b f(x)\,dx - \mathcal{L}\int_a^b f(x)\,dx \le U(P, f) - L(P, f) \le 2\varepsilon(b - a).$$

But this inequality must be satisfied for any strictly positive real number ε. Therefore

$$\mathcal{U}\int_a^b f(x)\,dx = \mathcal{L}\int_a^b f(x)\,dx,$$

and thus the function f is Riemann-integrable on $[a, b]$. ∎

5.3 The Fundamental Theorem of Calculus

Let a and b be real numbers satisfying $a < b$. One can show that all continuous functions on the interval $[a, b]$ are Riemann-integrable (see Theorem 5.6). However the task of calculating the Riemann integral of a continuous function directly from the definition is difficult if not impossible for all but the simplest functions. Thus to calculate such integrals one makes use of the Fundamental Theorem of Calculus.

Theorem 5.7 (The Fundamental Theorem of Calculus) *Let f be a continuous real-valued function on the interval $[a, b]$, where $a < b$. Then*

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

for all x satisfying $a < x < b$.

Proof Let $F(s) = \int_a^s f(t)\,dt$ for all $s \in (a, b)$. Now the function f is continuous at x, where $a < x < b$. Thus, given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(t) - f(x)| < \frac{1}{2}\varepsilon$ for all $t \in [a, b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x) = \frac{1}{h}\int_x^{x+h} (f(t) - f(x))\,dt.$$

But if $0 < |h| < \delta$ and $x + h \in [a, b]$ then $\left|\int_x^{x+h}(f(t) - f(x))\,dt\right| \leq \frac{1}{2}\varepsilon|h|$, and thus

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

It follows that

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = f(x),$$

as required. ∎

Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function on a closed interval $[a, b]$. We say that f is *continuously differentiable* on $[a, b]$ if the derivative $f'(x)$ of f exists for all x satisfying $a < x < b$, the one-sided derivatives

$$f'(a) = \lim_{h\to 0^+}\frac{f(a+h) - f(a)}{h}, \qquad f'(b) = \lim_{h\to 0^-}\frac{f(b+h) - f(b)}{h}$$

exist at the endpoints of $[a, b]$, and the function f' is continuous on $[a, b]$.

If $f\colon [a, b] \to \mathbb{R}$ is continuous, and if $F(x) = \int_a^x f(t)\,dt$ for all $x \in [a, b]$ then the one-sided derivatives of F at the endpoints of $[a, b]$ exist, and

$$\lim_{h\to 0^+}\frac{F(a+h) - F(a)}{h} = f(a), \qquad \lim_{h\to 0^-}\frac{F(b+h) - F(b)}{h} = f(b).$$

One can verify these results by adapting the proof of the Fundamental Theorem of Calculus.

Corollary 5.8 *Let f be a continuously differentiable real-valued function on the interval $[a, b]$. Then*

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

Proof Define $g: [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \int_a^x \frac{df(t)}{dt}\, dt.$$

Then $g(a) = 0$, and

$$\frac{dg(x)}{dx} = \frac{df(x)}{dx} - \frac{d}{dx}\left(\int_a^x \frac{df(t)}{dt}\, dt\right) = 0$$

for all x satisfing $a < x < b$, by the Fundamental Theorem of Calculus. Now it follows from the Mean Value Theorem (Theorem 4.6) that there exists some s satisfying $a < s < b$ for which $g(b) - g(a) = (b-a)g'(s)$. We deduce therefore that $g(b) = 0$, which yields the required result. ∎

Corollary 5.9 (Integration by Parts) *Let f and g be continuously differentiable real-valued functions on the interval $[a,b]$. Then*

$$\int_a^b f(t)\frac{dg(x)}{dx}\, dx = f(b)g(b) - f(a)g(a) - \int_a^b \frac{df(x)}{dx} g(x)\, dx.$$

Proof This result follows from Corollary 5.8 on integrating the identity

$$f(x)\frac{dg(x)}{dx} = \frac{d}{dx}\left(f(x)g(x)\right) - \frac{df(x)}{dx} g(x).$$ ∎

Corollary 5.10 (Integration by Substitution) *Let $u: [a,b] \to \mathbb{R}$ be a continuously differentiable monotonically increasing function on the interval $[a,b]$, and let $c = u(a)$ and $d = u(b)$. Then*

$$\int_c^d f(x)\, dx = \int_a^b f(u(t))\frac{du(t)}{dt}\, dt.$$

for all continuous real-valued functions f on $[c,d]$.

Proof Let F and G be the functions on $[a,b]$ defined by

$$F(x) = \int_c^{u(x)} f(y)dy, \qquad G(x) = \int_a^x f(u(t))\frac{du(t)}{dt} dt.$$

Then $F(a) = 0 = G(a)$. Moreover $F(x) = H(u(x))$, where

$$H(s) = \int_c^s f(y)\, dy,$$

and $H'(s) = f(s)$ for all $s \in [a,b]$. Using the Chain Rule and the Fundamental Theorem of Calculus, we deduce that

$$F'(x) = H'(u(x))u'(x) = f(u(x))u'(x) = G'(x)$$

for all $x \in (a,b)$. On applying the Mean Value Theorem (Theorem 4.6) to the function $F - G$ on the interval $[a,b]$, we see that $F(b) - G(b) = F(a) - G(a) = 0$. Thus $F(b) = G(b) = H(d)$, which yields the required identity. ∎

5.4 Interchanging Limits and Integrals, Uniform Convergence

Let $f_1, f_2, f_3, \ldots$ be a sequence of Riemann-integrable functions defined over the interval $[a, b]$, where a and b are real numbers satisfying $a \leq b$. Suppose that the sequence $f_1(x), f_2(x), f_3(x)$ converges for all $x \in [a, b]$. We wish to determine whether or not

$$\lim_{j \to +\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

The following example demonstrates that this identity can fail to hold, even when the functions involved are well-behaved polynomial functions.

Example Let $f_1, f_2, f_3, \ldots$ be the sequence of continuous functions on the interval $[0, 1]$ defined by $f_j(x) = j(x^j - x^{2j})$. Now

$$\lim_{j \to +\infty} \int_0^1 f_j(x)\,dx = \lim_{j \to +\infty} \left(\frac{j}{j+1} - \frac{j}{2j+1} \right) = \frac{1}{2}.$$

On the other hand, we shall show that $\lim_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$. Thus one cannot interchange limits and integrals in this case.

Suppose that $0 \leq x < 1$. We claim that $jx^j \to 0$ as $j \to +\infty$. To verify this, choose u satisfying $x < u < 1$. Then $0 \leq (j+1)u^{j+1} \leq nu^j$ for all positive integers j satisfying $j > u/(1-u)$. Therefore there exists some constant B with the property that $0 \leq nu^j \leq B$ for all positive integers j. But then $0 \leq jx^j \leq B(x/u)^j$ for all positive integers j, and $(x/u)^j \to 0$ as $j \to +\infty$. Therefore $jx^j \to 0$ as $j \to +\infty$, as claimed. It follows that

$$\lim_{j \to +\infty} f_j(x) = \left(\lim_{j \to +\infty} jx^j \right) \left(\lim_{j \to +\infty} (1 - x^j) \right) = 0$$

for all x satisfying $0 \leq x < 1$. Also $f_j(1) = 0$ for all positive integers j. We conclude that $\lim_{j \to +\infty} f_j(x) = 0$ for all $x \in [0, 1]$, which is what we set out to show.

We now introduce the concept of *uniform convergence*. Later shall show that, given a sequence $f_1, f_2, f_3, \ldots$ of Riemann-integrable functions on some interval $[a, b]$, the identity

$$\lim_{j \to +\infty} \int_a^b f_j(x)\,dx = \int_a^b \left(\lim_{j \to +\infty} f_j(x) \right) dx.$$

is valid, provided that the sequence $f_1, f_2, f_3, \ldots$ of functions converges *uniformly* on the interval $[a, b]$.

Definition Let $f_1, f_2, f_3, \ldots$ be a sequence of real-valued functions defined on some subset D of $\mathbb{R}$. The sequence (f_j) is said to converge *uniformly* to a function f on D as $j \to +\infty$ if and only if the following criterion is satisfied:

given any strictly positive real number ε, there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon$ for all $x \in D$ and for all positive integers j satisfying $j \geq N$ (where the value of N is independent of x).

Let $f_1, f_2, f_3, \ldots$ be a sequence of bounded real-valued functions on some subset D of $\mathbb{R}$ which converges uniformly on D to the zero function. For each positive integer j, let $M_j = \sup\{f_j(x) : x \in D\}$. We claim that $M_j \to 0$ as $j \to +\infty$. To prove this, let some strictly positive real number ε be given. Then there exists some positive integer N such that $|f_j(x)| < \frac{1}{2}\varepsilon$ for all $x \in D$ and $j \geq N$. Thus if $j \geq N$ then $M_j \leq \frac{1}{2}\varepsilon < \varepsilon$. This shows that $M_j \to 0$ as $j \to +\infty$, as claimed.

Example Let $(f_j : n \in \mathbb{N})$ be the sequence of continuous functions on the interval $[0,1]$ defined by $f_j(x) = j(x^j - x^{2j})$. We have already shown (in an earlier example) that $\lim\limits_{j \to +\infty} f_j(x) = 0$ for all $x \in [0,1]$. However a straightforward exercise in Calculus shows that the maximum value attained by the function f_j is $j/4$ (which is attained at $x = 1/2^{\frac{1}{j}}$), and $j/4 \to +\infty$ as $j \to +\infty$. It follows from this that the sequence $f_1, f_2, f_3, \ldots$ does not converge uniformly to the zero function on the interval $[0,1]$.

Proposition 5.11 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions defined on some subset D of $\mathbb{R}$. Suppose that this sequence converges uniformly on D to some real-valued function f. Then f is continuous on D.*

Proof Let s be an element of D, and let some strictly positive real number ε be given. If j is chosen sufficiently large then $|f(x) - f_j(x)| < \frac{1}{3}\varepsilon$ for all $x \in D$, since $f_j \to f$ uniformly on D as $j \to +\infty$. It then follows from the continuity of f_j that there exists some strictly positive real number δ such that $|f_j(x) - f_j(s)| < \frac{1}{3}\varepsilon$ for all $x \in D$ satisfying $|x - s| < \delta$. But then

$$|f(x)-f(s)| \leq |f(x)-f_j(x)|+|f_j(x)-f_j(s)|+|f_j(s)-f(s)| < \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon = \varepsilon$$

whenever $|x-s| < \delta$. Thus the function f is continuous at s, as required. ∎

Theorem 5.12 *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions which converges uniformly on the interval $[a,b]$ to some continuous real-valued function f. Then*

$$\lim_{j \to +\infty} \int_a^b f_j(x)\,dx = \int_a^b f(x)\,dx.$$

Proof Let some strictly positive real number ε. Choose ε_0 small enough to ensure that $0 < \varepsilon_0(b-a) < \varepsilon$. Then there exists some positive integer N such that $|f_j(x) - f(x)| < \varepsilon_0$ for all $x \in [a,b]$ and $j \geq N$, since the sequence $f_1, f_2, f_3, \ldots$ of functions converges uniformly to f on $[a,b]$. Now

$$-\int_a^b |f_j(x) - f(x)|\,dx \leq \int_a^b f_j(x)\,dx - \int_a^b f(x)\,dx \leq \int_a^b |f_j(x) - f(x)|\,dx.$$

It follows that

$$\left|\int_a^b f_j(x)\,dx - \int_a^b f(x)\,dx\right| \leq \int_a^b |f_j(x) - f(x)|\,dx \leq \varepsilon_0(b-a) < \varepsilon,$$

whenever $j \geq N$. The result follows. ∎

5.5 Integrals over Unbounded Intervals

We define integrals over unbounded intervals by appropriate limiting processes. Given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $[a, +\infty)$, we define

$$\int_a^{+\infty} f(x)\,dx = \lim_{b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. Similarly, given any function f that is bounded and Riemann-integrable over each closed bounded subinterval of $(-\infty, b]$, we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx,$$

provided that this limit is well-defined. If f is bounded and Riemann integrable over each closed bounded interval in $\mathbb{R}$ then we define

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty, b \to +\infty} \int_a^b f(x)\,dx,$$

provided that this limit exists.

Remark Using techniques of complex analysis, it can be shown that

$$\lim_{b \to +\infty} \int_0^b \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

However it can also be shown that

$$\int_0^b \frac{|\sin x|}{x}\,dx \to +\infty \text{ as } b \to +\infty.$$

Therefore, in the standard theory of the Riemann integral, the integral of the function $(\sin x)/x$ on the interval $[0,+\infty)$ is defined, and $\displaystyle\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

There is an alternative theory of integration, due to Lebesgue, which is generally more powerful. All bounded Riemann-integrable functions on a closed bounded interval are Lebesgue-integrable on that interval. But a real-valued function f on a "measure space" is Lebesgue-integrable if and only if $|f|$ is Lebesgue-integrable on that measure space. Let $f\colon [0,+\infty) \to \mathbb{R}$ be the real-valued function defined such that $f(0) = 1$ and $f(x) = (\sin x)/x$ for all positive real numbers x. Then the function $|f|$ is neither Riemann-integrable nor Lebesgue-integrable on $[0,+\infty)$. It follows that the function f itself is not Lebesgue-integrable on $[0,+\infty)$. But, as we have remarked, the theory of the Riemann integral assigns a value of $\frac{\pi}{2}$ to $\int\limits_0^{+\infty} f(x)\,dx$.

6 Euclidean Spaces, Continuity, and Open Sets

6.1 Basic Properties of Vectors and Norms

We denote by $\mathbb{R}^n$ the set consisting of all n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers. The set $\mathbb{R}^n$ represents n-dimensional *Euclidean space* (with respect to the standard Cartesian coordinate system). Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$, where

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \quad \mathbf{y} = (y_1, y_2, \ldots, y_n),$$

and let λ be a real number. We define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n), \\ \mathbf{x} - \mathbf{y} &= (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n), \\ \lambda\mathbf{x} &= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n), \\ \mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n, \\ |\mathbf{x}| &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \end{aligned}$$

The quantity $\mathbf{x} \cdot \mathbf{y}$ is the *scalar product* (or *inner product*) of $\mathbf{x}$ and $\mathbf{y}$, and the quantity $|\mathbf{x}|$ is the *Euclidean norm* of $\mathbf{x}$. Note that $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. The *Euclidean distance* between two points $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^n$ is defined to be the Euclidean norm $|\mathbf{y} - \mathbf{x}|$ of the vector $\mathbf{y} - \mathbf{x}$.

Proposition 6.1 (Schwarz's Inequality) *Let $\mathbf{x}$ and $\mathbf{y}$ be elements of $\mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$.*

Proof We note that $|\lambda\mathbf{x} + \mu\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. But

$$|\lambda\mathbf{x} + \mu\mathbf{y}|^2 = (\lambda\mathbf{x} + \mu\mathbf{y}).(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2.$$

Therefore $\lambda^2|\mathbf{x}|^2 + 2\lambda\mu\mathbf{x} \cdot \mathbf{y} + \mu^2|\mathbf{y}|^2 \geq 0$ for all real numbers λ and μ. In particular, suppose that $\lambda = |\mathbf{y}|^2$ and $\mu = -\mathbf{x} \cdot \mathbf{y}$. We conclude that

$$|\mathbf{y}|^4|\mathbf{x}|^2 - 2|\mathbf{y}|^2(\mathbf{x} \cdot \mathbf{y})^2 + (\mathbf{x} \cdot \mathbf{y})^2|\mathbf{y}|^2 \geq 0,$$

so that $\left(|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2\right)|\mathbf{y}|^2 \geq 0$. Thus if $\mathbf{y} \neq \mathbf{0}$ then $|\mathbf{y}| > 0$, and hence

$$|\mathbf{x}|^2|\mathbf{y}|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \geq 0.$$

But this inequality is trivially satisfied when $\mathbf{y} = \mathbf{0}$. Thus $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, as required. ∎

Corollary 6.2 (Triangle Inequality) *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be elements of* $\mathbb{R}^n$. *Then* $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

Proof Using Schwarz's Inequality, we see that

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}).(\mathbf{x} + \mathbf{y}) = |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2\mathbf{x} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + |\mathbf{y}|^2 + 2|\mathbf{x}||\mathbf{y}| = (|\mathbf{x}| + |\mathbf{y}|)^2. \end{aligned}$$

The result follows directly. ∎

It follows immediately from the Triangle Inequality (Corollary 6.2) that

$$|\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}|$$

for all points $\mathbf{x}$, $\mathbf{y}$ and $|\mathbf{z}|$ of $\mathbb{R}^n$. This important inequality expresses the geometric fact the the length of any triangle in a Euclidean space is less than or equal to the sum of the lengths of the other two sides.

6.2 Convergence of Sequences in Euclidean Spaces

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to *converge* to a point $\mathbf{p}$ if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{p} - \mathbf{x}_j| < \varepsilon$ whenever $j \geq N$.

We refer to $\mathbf{p}$ as the *limit* $\lim\limits_{j \to +\infty} \mathbf{x}_j$ of the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$.

Lemma 6.3 *Let* $\mathbf{p}$ *be a point of* $\mathbb{R}^n$, *where* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. *Then a sequence* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *of points in* $\mathbb{R}^n$ *converges to* $\mathbf{p}$ *if and only if the* i*th components of the elements of this sequence converge to* p_i *for* $i = 1, 2, \ldots, n$.

Proof Let x_{ji} and p_i denote the ith components of $\mathbf{x}_j$ and $\mathbf{p}$, where $\mathbf{p} = \lim\limits_{j \to +\infty} \mathbf{x}_j$. Then $|x_{ji} - p_i| \leq |\mathbf{x}_j - \mathbf{p}|$ for all j. It follows directly from the definition of convergence that if $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$ then $x_{ji} \to p_i$ as $j \to +\infty$.

Conversely suppose that, for each i, $x_{ji} \to p_i$ as $j \to +\infty$. Let $\varepsilon > 0$ be given. Then there exist positive integers $N_1, N_2, \ldots, N_n$ such that $|x_{ji} - p_i| < \varepsilon/\sqrt{n}$ whenever $j \geq N_i$. Let N be the maximum of $N_1, N_2, \ldots, N_n$. If $j \geq N$ then

$$|\mathbf{x}_j - \mathbf{p}|^2 = \sum_{i=1}^{n} (x_{ji} - p_i)^2 < n(\varepsilon/\sqrt{n})^2 = \varepsilon^2,$$

so that $\mathbf{x}_j \to \mathbf{p}$ as $j \to +\infty$. ∎

Definition A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be a *Cauchy sequence* if and only if the following criterion is satisfied:—

> given any real number ε satisfying $\varepsilon > 0$ there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{x}_k| < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

Lemma 6.4 *A sequence of points in $\mathbb{R}^n$ is convergent if and only if it is a Cauchy sequence.*

Proof Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of $\mathbb{R}^n$ converging to some point $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \frac{1}{2}\varepsilon$ whenever $j \geq N$. If $j \geq N$ and $k \geq N$ then

$$|\mathbf{x}_j - \mathbf{x}_k| \leq |\mathbf{x}_j - \mathbf{p}| + |\mathbf{p} - \mathbf{x}_k| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

by the Triangle Inequality. Thus every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Now let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a Cauchy sequence in $\mathbb{R}^n$. Then the ith components of the elements of this sequence constitute a Cauchy sequence of real numbers. This Cauchy sequence must converge to some real number p_i, by Cauchy's Criterion for Convergence (Theorem 2.7). It follows from Lemma 6.3 that the Cauchy sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to the point $\mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. ∎

6.3 Continuity of Functions of Several Real Variables

Definition Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. A function $f: X \to Y$ from X to Y is said to be *continuous* at a point $\mathbf{p}$ of X if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$.

The function $f: X \to Y$ is said to be continuous on X if and only if it is continuous at every point $\mathbf{p}$ of X.

Lemma 6.5 *Let X, Y and Z be subsets of $\mathbb{R}^m$, $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively, and let $f: X \to Y$ and $g: Y \to Z$ be functions satisfying $f(X) \subset Y$. Suppose that f is continuous at some point $\mathbf{p}$ of X and that g is continuous at $f(\mathbf{p})$. Then the composition function $g \circ f: X \to Z$ is continuous at $\mathbf{p}$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\eta > 0$ such that $|g(\mathbf{y}) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{y} \in Y$ satisfying $|\mathbf{y} - f(\mathbf{p})| < \eta$. But then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \eta$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. It follows that $|g(f(\mathbf{x})) - g(f(\mathbf{p}))| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, and thus $g \circ f$ is continuous at $\mathbf{p}$, as required. ∎

Lemma 6.6 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a continuous function from X to Y. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a sequence of points of X which converges to some point $\mathbf{p}$ of X. Then the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$.*

Proof Let $\varepsilon > 0$ be given. Then there exists some $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, since the function f is continuous at $\mathbf{p}$. Also there exists some positive integer N such that $|\mathbf{x}_j - \mathbf{p}| < \delta$ whenever $j \geq N$, since the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Thus if $j \geq N$ then $|f(\mathbf{x}_j) - f(\mathbf{p})| < \varepsilon$. Thus the sequence $f(\mathbf{x}_1), f(\mathbf{x}_2), f(\mathbf{x}_3), \ldots$ converges to $f(\mathbf{p})$, as required. ∎

Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $f: X \to Y$ be a function from X to Y. Then

$$f(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in X$, where $f_1, f_2, \ldots, f_n$ are functions from X to $\mathbb{R}$, referred to as the *components* of the function f.

Proposition 6.7 *Let X and Y be a subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathbf{p} \in X$. A function $f: X \to Y$ is continuous at the point $\mathbf{p}$ if and only if its components are all continuous at $\mathbf{p}$.*

Proof Note that the ith component f_i of f is given by $f_i = \pi_i \circ f$, where $\pi_i: \mathbb{R}^n \to \mathbb{R}$ is the continuous function which maps $(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ onto its ith coordinate y_i. Now any composition of continuous functions is continuous, by Lemma 6.5. Thus if f is continuous at $\mathbf{p}$, then so are the components of f.

Conversely suppose that the components of f are continuous at $\mathbf{p} \in X$. Let $\varepsilon > 0$ be given. Then there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that $|f_i(\mathbf{x}) - f_i(\mathbf{p})| < \varepsilon/\sqrt{n}$ for $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta_i$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_n$. If $\mathbf{x} \in X$ satisfies $|\mathbf{x} - \mathbf{p}| < \delta$ then

$$|f(\mathbf{x}) - f(\mathbf{p})|^2 = \sum_{i=1}^{n} |f_i(\mathbf{x}) - f_i(\mathbf{p})|^2 < \varepsilon^2,$$

and hence $|f(\mathbf{x}) - f(\mathbf{p})| < \varepsilon$. Thus the function f is continuous at $\mathbf{p}$, as required. ∎

Lemma 6.8 *The functions $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $s(x,y) = x+y$ and $m(x,y) = xy$ are continuous.*

Proof Let $(u,v) \in \mathbb{R}^2$. We first show that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Let $\varepsilon > 0$ be given. Let $\delta = \frac{1}{2}\varepsilon$. If (x,y) is any point of $\mathbb{R}^2$ whose distance from (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence

$$|s(x,y) - s(u,v)| = |x+y-u-v| \leq |x-u| + |y-v| < 2\delta = \varepsilon.$$

This shows that $s\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v).

Next we show that $m\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). Now

$$m(x,y) - m(u,v) = xy - uv = (x-u)(y-v) + u(y-v) + (x-u)v.$$

for all points (x,y) of $\mathbb{R}^2$. Thus if the distance from (x,y) to (u,v) is less than δ then $|x-u| < \delta$ and $|y-v| < \delta$, and hence $|m(x,y) - m(u,v)| < \delta^2 + (|u| + |v|)\delta$. Let $\varepsilon > 0$ is given. If $\delta > 0$ is chosen to be the minimum of 1 and $\varepsilon/(1 + |u| + |v|)$ then $\delta^2 + (|u| + |v|)\delta < (1 + |u| + |v|)\delta < \varepsilon$, and thus $|m(x,y) - m(u,v)| < \varepsilon$ for all points (x,y) of $\mathbb{R}^2$ whose distance from (u,v) is less than δ. This shows that $p\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at (u,v). ∎

Proposition 6.9 *Let X be a subset of $\mathbb{R}^n$, and let $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ be continuous functions from X to $\mathbb{R}$. Then the functions $f+g$, $f-g$ and $f \cdot g$ are continuous. If in addition $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ then the quotient function f/g is continuous.*

Proof Note that $f + g = s \circ h$ and $f \cdot g = m \circ h$, where $h\colon X \to \mathbb{R}^2$, $s\colon \mathbb{R}^2 \to \mathbb{R}$ and $m\colon \mathbb{R}^2 \to \mathbb{R}$ are given by $h(\mathbf{x}) = (f(\mathbf{x}), g(\mathbf{x}))$, $s(u,v) = u+v$ and $m(u,v) = uv$ for all $\mathbf{x} \in X$ and $u, v \in \mathbb{R}$. It follows from Proposition 6.7, Lemma 6.8 and Lemma 6.5 that $f+g$ and $f \cdot g$ are continuous, being compositions of continuous functions. Now $f - g = f + (-g)$, and both f and $-g$ are continuous. Therefore $f - g$ is continuous.

Now suppose that $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$. Note that $1/g = r \circ g$, where $r\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is the reciprocal function, defined by $r(t) = 1/t$. Now the reciprocal function r is continuous. Thus the function $1/g$ is a composition of continuous functions and is thus continuous. But then, using the fact that a product of continuous real-valued functions is continuous, we deduce that f/g is continuous. ∎

Example Consider the function $f\colon \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ defined by

$$f(x,y) = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right).$$

The continuity of the components of the function f follows from straightforward applications of Proposition 6.9. It then follows from Proposition 6.7 that the function f is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.

6.4 Limits of Functions of Several Real Variables

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p} \in \mathbb{R}^m$. The point $\mathbf{p}$ is said to be a *limit point* of the set X if, given any $\delta > 0$, there exists some point $\mathbf{x}$ of X such that $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

It follows easily from the definition of convergence of sequences of points in Euclidean space that if X is a subset of m-dimensional Euclidean space $\mathbb{R}^m$ and if $\mathbf{p}$ is a point of $\mathbb{R}^m$ then the point $\mathbf{p}$ is a limit point of the set X if and only if there exists an infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points of X, all distinct from the point $\mathbf{p}$, such that $\lim\limits_{j \to +\infty} \mathbf{x}_j = \mathbf{p}$.

Definition Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. The point $\mathbf{q}$ is said to be the *limit* of $f(\mathbf{x})$, as $\mathbf{x}$ tends to $\mathbf{p}$ in X, if and only if the following criterion is satisfied:—

> given any strictly positive real number ε, there exists some strictly positive real number δ such that $|f(\mathbf{x}) - \mathbf{q}| < \varepsilon$ whenever $\mathbf{x} \in X$ satisfies $0 < |\mathbf{x} - \mathbf{p}| < \delta$.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. If $\mathbf{q}$ is the limit of $f(\mathbf{x})$ as $\mathbf{x}$ tends to $\mathbf{p}$ in X then we can denote this fact by writing $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = \mathbf{q}$.

Proposition 6.10 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, let $\mathbf{p}$ be a limit point of the set X, and let $\mathbf{q}$ be a point $\mathbb{R}^n$. Let $\tilde{X} = X \cup \{\mathbf{p}\}$, and let $\tilde{f}\colon \tilde{X} \to \mathbb{R}^n$ be defined such that*

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p}; \\ \mathbf{q} & \text{if } \mathbf{x} = \mathbf{p}. \end{cases}$$

Then $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(x) = \mathbf{q}$ if and only if the function $\tilde{f}$ is continuous at $\mathbf{p}$.

Proof The result follows directly on comparing the relevant definitions. ∎

Corollary 6.11 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f\colon X \to \mathbb{R}^n$ be a function mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of the set X that is also a limit point of X. Then the function f is continuous at the point $\mathbf{p}$ if and only if $\lim\limits_{\mathbf{x} \to \mathbf{p}} f(\mathbf{x}) = f(\mathbf{p})$.*

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, and let $\mathbf{p}$ be a point of the set X. Suppose that the point $\mathbf{p}$ is not a limit point of the set X. Then there exists some strictly positive real number δ_0 such that $|\mathbf{x}-\mathbf{p}| \geq \delta_0$ for all $\mathbf{x} \in X$. The point $\mathbf{p}$ is then said to be an *isolated point* of X.

Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$. The definition of continuity then ensures that any function $f: X \to \mathbb{R}^n$ mapping the set X into n-dimensional Euclidean space $\mathbb{R}^n$ is continuous at any isolated point of its domain X.

Corollary 6.12 *Let X be a subset of m-dimensional Euclidean space $\mathbb{R}^m$, let $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be real-valued functions on X, and let $\mathbf{p}$ be a limit point of the set X. Suppose that $\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$ both exist. Then so do $\lim\limits_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})+g(\mathbf{x}))$, $\lim\limits_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})-g(\mathbf{x}))$ and $\lim\limits_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x}))$, and moreover*

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})+g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) + \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})-g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) - \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}), \\
\lim_{\mathbf{x}\to\mathbf{p}} (f(\mathbf{x})g(\mathbf{x})) &= \lim_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x}) \times \lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}),
\end{aligned}$$

If moreover $g(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in X$ and $\lim_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x}) \neq 0$ then

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{f(\mathbf{x})}{g(\mathbf{x})} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})}{\lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})}.$$

Proof Let $\tilde{X} = X \cup \{\mathbf{p}\}$, and let $\tilde{f}: \tilde{X} \to \mathbb{R}$ and $\tilde{g}: \tilde{X} \to \mathbb{R}$ be defined such that

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p}; \\ l & \text{if } \mathbf{x} = \mathbf{p}. \end{cases} \qquad \tilde{g}(\mathbf{x}) = \begin{cases} g(\mathbf{x}) & \text{if } \mathbf{x} \neq \mathbf{p}; \\ m & \text{if } \mathbf{x} = \mathbf{p}. \end{cases},$$

where $l = \lim\limits_{\mathbf{x}\to\mathbf{p}} f(\mathbf{x})$ and $m = \lim\limits_{\mathbf{x}\to\mathbf{p}} g(\mathbf{x})$. Then the functions $\tilde{f}$ and $\tilde{g}$ are continuous at $\mathbf{p}$. The result therefore follows on applying Proposition 6.9. ∎

6.5 Open Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. Given a point $\mathbf{p}$ of X and a non-negative real number r, the *open ball* $B_X(\mathbf{p}, r)$ in X of *radius* r about $\mathbf{p}$ is defined to be the subset of X given by

$$B_X(\mathbf{p}, r) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < r\}.$$

(Thus $B_X(\mathbf{p}, r)$ is the set consisting of all points of X that lie within a sphere of radius r centred on the point $\mathbf{p}$.)

Definition Let X be a subset of $\mathbb{R}^n$. A subset V of X is said to be *open* in X if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in the case when V is the empty set.)

In particular, a subset V of $\mathbb{R}^n$ is said to be an *open set* (in $\mathbb{R}^n$) if and only if, given any point $\mathbf{p}$ of V, there exists some $\delta > 0$ such that $B(\mathbf{p}, \delta) \subset V$, where $B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{p}| < r\}$.

Example Let $H = \{(x, y, z) \in \mathbb{R}^3 : z > c\}$, where c is some real number. Then H is an open set in $\mathbb{R}^3$. Indeed let $\mathbf{p}$ be a point of H. Then $\mathbf{p} = (u, v, w)$, where $w > c$. Let $\delta = w - c$. If the distance from a point (x, y, z) to the point (u, v, w) is less than δ then $|z - w| < \delta$, and hence $z > c$, so that $(x, y, z) \in H$. Thus $B(\mathbf{p}, \delta) \subset H$, and therefore H is an open set.

The previous example can be generalized. Given any integer i between 1 and n, and given any real number c_i, the sets

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > c_i\}, \qquad \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i < c_i\}$$

are open sets in $\mathbb{R}^n$.

Example Let U be an open set in $\mathbb{R}^n$. Then for any subset X of $\mathbb{R}^n$, the intersection $U \cap X$ is open in X. (This follows directly from the definitions.) Thus for example, let S^2 be the unit sphere in $\mathbb{R}^3$, given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

and let N be the subset of S^2 given by

$$N = \{(x, y, z) \in \mathbb{R}^n : x^2 + y^2 + z^2 = 1 \text{ and } z > 0\}.$$

Then N is open in S^2, since $N = H \cap S^2$, where H is the open set in $\mathbb{R}^3$ given by

$$H = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}.$$

Note that N is not itself an open set in $\mathbb{R}^3$. Indeed the point $(0, 0, 1)$ belongs to N, but, for any $\delta > 0$, the open ball (in $\mathbb{R}^3$ of radius δ about $(0, 0, 1)$ contains points (x, y, z) for which $x^2 + y^2 + z^2 \neq 1$. Thus the open ball of radius δ about the point $(0, 0, 1)$ is not a subset of N.

Lemma 6.13 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any positive real number r, the open ball $B_X(\mathbf{p}, r)$ in X of radius r about $\mathbf{p}$ is open in X.*

Proof Let $\mathbf{x}$ be an element of $B_X(\mathbf{p}, r)$. We must show that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. Let $\delta = r - |\mathbf{x} - \mathbf{p}|$. Then $\delta > 0$, since $|\mathbf{x} - \mathbf{p}| < r$. Moreover if $\mathbf{y} \in B_X(\mathbf{x}, \delta)$ then

$$|\mathbf{y} - \mathbf{p}| \leq |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{p}| < \delta + |\mathbf{x} - \mathbf{p}| = r,$$

by the Triangle Inequality, and hence $\mathbf{y} \in B_X(\mathbf{p}, r)$. Thus $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{p}, r)$. This shows that $B_X(\mathbf{p}, r)$ is an open set, as required. ∎

Lemma 6.14 *Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of X. Then, for any non-negative real number r, the set $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| > r\}$ is an open set in X.*

Proof Let $\mathbf{x}$ be a point of X satisfying $|\mathbf{x} - \mathbf{p}| > r$, and let $\mathbf{y}$ be any point of X satisfying $|\mathbf{y} - \mathbf{x}| < \delta$, where $\delta = |\mathbf{x} - \mathbf{p}| - r$. Then

$$|\mathbf{x} - \mathbf{p}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{p}|,$$

by the Triangle Inequality, and therefore

$$|\mathbf{y} - \mathbf{p}| \geq |\mathbf{x} - \mathbf{p}| - |\mathbf{y} - \mathbf{x}| > |\mathbf{x} - \mathbf{p}| - \delta = r.$$

Thus $B_X(\mathbf{x}, \delta)$ is contained in the given set. The result follows. ∎

Proposition 6.15 *Let X be a subset of $\mathbb{R}^n$. The collection of open sets in X has the following properties:—*

(i) *the empty set $\emptyset$ and the whole set X are both open in X;*

(ii) *the union of any collection of open sets in X is itself open in X;*

(iii) *the intersection of any* finite *collection of open sets in X is itself open in X.*

Proof The empty set $\emptyset$ is an open set by convention. Moreover the definition of an open set is satisfied trivially by the whole set X. This proves (i).

Let $\mathcal{A}$ be any collection of open sets in X, and let U denote the union of all the open sets belonging to $\mathcal{A}$. We must show that U is itself open in X. Let $\mathbf{x} \in U$. Then $\mathbf{x} \in V$ for some set V belonging to the collection $\mathcal{A}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{x}, \delta) \subset V$. But $V \subset U$, and thus $B_X(\mathbf{x}, \delta) \subset U$. This shows that U is open in X. This proves (ii).

Finally let $V_1, V_2, V_3, \ldots, V_k$ be a *finite* collection of subsets of X that are open in X, and let V denote the intersection $V_1 \cap V_2 \cap \cdots \cap V_k$ of these sets. Let $\mathbf{x} \in V$. Now $\mathbf{x} \in V_j$ for $j = 1, 2, \ldots, k$, and therefore there exist strictly positive real numbers $\delta_1, \delta_2, \ldots, \delta_k$ such that $B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$. Let δ be the minimum of $\delta_1, \delta_2, \ldots, \delta_k$. Then $\delta > 0$. (This is where we need the fact that we are dealing with a finite collection of sets.) Now $B_X(\mathbf{x}, \delta) \subset B_X(\mathbf{x}, \delta_j) \subset V_j$ for $j = 1, 2, \ldots, k$, and thus $B_X(\mathbf{x}, \delta) \subset V$. Thus the intersection V of the sets $V_1, V_2, \ldots, V_k$ is itself open in X. This proves (iii). ∎

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ and } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the intersection of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 4 \text{ or } z > 1\}$ is an open set in $\mathbb{R}^3$, since it is the union of the open ball of radius 2 about the origin with the open set $\{(x, y, z) \in \mathbb{R}^3 : z > 1\}$.

Example The set

$$\{(x, y, z) \in \mathbb{R}^3 : (x - n)^2 + y^2 + z^2 < \tfrac{1}{4} \text{ for some } n \in \mathbb{Z}\}$$

is an open set in $\mathbb{R}^3$, since it is the union of the open balls of radius $\frac{1}{2}$ about the points $(n, 0, 0)$ for all integers n.

Example For each positive integer k, let

$$V_k = \{(x, y, z) \in \mathbb{R}^3 : k^2(x^2 + y^2 + z^2) < 1\}.$$

Now each set V_k is an open ball of radius $1/k$ about the origin, and is therefore an open set in $\mathbb{R}^3$. However the intersection of the sets V_k for all positive integers k is the set $\{(0, 0, 0)\}$, and thus the intersection of the sets V_k for all positive integers k is not itself an open set in $\mathbb{R}^3$. This example demonstrates that infinite intersections of open sets need not be open.

Lemma 6.16 *A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ converges to a point $\mathbf{p}$ if and only if, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ for all j satisfying $j \geq N$.*

Proof Suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ has the property that, given any open set U which contains $\mathbf{p}$, there exists some positive integer N such that $\mathbf{x}_j \in U$ whenever $j \geq N$. Let $\varepsilon > 0$ be given. The open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is an open set by Lemma 6.13. Therefore there exists some positive integer N such that $\mathbf{x}_j \in B(\mathbf{p}, \varepsilon)$ whenever $j \geq N$. Thus $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$. This shows that the sequence converges to $\mathbf{p}$.

Conversely, suppose that the sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ converges to $\mathbf{p}$. Let U be an open set which contains $\mathbf{p}$. Then there exists some $\varepsilon > 0$ such that the open ball $B(\mathbf{p}, \varepsilon)$ of radius ε about $\mathbf{p}$ is a subset of U. Thus there exists some $\varepsilon > 0$ such that U contains all points $\mathbf{x}$ of X that satisfy $|\mathbf{x} - \mathbf{p}| < \varepsilon$. But there exists some positive integer N with the property that $|\mathbf{x}_j - \mathbf{p}| < \varepsilon$ whenever $j \geq N$, since the sequence converges to $\mathbf{p}$. Therefore $\mathbf{x}_j \in U$ whenever $j \geq N$, as required. ∎

6.6 Closed Sets in Euclidean Spaces

Let X be a subset of $\mathbb{R}^n$. A subset F of X is said to be *closed* in X if and only if its complement $X \setminus F$ in X is open in X. (Recall that $X \setminus F = \{\mathbf{x} \in X : \mathbf{x} \notin F\}$.)

Example The sets $\{(x, y, z) \in \mathbb{R}^3 : z \geq c\}$, $\{(x, y, z) \in \mathbb{R}^3 : z \leq c\}$, and $\{(x, y, z) \in \mathbb{R}^3 : z = c\}$ are closed sets in $\mathbb{R}^3$ for each real number c, since the complements of these sets are open in $\mathbb{R}^3$.

Example Let X be a subset of $\mathbb{R}^n$, and let $\mathbf{x}_0$ be a point of X. Then the sets $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \leq r\}$ and $\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{x}_0| \geq r\}$ are closed for each non-negative real number r. In particular, the set $\{\mathbf{x}_0\}$ consisting of the single point $\mathbf{x}_0$ is a closed set in X. (These results follow immediately using Lemma 6.13 and Lemma 6.14 and the definition of closed sets.)

Let $\mathcal{A}$ be some collection of subsets of a set X. Then

$$X \setminus \bigcup_{S \in \mathcal{A}} S = \bigcap_{S \in \mathcal{A}} (X \setminus S), \qquad X \setminus \bigcap_{S \in \mathcal{A}} S = \bigcup_{S \in \mathcal{A}} (X \setminus S)$$

(i.e., the complement of the union of some collection of subsets of X is the intersection of the complements of those sets, and the complement of the intersection of some collection of subsets of X is the union of the complements of those sets). The following result therefore follows directly from Proposition 6.15.

Proposition 6.17 *Let X be a subset of $\mathbb{R}^n$. The collection of closed sets in X has the following properties:—*

(i) *the empty set* $\emptyset$ *and the whole set* X *are both closed in* X*;*

(ii) *the intersection of any collection of closed sets in* X *is itself closed in* X*;*

(iii) *the union of any* finite *collection of closed sets in* X *is itself closed in* X*.*

Lemma 6.18 *Let* X *be a subset of* $\mathbb{R}^n$*, and let* F *be a subset of* X *which is closed in* X*. Let* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ *be a sequence of points of* F *which converges to a point* $\mathbf{p}$ *of* X*. Then* $\mathbf{p} \in F$*.*

Proof The complement $X \setminus F$ of F in X is open, since F is closed. Suppose that $\mathbf{p}$ were a point belonging to $X \setminus F$. It would then follow from Lemma 6.16 that $\mathbf{x}_j \in X \setminus F$ for all values of j greater than some positive integer N, contradicting the fact that $\mathbf{x}_j \in F$ for all j. This contradiction shows that $\mathbf{p}$ must belong to F, as required. ∎

6.7 Continuous Functions and Open Sets

Let X and Y be subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, and let $f: X \to Y$ be a function from X to Y. We recall that the function f is continuous at a point $\mathbf{p}$ of X if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that $|f(\mathbf{u}) - f(\mathbf{p})| < \varepsilon$ for all points $\mathbf{u}$ of X satisfying $|\mathbf{u} - \mathbf{p}| < \delta$. Thus the function $f: X \to Y$ is continuous at $\mathbf{p}$ if and only if, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that the function f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (where $B_X(\mathbf{p}, \delta)$ and $B_Y(f(\mathbf{p}), \varepsilon)$ denote the open balls in X and Y of radius δ and ε about $\mathbf{p}$ and $f(\mathbf{p})$ respectively).

Given any function $f: X \to Y$, we denote by $f^{-1}(V)$ the *preimage* of a subset V of Y under the map f, defined by $f^{-1}(V) = \{\mathbf{x} \in X : f(\mathbf{x}) \in V\}$.

Proposition 6.19 *Let* X *and* Y *be subsets of* $\mathbb{R}^m$ *and* $\mathbb{R}^n$*, and let* $f: X \to Y$ *be a function from* X *to* Y*. The function* f *is continuous if and only if* $f^{-1}(V)$ *is open in* X *for every open subset* V *of* Y*.*

Proof Suppose that $f: X \to Y$ is continuous. Let V be an open set in Y. We must show that $f^{-1}(V)$ is open in X. Let $\mathbf{p} \in f^{-1}(V)$. Then $f(\mathbf{p}) \in V$. But V is open, hence there exists some $\varepsilon > 0$ with the property that $B_Y(f(\mathbf{p}), \varepsilon) \subset V$. But f is continuous at $\mathbf{p}$. Therefore there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$ (see the remarks above). Thus $f(\mathbf{x}) \in V$ for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, showing that $B_X(\mathbf{p}, \delta) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is open in X for every open set V in Y.

Conversely suppose that $f\colon X \to Y$ is a function with the property that $f^{-1}(V)$ is open in X for every open set V in Y. Let $\mathbf{p} \in X$. We must show that f is continuous at $\mathbf{p}$. Let $\varepsilon > 0$ be given. Then $B_X(f(\mathbf{p}), \varepsilon)$ is an open set in Y, by Lemma 6.13, hence $f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$ is an open set in X which contains $\mathbf{p}$. It follows that there exists some $\delta > 0$ such that $B_X(\mathbf{p}, \delta) \subset f^{-1}(B_Y(f(\mathbf{p}), \varepsilon))$. Thus, given any $\varepsilon > 0$, there exists some $\delta > 0$ such that f maps $B_X(\mathbf{p}, \delta)$ into $B_Y(f(\mathbf{p}), \varepsilon)$. We conclude that f is continuous at $\mathbf{p}$, as required. ∎

Let X be a subset of $\mathbb{R}^n$, let $f\colon X \to \mathbb{R}$ be continuous, and let c be some real number. Then the sets $\{\mathbf{x} \in X : f(\mathbf{x}) > c\}$ and $\{\mathbf{x} \in X : f(\mathbf{x}) < c\}$ are open in X, and, given real numbers a and b satisfying $a < b$, the set $\{\mathbf{x} \in X : a < f(\mathbf{x}) < b\}$ is open in X.

6.8 The Multidimensional Bolzano-Weierstrass Theorem

A sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ of points in $\mathbb{R}^n$ is said to be *bounded* if there exists some constant K such that $|\mathbf{x}_j| \leq K$ for all j.

Example Let

$$(x_j, y_j, z_j) = \left(\sin(\pi\sqrt{j}),\ (-1)^j, \cos\left(\frac{2\pi \log j}{\log 2}\right)\right)$$

for $j = 1, 2, 3, \ldots$. This sequence of points in $\mathbb{R}^3$ is bounded, because the components of its members all take values between -1 and 1. Moreover $x_j = 0$ whenever j is the square of a positive integer, $y_j = 1$ whenever j is even and $z_j = 1$ whenever j is a power of two.

The infinite sequence $x_1, x_2, x_3, \ldots$ has a convergent subsequence

$$x_1, x_4, x_9, x_{16}, x_{25}, \ldots$$

which includes those x_j for which j is the square of a positive integer. The corresponding subsequence $y_1, y_4, y_9, \ldots$ of $y_1, y_2, y_3, \ldots$ is not convergent, because its values alternate between 1 and -1. However this subsequence is bounded, and we can extract from this sequence a convergent subsequence

$$y_4, y_{16}, y_{36}, y_{64}, y_{100}, \ldots$$

which includes those x_j for which j is the square of an even positive integer. The subsequence

$$x_4, x_{16}, x_{36}, y_{64}, y_{100}, \ldots$$

is also convergent, because it is a subsequence of a convergent subsequence. However the corresponding subsequence

$$z_4, z_{16}, z_{36}, z_{64}, z_{100}, \ldots$$

does not converge. (Indeed $z_j = 1$ when j is an even power of 2, but $z_j = \cos(2\pi \log(9)/\log(2))$ when $j = 9 \times 2^{2p}$ for some positive integer p.) However this subsequence is bounded, and we can extract from it a convergent subsequence

$$z_4, z_{16}, z_{64}, z_{256}, z_{1024}, \ldots$$

which includes those x_j for which j is equal to two raised to the power of an even positive integer. Then the first, second and third components of the following subsequence

$$(x_4, y_4, z_4), \;\; (x_{16}, y_{16}, z_{16}), \;\; (x_{64}, y_{64}, z_{64}), \;\; (x_{256}, y_{256}, z_{256}), \ldots$$

of the original sequence of points in $\mathbb{R}^3$ converge, and it therefore follows from Lemma 6.3 that this sequence is a convergent subsequence of the given sequence of points in $\mathbb{R}^3$.

Example Let

$$x_j = \begin{cases} 1 & \text{if } j = 4k \text{ for some integer } k \\ 0 & \text{if } j = 4k+1 \text{ for some integer } k \\ -1 & \text{if } j = 4k+2 \text{ for some integer } k \\ 0 & \text{if } j = 4k+3 \text{ for some integer } k \end{cases}$$

and

$$y_j = \begin{cases} 0 & \text{if } j = 4k \text{ for some integer } k, \\ 1 & \text{if } j = 4k+1 \text{ for some integer } k, \\ 0 & \text{if } j = 4k+2 \text{ for some integer } k, \\ -1 & \text{if } j = 4k+3 \text{ for some integer } k, \end{cases},$$

and let $\mathbf{u}_j = (x_j, y_j)$ for $j = 1, 2, 3, 4, \ldots$. Then the first components x_j for which the index j is odd constitute a convergent sequence $x_1, x_3, x_5, x_7, \ldots$ of real numbers, and the second components y_j for which the index j is even also constitute a convergent sequence $y_2, y_4, y_6, y_8, \ldots$ of real numbers.

However one would not obtain a convergent subsequence of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ simply by selecting those indices j for which x_j is in the convergent subsequence $x_1, x_3, x_5, \ldots$ and y_j is in the convergent subsequence $y_2, y_4, y_6, \ldots$, because there no values of the index j for which x_j and y_j both belong to the respective subsequences. However the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) guarantees that there is a convergent subsequence of $y_1, y_3, y_5, y_7, \ldots$, and indeed $y_1, y_5, y_9, y_{13}, \ldots$ is such a convergent subsequence. This yields a convergent subsequence $\mathbf{u}_1, \mathbf{u}_5, \mathbf{u}_9, \mathbf{u}_{13}, \ldots$ of the given bounded sequence of points in $\mathbb{R}^2$.

Theorem 6.20 *Every bounded sequence of points in $\mathbb{R}^n$ has a convergent subsequence.*

Proof We prove the result by induction on the dimension n of the Euclidean space $\mathbb{R}^n$ that contains the infinite sequence in question. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that the theorem is true when $n = 1$. Suppose that $n > 1$, and that every bounded sequence in $\mathbb{R}^{n-1}$ has a convergent subsequence. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be a bounded infinite sequence of elements of $\mathbb{R}^n$, and let $x_{j,i}$ denote the ith component of $\mathbf{x}_j$ for $i = 1, 2, \ldots, n$ and for all positive integers j. The induction hypothesis requires that all bounded sequences in $\mathbb{R}^{n-1}$ contain convergent subsequences. Therefore there exist real numbers $p_1, p_2, \ldots, p_{n-1}$ and an increasing sequence $m_1, m_2, m_3, \ldots$ of positive integers such that $\lim\limits_{k \to +\infty} x_{m_k,i} = p_i$ for $i = 1, 2, \ldots, n-1$. The nth components $x_{m_1,n}, x_{m_2,n}, x_{m_3,n}, \ldots$ of the members of the subsequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ then constitute a bounded sequence of real numbers. It follows from the one-dimensional Bolzano-Weierstrass Theorem (Theorem 2.5) that there exists an increasing sequence $k_1, k_2, k_3, \ldots$ of positive integers for which the sequence $x_{m_{k_1},n}, x_{m_{k_2},n}, x_{m_{k_3},n}, \ldots$ converges. Let $s_j = m_{k_j}$ for all positive integers j, and let $p_n = \lim\limits_{j \to +\infty} x_{m_{k_j},n} = \lim\limits_{j \to +\infty} x_{s_j,n}$. Then the sequence $x_{s_1,i}, x_{s_2,i}, x_{s_3,i}, \ldots$ converges for values of i between 1 and $n-1$, because it is a subquence of the convergent sequence $x_{m_1,i}, x_{m_2,i}, x_{m_3,i}, \ldots$. Moreover $x_{s_1,n}, x_{s_2,n}, x_{s_3,n}, \ldots$ also converges. Thus the ith components of the infinite sequence $\mathbf{x}_{m_1}, \mathbf{x}_{m_2}, \mathbf{x}_{m_3}, \ldots$ converge for $i = 1, 2, \ldots, n$. It then follows from Lemma 6.3 that $\lim\limits_{j \to +\infty} \mathbf{x}_{s_k} = \mathbf{p}$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. The result follows. ∎

6.9 The Extreme Value Theorem for Functions of Several Real Variables

Theorem 6.21 (The Extreme Value Theorem for Continuous Functions on Closed Bounded Sets) *Let X be a closed bounded set in m-dimensional Euclidean space, and let $f \colon X \to \mathbb{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.*

Proof We prove the result for an arbitrary continuous real-valued function $f \colon X \to \mathbb{R}$ by showing that the result holds for a related continuous function

$g: X \to \mathbb{R}$ that is known to be bounded above and below on X. Let $h: \mathbb{R} \to \mathbb{R}$ be the continuous function defined such that

$$h(t) = \frac{t}{1 + |t|}$$

for all $t \in \mathbb{R}$. Then the continuous function $h: \mathbb{R} \to \mathbb{R}$ is increasing. Moreover $-1 \leq h(t) \leq 1$ for all $t \in \mathbb{R}$ (see the proof of Theorem 3.15).

Let $f: X \to \mathbb{R}$ be a continuous real-valued function on the closed bounded set X, and let $g: X \to \mathbb{R}$ be the continuous real-valued function defined on X such that

$$g(\mathbf{x}) = h(f(\mathbf{x})) = \frac{f(\mathbf{x})}{1 + |f(\mathbf{x})|}$$

for all $\mathbf{x} \in X$. Then $-1 \leq g(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in X$. The set of values of the function g is then non-empty and bounded above, and therefore has a least upper bound. Let

$$M = \sup\{g(\mathbf{x}) : \mathbf{x} \in X\}.$$

Then, for each positive integer j, the real number $M - j^{-1}$ is not an upper bound for the set of values of the function g, and therefore there exists some point $\mathbf{x}_j$ in the set X for which $M - j^{-1} < g(\mathbf{x}_j) \leq M$. The sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is then a bounded sequence of points in $\mathbb{R}^m$, because the set X is bounded. It follows from the multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) that this sequence has a subsequence $\mathbf{x}_{k_1}, \mathbf{x}_{k_2}, \mathbf{x}_{k_3}, \ldots$ which converges to some point $\mathbf{v}$ of $\mathbb{R}^n$. Moreover this point $\mathbf{v}$ belongs to the set X because X is closed (see Lemma 6.18). Now

$$M - \frac{1}{k_j} < g(\mathbf{x}_{k_j}) \leq M$$

for all positive integers j, and therefore $g(\mathbf{x}_{k_j}) \to M$ as $j \to +\infty$. It then follows from Lemma 6.6 that

$$g(\mathbf{v}) = g\left(\lim_{j \to +\infty} \mathbf{x}_{k_j}\right) = \lim_{j \to +\infty} g(\mathbf{x}_{k_j}) = M.$$

But $g(\mathbf{x}) \leq M$ for all $\mathbf{x} \in X$. It follows that $h(f(\mathbf{x})) = g(\mathbf{x}) \leq g(\mathbf{v}) = h(f(\mathbf{v}))$ for all $\mathbf{x} \in X$. Moreover $h: \mathbb{R} \to \mathbb{R}$ is an increasing function. It follows therefore that $f(\mathbf{x}) \leq f(\mathbf{v})$ for all $\mathbf{x} \in X$.

On applying this result with the continuous function f replaced by the function $-f$, we conclude also that there exists some point $\mathbf{u}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in X$. The result follows. ∎

6.10 Uniform Continuity for Functions of Several Real Variables

Definition Let X be a subset of $\mathbb{R}^m$. A function $f: X \to \mathbb{R}^n$ from X to $\mathbb{R}^n$ is said to be *uniformly continuous* if, given any $\varepsilon > 0$, there exists some $\delta > 0$ (which does not depend on either $\mathbf{x}'$ or $\mathbf{x}$) such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}'$ and $\mathbf{x}$ of X satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$.

Theorem 6.22 *Let X be a subset of $\mathbb{R}^m$ that is both closed and bounded. Then any continuous function $f: X \to \mathbb{R}^n$ is uniformly continuous.*

Proof Let $\varepsilon > 0$ be given. Suppose that there did not exist any $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$. Then, for each positive integer j, there would exist points $\mathbf{u}_j$ and $\mathbf{v}_j$ in X such that $|\mathbf{u}_j - \mathbf{v}_j| < 1/j$ and $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$. But the sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, since X is bounded, and thus would possess a subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ converging to some point $\mathbf{p}$ (Theorem 6.20). Moreover $\mathbf{p} \in X$, since X is closed. The sequence $\mathbf{v}_{j_1}, \mathbf{v}_{j_2}, \mathbf{v}_{j_3}, \ldots$ would also converge to $\mathbf{p}$, since $\lim\limits_{k \to +\infty} |\mathbf{v}_{j_k} - \mathbf{u}_{j_k}| = 0$. But then the sequences $f(\mathbf{u}_{j_1}), f(\mathbf{u}_{j_2}), f(\mathbf{u}_{j_3}), \ldots$ and $f(\mathbf{v}_{j_1}), f(\mathbf{v}_{j_2}), f(\mathbf{v}_{j_3}), \ldots$ would converge to $f(\mathbf{p})$, since f is continuous (Lemma 6.6), and thus $\lim\limits_{k \to +\infty} |f(\mathbf{u}_{j_k}) - f(\mathbf{v}_{j_k})| = 0$. But this is impossible, since $\mathbf{u}_j$ and $\mathbf{v}_j$ have been chosen so that $|f(\mathbf{u}_j) - f(\mathbf{v}_j)| \geq \varepsilon$ for all j. We conclude therefore that there must exist some $\delta > 0$ such that $|f(\mathbf{x}') - f(\mathbf{x})| < \varepsilon$ for all points $\mathbf{x}', \mathbf{x} \in X$ satisfying $|\mathbf{x}' - \mathbf{x}| < \delta$, as required. ∎

7 Differentiation of Functions of Several Real Variables

7.1 Linear Transformations

The space $\mathbb{R}^n$ consisting of all n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers is a vector space over the field $\mathbb{R}$ of real numbers, where addition and multiplication by scalars are defined by

$$\begin{aligned} (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) &= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \\ t(x_1, x_2, \dots, x_n) &= (tx_1, tx_2, \dots, tx_n) \end{aligned}$$

for all $(x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Definition A map $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be a *linear transformation* if

$$T(\mathbf{x} + \mathbf{y}) = T\mathbf{x} + T\mathbf{y}, \qquad T(t\mathbf{x}) = tT\mathbf{x}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Every linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is represented by an $m \times n$ matrix $(T_{i,j})$. Indeed let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ be the standard basis vectors of $\mathbb{R}^n$ defined by

$$\mathbf{e}_1 = (1, 0, \dots, 0), \quad \mathbf{e}_2 = (0, 1, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 1).$$

Thus if $\mathbf{x} \in \mathbb{R}^n$ is represented by the n-tuple $(x_1, x_2, \dots, x_n)$ then

$$\mathbf{x} = \sum_{j=1}^{n} x_j \mathbf{e}_j.$$

Similarly let $\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_m$ be the standard basis vectors of $\mathbb{R}^m$ defined by

$$\mathbf{f}_1 = (1, 0, \dots, 0), \quad \mathbf{f}_2 = (0, 1, \dots, 0), \dots, \mathbf{f}_m = (0, 0, \dots, 1).$$

Thus if $\mathbf{v} \in \mathbb{R}^m$ is represented by the n-tuple $(v_1, v_2, \dots, v_m)$ then

$$\mathbf{v} = \sum_{i=1}^{m} v_i \mathbf{f}_i.$$

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Define $T_{i,j}$ for all integers i between 1 and m and for all integers j between 1 and n such that

$$T\mathbf{e}_j = \sum_{i=1}^{m} T_{i,j} \mathbf{f}_i.$$

Using the linearity of T, we see that if $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ then

$$T\mathbf{x} = T\left(\sum_{j=1}^{n} x_j \mathbf{e}_j\right) = \sum_{j=1}^{n} (x_j\, T\mathbf{e}_j) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} T_{i,j} x_j\right)\mathbf{f}_i.$$

Thus the ith component of $T\mathbf{x}$ is

$$T_{i1}x_1 + T_{i2}x_2 + \cdots + T_{in}x_n.$$

Writing out this identity in matrix notation, we see that if $T\mathbf{x} = \mathbf{v}$, where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix},$$

then

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ T_{21} & T_{22} & \ldots & T_{2n} \\ \vdots & \vdots & & \vdots \\ T_{m1} & T_{m2} & \ldots & T_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Recall that the *length* (or *norm*) of an element $\mathbf{x} \in \mathbb{R}^n$ is defined such that

$$|\mathbf{x}|^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Definition Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let $(T_{i,j})$ be the $m \times n$ matrix representing this linear transformation with respect to the standard bases of $\mathbb{R}^m$ and $\mathbb{R}^m$. The *Hilbert-Schmidt norm* $\|T\|_{\mathrm{HS}}$ of the linear transformation is then defined so that

$$\|T\|_{\mathrm{HS}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} T_{i,j}^2}.$$

Note that the Hilbert-Schmidt norm is just the Euclidean norm on the real vector space of dimension mn whose elements are $m \times n$ matrices representing linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ with respect to the standard bases of these vector spaces. Therefore it has the standard properties of the Euclidean norm. In particular it follows from the Triangle Inequality (Lemma 6.2) that

$$\|T + U\|_{\mathrm{HS}} \leq \|T\|_{\mathrm{HS}} + \|U\|_{\mathrm{HS}} \quad \text{and} \quad \|sT\|_{\mathrm{HS}} = |s|\,\|T\|_{\mathrm{HS}}$$

for all linear transformations T and U from $\mathbb{R}^n$ to $\mathbb{R}^m$ and for all real numbers s.

Lemma 7.1 *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then T is uniformly continuous on $\mathbb{R}^n$. Moreover*

$$|T\mathbf{x} - T\mathbf{y}| \leq \|T\|_{\mathrm{HS}} |\mathbf{x} - \mathbf{y}|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, where $\|T\|_{HS}$ is the Hilbert-Schmidt norm of the linear transformation T.

Proof Let $\mathbf{v} = T\mathbf{x} - T\mathbf{y}$, where $\mathbf{v} \in \mathbb{R}^m$ is represented by the m-tuple $(v_1, v_2, \ldots, v_m)$. Then

$$v_i = T_{i1}(x_1 - y_1) + T_{i2}(x_2 - y_2) + \cdots + T_{in}(x_n - y_n)$$

for all integers i between 1 and m. It follows from Schwarz' Inequality (Lemma 6.1) that

$$v_i^2 \leq \left(\sum_{j=1}^n T_{i,j}^2 \right) \left(\sum_{j=1}^n (x_j - y_j)^2 \right) = \left(\sum_{j=1}^n T_{i,j}^2 \right) |\mathbf{x} - \mathbf{y}|^2.$$

Hence

$$|\mathbf{v}|^2 = \sum_{i=1}^m v_i^2 \leq \left(\sum_{i=1}^m \sum_{j=1}^n T_{i,j}^2 \right) |\mathbf{x} - \mathbf{y}|^2 = \|T\|_{\mathrm{HS}} |\mathbf{x} - \mathbf{y}|^2.$$

Thus $|T\mathbf{x} - T\mathbf{y}| \leq \|T\|_{\mathrm{HS}} |\mathbf{x} - \mathbf{y}|$. It follows from this that T is uniformly continuous. Indeed let some positive real number ε be given. We can then choose δ so that $\|T\|_{\mathrm{HS}}\, \delta < \varepsilon$. If $\mathbf{x}$ and $\mathbf{y}$ are elements of $\mathbb{R}^n$ which satisfy the condition $|\mathbf{x} - \mathbf{y}| < \delta$ then $|T\mathbf{x} - T\mathbf{y}| < \varepsilon$. This shows that $T: \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous on $\mathbb{R}^n$, as required. ∎

Lemma 7.2 *Let $T: \mathbb{R}^n \to \mathbf{R}^m$ be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and let $S: \mathbf{R}^m \to \mathbf{R}^p$ be a linear transformation from $\mathbf{R}^m$ to $\mathbf{R}^p$. Then the Hilbert-Schmidt norm of the composition of the linear operators T and S satisfies the inequality $\|ST\|_{\mathrm{HS}} \leq \|S\|_{\mathrm{HS}}\, \|T\|_{\mathrm{HS}}$.*

Proof The composition ST of the linear operators is represented by the product of the corresponding matrices. Thus the component $(ST)_{k,j}$ in the kth row and the jth column of the $p \times n$ matrix representing the linear transformation ST satisfies

$$(ST)_{k,j} = \sum_{i=1}^m S_{k,i} T_{i,j}.$$

It follows from Schwarz' Inequality (Lemma 6.1) that

$$(ST)_{k,j}^2 \leq \left(\sum_{i=1}^m S_{k,i}^2 \right) \left(\sum_{i=1}^m T_{i,j}^2 \right).$$

Summing over k, we find that

$$\sum_{k=1}^p (ST)_{k,j}^2 \leq \left(\sum_{k=1}^p \sum_{i=1}^m S_{k,i}^2 \right) \left(\sum_{i=1}^m T_{i,j}^2 \right) = \|S\|_{\mathrm{HS}}^2 \left(\sum_{i=1}^m T_{i,j}^2 \right).$$

Then summing over j, we find that

$$\|ST\|_{\mathrm{HS}}^2 = \sum_{k=1}^p \sum_{j=1}^n (ST)_{k,j}^2 \leq \|S\|_{\mathrm{HS}}^2 \left(\sum_{i=1}^m \sum_{j=1}^n T_{i,j}^2 \right) \leq \|S\|_{\mathrm{HS}}\|^2 \|T\|_{\mathrm{HS}}\|^2.$$

On taking square roots, we find that $\|ST\|_{\mathrm{HS}} \leq \|S\|_{\mathrm{HS}} \|T\|_{\mathrm{HS}}$, as required. ∎

7.2 Review of Differentiability for Functions of One Real Variable

Let $f: I \to \mathbb{R}$ be a real-valued function defined on some open interval I in $\mathbb{R}$. Let a be an element of I. Recall that the function f is *differentiable* at a if and only if

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists, and the value of this limit (if it exists) is known as the *derivative* of f at a (denoted by $f'(a)$).

We wish to define the notion of differentiability for functions of more than one variable. However we cannot immediately generalize the above definition as it stands (because this would require us to divide one element in $\mathbb{R}^n$ by another, which we cannot do since the operation of division is not defined on $\mathbb{R}^n$). We shall therefore reformulate the above definition of differentiability for functions of one real variable, exhibiting a criterion which is equivalent to the definition of differentiability given above and which can be easily generalized to functions of more than one real variable. This criterion is provided by the following lemma.

Lemma 7.3 *Let $f: I \to \mathbb{R}$ be a real-valued function defined on some open interval I in $\mathbb{R}$. Let a be an element of I. The function f is differentiable at a with derivative $f'(a)$ (where $f'(a)$ is some real number) if and only if*

$$\lim_{h \to 0} \frac{1}{|h|} \left(f(a+h) - f(a) - f'(a)h \right) = 0.$$

Proof It follows directly from the definition of the limit of a function that

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

if and only if

$$\lim_{h\to 0} \left| \frac{f(a+h)-f(a)}{h} - f'(a) \right| = 0.$$

But

$$\left| \frac{f(a+h)-f(a)}{h} - f'(a) \right| = \left| \frac{1}{|h|} \left(f(a+h) - f(a) - f'(a)h \right) \right|.$$

It follows immediately from this that the function f is differentiable at a with derivative $f'(a)$ if and only if

$$\lim_{h\to 0} \frac{1}{|h|} \left(f(a+h) - f(a) - f'(a)h \right) = 0. \quad \blacksquare$$

Now let us observe that, for any real number c, the map $h \mapsto ch$ defines a linear transformation from $\mathbb{R}$ to $\mathbb{R}$. Conversely, every linear transformation from $\mathbb{R}$ to $\mathbb{R}$ is of the form $h \mapsto ch$ for some $c \in \mathbb{R}$. Because of this, we may regard the derivative $f'(a)$ of f at a as representing a linear transformation $h \mapsto f'(a)h$, characterized by the property that the map

$$x \mapsto f(a) + f'(a)(x-a)$$

provides a 'good' approximation to f around a in the sense that

$$\lim_{h\to 0} \frac{e(a,h)}{|h|} = 0,$$

where

$$e(a,h) = f(a+h) - f(a) - f'(a)h$$

(i.e., e(a,h) measures the difference between $f(a+h)$ and the value $f(a) + f'(a)h$ of the approximation at $a+h$, and thus provides a measure of the error of this approximation). We shall generalize the notion of differentiability to functions f from $\mathbb{R}^n$ to $\mathbb{R}^m$ by defining the derivative $(Df)_{\mathbf{p}}$ of f at $\mathbf{p}$ to be a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ characterized by the property that the map

$$\mathbf{x} \mapsto f(\mathbf{p}) + (Df)_{\mathbf{p}} (\mathbf{x} - \mathbf{p})$$

provides a 'good' approximation to f around $\mathbf{p}$.

7.3 Derivatives of Functions of Several Variables

Definition Let V be an open subset of $\mathbb{R}^n$ and let $\varphi: V \to \mathbb{R}^m$ be a map from V into $\mathbb{R}^m$. Let $\mathbf{p}$ be a point of V. The function φ is said to be *differentiable* at $\mathbf{p}$, with *derivative* $T: \mathbb{R}^n \to \mathbb{R}^m$ if and only if there exists a linear transformation $T: R^n \to R^m$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ with the property that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \mathbf{0}.$$

If φ is differentiable at $\mathbf{p}$ then the derivative $T: \mathbb{R}^n \to \mathbb{R}^m$ of φ at $\mathbf{p}$ may be denoted by $(D\varphi)_{\mathbf{p}}$, or by $(D\varphi)(\mathbf{p})$, or by $f'(\mathbf{p})$.

The derivative $(D\varphi)_{\mathbf{p}}$ of φ at $\mathbf{p}$ is sometimes referred to as the *total derivative* of φ at $\mathbf{p}$. If φ is differentiable at every point of V then we say that φ is differentiable on V.

Lemma 7.4 *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. Then T is differentiable at each point $\mathbf{p}$ of $\mathbb{R}^n$, and $(DT)_{\mathbf{p}} = T$.*

Proof This follows immediately from the identity $T(\mathbf{p}+\mathbf{h}) - T\mathbf{p} - T\mathbf{h} = \mathbf{0}$. ∎

Lemma 7.5 *Let V be an open subset of $\mathbb{R}^n$, let $\varphi: V \to \mathbb{R}^m$ be a map from V into $\mathbb{R}^m$, let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\mathbf{p}$ be a point of V. Then φ is differentiable at $\mathbf{p}$, with derivative T, if and only if, given any strictly positive real number ε, there exists some strictly positive real number δ such that $\mathbf{p}+\mathbf{h} \in V$ and*

$$|\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}| \leq \varepsilon|\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$.

Proof Suppose that the function $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ satisfies the criterion described in the statement of the lemma. Let some strictly positive real number ε be given. Take some real number ε' satisfying $0 < \varepsilon' < \varepsilon$. Then there exists some strictly positive real number δ such that $\mathbf{p}+\mathbf{h} \in V$ and

$$|\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\,\mathbf{h}| \leq \varepsilon'|\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$. Then

$$\frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) \leq \varepsilon' < \varepsilon$$

whenever $0 < |\mathbf{h}| < \delta$, and therefore

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \mathbf{0}.$$

It then follows that the function φ is differentiable at $\mathbf{p}$, with derivative T.

Conversely, the function φ is differentiable at $\mathbf{p}$, with derivative T, then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} (\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - T\mathbf{h}) = \mathbf{0}$$

then it follows from the definition of limits that, given any strictly positive real number ε, there exists some strictly positive real number δ such that the condition set out in the statement of the lemma is satisfied, as required. ∎

It follows from Lemma 7.5 that if a function $\varphi: V \to \mathbb{R}^m$ defined over an open set V in $\mathbb{R}^n$ is differentiable at a point $\mathbf{p}$ of V, then, given any positive real number ε there exists a positive real number δ such that

$$|\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,\mathbf{h}| \leq \varepsilon|\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$, where $(D\varphi)_{\mathbf{p}}: \mathbf{R}^n \to \mathbb{R}^m$ denotes the derivative of φ at the point $\mathbf{p}$. In that case

$$\varphi(\mathbf{p}+\mathbf{h}) = \varphi(\mathbf{p}) + (D\varphi)_{\mathbf{p}}\,\mathbf{h} + e(\mathbf{p},\mathbf{h}),$$

where

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{e(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = \mathbf{0}.$$

Thus if φ is differentiable at $\mathbf{p}$ then the map $\lambda: V \to \mathbb{R}$ defined by

$$\lambda(\mathbf{x}) = \varphi(\mathbf{p}) + (D\varphi)_{\mathbf{p}}\,(\mathbf{x}-\mathbf{p})$$

provides a good approximation to the function around $\mathbf{p}$. The difference between $\varphi(\mathbf{x})$ and $\lambda(\mathbf{x})$ is equal to $e(\mathbf{p}, \mathbf{x}-\mathbf{p})$, and this quantity tends to $\mathbf{0}$ faster than $|\mathbf{x}-\mathbf{p}|$ as $\mathbf{x}$ tends to $\mathbf{p}$.

Example Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be defined so that

$$\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

for all real numbers x and y. Let p, q, h and k be real numbers. Then

$$\begin{aligned}
\varphi\left(\begin{pmatrix} p+h \\ q+k \end{pmatrix}\right) &= \begin{pmatrix} (p+h)^2-(q+k)^2 \\ 2(p+h)(q+k) \end{pmatrix} \\
&= \begin{pmatrix} p^2-q^2+2(ph-qk)+h^2-k^2 \\ 2pq+2(qh+pk)+2hk \end{pmatrix} \\
&= \begin{pmatrix} p^2-q^2 \\ 2pq \end{pmatrix} + \begin{pmatrix} 2(ph-qk) \\ 2(qh+pk) \end{pmatrix} + \begin{pmatrix} h^2-k^2 \\ 2hk \end{pmatrix} \\
&= \varphi\left(\begin{pmatrix} p \\ q \end{pmatrix}\right) + \begin{pmatrix} 2p & -2q \\ 2q & 2p \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} + \begin{pmatrix} h^2-k^2 \\ 2hk \end{pmatrix}.
\end{aligned}$$

Now $|(h,k)| = \sqrt{h^2+k^2}$, and

$$\frac{1}{h^2+k^2}\left|\begin{pmatrix} h^2-k^2 \\ 2hk \end{pmatrix}\right|^2 = \frac{(h^2-k^2)^2+4h^2k^2}{h^2+k^2}$$

for $(h,k) \neq (0,0)$. Note that if h and k are both multiplied by some positive real number t then the right hand side of the above equality is multiplied by t^2. It follows that if K is the maximum value of the right hand side of this equality on the circle $\{(h,k) : h^2+k^2 = 1\}$ then

$$\frac{1}{h^2+k^2}\left|\begin{pmatrix} h^2-k^2 \\ 2hk \end{pmatrix}\right|^2 \leq K(h^2+k^2).$$

Therefore

$$\frac{1}{\sqrt{h^2+k^2}}\left|\begin{pmatrix} h^2-k^2 \\ 2hk \end{pmatrix}\right| \to 0 \text{ as } (h,k) \to (0,0).$$

It follows that the function $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable, and the derivative $(D\varphi)_{(p,q)}$ of this function at the point (p,q) is the linear transformation represented as a matrix with respect to the standard bases as follows:

$$(D\varphi)_{(p,q)} = \begin{pmatrix} 2p & -2q \\ 2q & 2p \end{pmatrix}.$$

Example Let $M_n(\mathbb{R})$ denote the real vector space consisting of all $n \times n$ matrices with real coefficients. $M_n(\mathbb{R})$ may be regarded as a Euclidean space, where the Euclidean distance between two $n \times n$ matrices A and B is the Hilbert-Schmidt norm of $\|A-B\|_{\mathrm{HS}}$ of $A-B$, defined such that

$$\|A-B\|_{\mathrm{HS}} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(A_{i,j}-B_{i,j})^2}.$$

Let $\mathrm{GL}_(n, \mathbb{R})$ denote the set of invertible $n \times n$ matrices with real coefficients. Then

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}.$$

Now the determinant $\det A$ of a square $n \times n$ matrix A is a continuous function of the coefficients of the matrix. It follows from this that $\mathrm{GL}(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R})$. We denote the identity $n \times n$ matrix by I. Then $\|I\|_{\mathrm{HS}} = \sqrt{n}$, because the square of the Hilbert-Schmidt norm $\|I\|_{\mathrm{HS}}$ is the sum of the squares of the components of the identity matrix, and is therefore equal to n.

Let $\varphi\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be the function defined so that $\varphi(A) = A^{-1}$ for all invertible $n \times n$ matrices A. We show that this function $\varphi\colon \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is differentiable.

Let A be an invertible $n \times n$ matrix. Then for all $n \times n$ matrices H. Now the matrix $I + A^{-1}H$ is invertible if and only if $\det(I + A^{-1}H) \neq 0$. Moreover this determinant is a continuous function of the coefficients of the matrix H. It follows that there exists some positive number δ_0 such that $I + A^{-1}H$ is invertible whenever $\|H\|_{\mathrm{HS}} < \delta_0$. Moreover the function mapping the matrix H to $\|(I + A^{-1}H)^{-1}\|_{\mathrm{HS}}$ is continuous and takes the value $\sqrt{n}$ when H is the zero matrix. We can therefore choose a positive number δ_0 small enough to ensure that $I + A^{-1}H$ is invertible and $\|(I + A^{-1}H)^{-1}\|_{\mathrm{HS}} < 2\sqrt{n}$ whenever $\|H\|_{\mathrm{HS}} < \delta_0$.

Let the $n \times n$ matrix H satisfy $\|H\|_{\mathrm{HS}} < \delta_0$. Then

$$(I - A^{-1}H)(I + A^{-1}H) = I - A^{-1}HA^{-1}H,$$

and therefore

$$I = (I - A^{-1}H)(I + A^{-1}H) + A^{-1}HA^{-1}H.$$

Multiplying this identity on the right by the matrix $(I + A^{-1}H)^{-1}$, we find that

$$(I + A^{-1}H)^{-1} = I - A^{-1}H + A^{-1}HA^{-1}H(I + A^{-1}H)^{-1}.$$

It follows that

$$\begin{aligned} (A + H)^{-1} &= (A(I + A^{-1}H))^{-1} = (I + A^{-1}H)^{-1}A^{-1} \\ &= A^{-1} - A^{-1}HA^{-1} + A^{-1}HA^{-1}H(I + A^{-1}H)^{-1}A^{-1}. \end{aligned}$$

The Hilbert-Schmidt norm of a product of $n \times n$ matrices is bounded above by the product of the Hilbert-Schmidt norms of those matrices. Therefore if $\|H\|_{\mathrm{HS}} < \delta_0$ then

$$\|A^{-1}HA^{-1}H(I + A^{-1}H)^{-1}A^{-1}\|_{\mathrm{HS}} \leq \|A^{-1}\|_{\mathrm{HS}}^3 \|(I + A^{-1}H)^{-1}\|_{\mathrm{HS}} \|H\|_{\mathrm{HS}}^2,$$

where $\|(I + A^{-1}H)^{-1}\|_{\mathrm{HS}} < 2\sqrt{n}$, and therefore

$$\left\|(A+H)^{-1} - A^{-1} + A^{-1}HA^{-1}\right\|_{\mathrm{HS}} \leq 2\sqrt{n}\|A^{-1}\|_{\mathrm{HS}}^3 \|H\|_{\mathrm{HS}}^2.$$

It follows that

$$\lim_{H\to 0} \frac{1}{\|H\|_{\mathrm{HS}}} \left\|(A+H)^{-1} - A^{-1} + A^{-1}HA^{-1}\right\|_{\mathrm{HS}} = 0.$$

Therefore the function $\varphi\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is differentiable, where $\varphi(A) = A^{-1}$ for all invertible $n \times n$ matrices A with real coefficients, and moreover

$$(D\varphi)_A(H) = -A^{-1}HA^{-1}.$$

Lemma 7.6 *Let $\varphi\colon V \to \mathbb{R}^m$ be a function which maps an open subset V of $\mathbb{R}^n$ into $\mathbb{R}^m$ which is differentiable at some point $\mathbf{p}$ of V. Then φ is continuous at $\mathbf{p}$.*

Proof If we define

$$e(\mathbf{p},\mathbf{h}) = \varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,\mathbf{h}$$

then

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{e(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = \mathbf{0}$$

(because φ is differentiable at $\mathbf{p}$), and hence

$$\lim_{\mathbf{h}\to\mathbf{0}} e(\mathbf{p},\mathbf{h}) = \left(\lim_{\mathbf{h}\to\mathbf{0}} |\mathbf{h}|\right)\left(\lim_{\mathbf{h}\to\mathbf{0}} \frac{e(\mathbf{p},\mathbf{h})}{|\mathbf{h}|}\right) = \mathbf{0}.$$

But

$$\lim_{\mathbf{h}\to\mathbf{0}} e(\mathbf{p},\mathbf{h}) = \lim_{\mathbf{h}\to\mathbf{0}} \varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}),$$

since

$$\lim_{\mathbf{h}\to\mathbf{0}} (D\varphi)_{\mathbf{p}}\,\mathbf{h} = (D\varphi)_{\mathbf{p}}\left(\lim_{\mathbf{h}\to\mathbf{0}} \mathbf{h}\right) = \mathbf{0}$$

(on account of the fact that every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous). We conclude therefore that

$$\lim_{\mathbf{h}\to\mathbf{0}} \varphi(\mathbf{p}+\mathbf{h}) = \varphi(\mathbf{p}),$$

showing that φ is continuous at $\mathbf{p}$. ∎

Lemma 7.7 *Let $\varphi: V \to \mathbb{R}^m$ be a function which maps an open subset V of $\mathbb{R}^n$ into $\mathbb{R}^m$ which is differentiable at some point $\mathbf{p}$ of V. Let $(D\varphi)_\mathbf{p}: \mathbb{R}^n \to \mathbb{R}^m$ be the derivative of φ at $\mathbf{p}$. Let $\mathbf{u}$ be an element of $\mathbb{R}^n$. Then*

$$(D\varphi)_\mathbf{p}\mathbf{u} = \lim_{t \to 0} \frac{1}{t} \left(\varphi(\mathbf{p} + t\mathbf{u}) - \varphi(\mathbf{p})\right).$$

Thus the derivative $(D\varphi)_\mathbf{p}$ of φ at $\mathbf{p}$ is uniquely determined by the map φ.

Proof It follows from the differentiability of φ at $\mathbf{p}$ that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{1}{|\mathbf{h}|} \left(\varphi(\mathbf{p} + \mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_\mathbf{p}\,\mathbf{h}\right) = \mathbf{0}.$$

In particular, if we set $\mathbf{h} = t\mathbf{u}$, and $\mathbf{h} = -t\mathbf{u}$, where t is a real variable, we can conclude that

$$\lim_{t \to 0^+} \frac{1}{t} \left(\varphi(\mathbf{p} + t\mathbf{u}) - \varphi(\mathbf{p}) - t(D\varphi)_\mathbf{p}\mathbf{u}\right) = \mathbf{0},$$

$$\lim_{t \to 0^-} \frac{1}{t} \left(\varphi(\mathbf{p} + t\mathbf{u}) - \varphi(\mathbf{p}) - t(D\varphi)_\mathbf{p}\mathbf{u}\right) = \mathbf{0},$$

It follows that

$$\lim_{t \to 0} \frac{1}{t} \left(\varphi(\mathbf{p} + t\mathbf{u}) - \varphi(\mathbf{p}) - t(D\varphi)_\mathbf{p}\mathbf{u}\right) = \mathbf{0},$$

as required. ∎

We now show that given two differentiable functions mapping V into $\mathbb{R}$, where V is an open set in $\mathbb{R}^n$, the sum, difference and product of these functions are also differentiable.

Theorem 7.8 *Let V be an open set in $\mathbb{R}^n$, and let $f: V \to \mathbb{R}$ and $g: V \to \mathbb{R}$ be functions mapping V into $\mathbb{R}$. Let $\mathbf{p}$ be a point of V. Suppose that f and g are differentiable at $\mathbf{p}$. Then the functions $f + g$, $f - g$ and $f.g$ are differentiable at $\mathbf{p}$, and*

$$\begin{aligned}
(D(f+g))_\mathbf{p} &= (Df)_\mathbf{p} + (Dg)_\mathbf{p}, \\
D(f-g)_\mathbf{p} &= (Df)_\mathbf{p} - (Dg)_\mathbf{p}, \\
D(f.g)_\mathbf{p} &= g(\mathbf{p})(Df)_\mathbf{p} + f(\mathbf{p})(Dg)_\mathbf{p}.
\end{aligned}$$

Proof We can write

$$\begin{aligned} f(\mathbf{p}+\mathbf{h}) &= f(\mathbf{p}) + (Df)_{\mathbf{p}}\,\mathbf{h} + e_1(\mathbf{p},\mathbf{h}), \\ g(\mathbf{p}+\mathbf{h}) &= g(\mathbf{p}) + (Dg)_{\mathbf{p}}\,\mathbf{h} + e_2(\mathbf{p},\mathbf{h}), \end{aligned}$$

for all sufficiently small h, where

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{e_1(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = \mathbf{0}, \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_2(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = \mathbf{0},$$

on account of the fact that f and g are differentiable at $\mathbf{p}$. Then

$$\begin{aligned} &\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} \left| f(\mathbf{p}+\mathbf{h}) + g(\mathbf{p}+\mathbf{h}) - (f(\mathbf{p}) + g(\mathbf{p})) - ((Df)_{\mathbf{p}} + (Dg)_{\mathbf{p}})\,\mathbf{h} \right| \\ &\qquad = \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_1(\mathbf{p},\mathbf{h}) + e_2(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0, \\ &\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|} \left| f(\mathbf{p}+\mathbf{h}) - g(\mathbf{p}+\mathbf{h}) - (f(\mathbf{p}) - g(\mathbf{p})) - ((Df)_{\mathbf{p}} - (Dg)_{\mathbf{p}})\,\mathbf{h} \right| \\ &\qquad = \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_1(\mathbf{p},\mathbf{h}) - e_2(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0. \end{aligned}$$

Thus $f+g$ and $f-g$ are differentiable at $\mathbf{p}$. Also

$$f(\mathbf{p}+\mathbf{h})g(\mathbf{p}+\mathbf{h}) = f(\mathbf{p})g(\mathbf{p}) + g(\mathbf{p})(Df)_{\mathbf{p}}\,\mathbf{h} + f(\mathbf{p})(Dg)_{\mathbf{p}}\,\mathbf{h} + e(\mathbf{p},\mathbf{h}),$$

where

$$\begin{aligned} e(\mathbf{p},\mathbf{h}) &= (f(\mathbf{p}) + (Df)_{\mathbf{p}}\,\mathbf{h})e_2(\mathbf{p},\mathbf{h}) + (g(\mathbf{p}) + (Dg)_{\mathbf{p}}\,\mathbf{h})e_1(\mathbf{p},\mathbf{h}) \\ &\quad + ((Df)_{\mathbf{p}}\,\mathbf{h})((Dg)_{\mathbf{p}}\,\mathbf{h}) + e_1(\mathbf{p},\mathbf{h})e_2(\mathbf{p},\mathbf{h}). \end{aligned}$$

It follows from Lemma 7.1 that there exist constants M_1 and M_2 such that

$$|(Df)_{\mathbf{p}}\,\mathbf{h}| \leq M_1|\mathbf{h}|, \qquad |(Dg)_{\mathbf{p}}\,\mathbf{h}| \leq M_2|\mathbf{h}|.$$

Therefore

$$|((Df)_{\mathbf{p}}\,\mathbf{h})((Dg)_{\mathbf{p}}\,\mathbf{h})| \leq M_1M_2|\mathbf{h}|^2,$$

so that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}((Df)_{\mathbf{p}}\,\mathbf{h})((Dg)_{\mathbf{p}}\,\mathbf{h}) = 0.$$

Also

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}((f(\mathbf{p}) + (Df)_{\mathbf{p}}\,\mathbf{h})e_2(\mathbf{p},\mathbf{h}))$$

$$
\begin{aligned}
&= \lim_{\mathbf{h}\to\mathbf{0}} (f(\mathbf{p}) + (Df)_{\mathbf{p}}\,\mathbf{h}) \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_2(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0, \\
\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}((g(\mathbf{p}) + (Dg)_{\mathbf{p}}\,\mathbf{h})e_1(\mathbf{p},\mathbf{h}))& \\
&= \lim_{\mathbf{h}\to\mathbf{0}} (g(\mathbf{p}) + (Dg)_{\mathbf{p}}\,\mathbf{h}) \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_1(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0, \\
\lim_{\mathbf{h}\to\mathbf{0}} \frac{1}{|\mathbf{h}|}(e_1(\mathbf{p},\mathbf{h})e_2(\mathbf{p},\mathbf{h})) &= \lim_{\mathbf{h}\to\mathbf{0}} e_1(\mathbf{p},\mathbf{h}) \lim_{\mathbf{h}\to\mathbf{0}} \frac{e_2(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0.
\end{aligned}
$$

Therefore

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{e(\mathbf{p},\mathbf{h})}{|\mathbf{h}|} = 0,$$

showing that the function $f.g$ is differentiable at $\mathbf{p}$ and that

$$D(f.g)_{\mathbf{p}} = g(\mathbf{p})(Df)_{\mathbf{p}} + f(\mathbf{p})(Dg)_{\mathbf{p}}. \quad \blacksquare$$

Theorem 7.9 (Chain Rule) *Let V be an open set in $\mathbb{R}^n$, and let $\varphi\colon V \to \mathbb{R}^m$ be a function mapping V into $\mathbb{R}^m$. Let W be an open set in $\mathbb{R}^m$ which contains $\varphi(V)$, and let $\psi\colon W \to \mathbb{R}^l$ be a function mapping W into $\mathbb{R}^l$. Let $\mathbf{p}$ be a point of V. Suppose that φ is differentiable at $\mathbf{p}$ and that ψ is differentiable at $\varphi(\mathbf{p})$. Then the composition $\psi \circ \varphi\colon \mathbb{R}^n \to \mathbb{R}^l$ (i.e., φ followed by ψ) is differentiable at $\mathbf{p}$. Moreover*

$$D(\psi \circ \varphi)_{\mathbf{p}} = (D\psi)_{\varphi(\mathbf{p})} \circ (D\varphi)_{\mathbf{p}}.$$

Thus the derivative of the composition $\psi\circ\varphi$ of the functions at the given point is the composition of the derivatives of those functions at the appropriate points.

Proof Let $\mathbf{q} = \varphi(\mathbf{p})$. First we note that there exist positive real numbers L, and M such that $|(D\varphi)_{\mathbf{p}}\mathbf{h}| \leq L|\mathbf{h}|$ for all $\mathbf{h} \in \mathbb{R}^m$ and $|(D\psi)_{\mathbf{q}}\mathbf{k}| \leq M|\mathbf{k}|$ for all $\mathbf{k} \in \mathbb{R}^n$. Indeed it follows from Lemma 7.1 that we can take $L = \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{HS}}$ and $M = \|(D\psi)_{\mathbf{q}}\|_{\mathrm{HS}}$, where $\|(D\varphi)_{\mathbf{p}}\|_{\mathrm{HS}}$ and $M = \|(D\psi)_{\mathbf{q}}\|_{\mathrm{HS}}$ denote the Hilbert-Schmidt norms of the linear transformations $(D\varphi)_{\mathbf{p}}$ and $(D\psi)_{\mathbf{q}}$.

Let some strictly positive number ε be given. The function ψ is differentiable at $\mathbf{q}$, with derivative $(D\psi)_{\mathbf{q}}$, and therefore there exists a strictly positive real number η such that $\mathbf{q} + \mathbf{k} \in W$ and

$$|\psi(\mathbf{q} + \mathbf{k}) - \psi(\mathbf{q}) - (D\psi)_{\mathbf{q}}\,\mathbf{k}| \leq \frac{1}{2(L+1)}\varepsilon|\mathbf{k}|$$

for all $\mathbf{k} \in \mathbb{R}^m$ satisfying $|\mathbf{k}| < \eta$ (see Lemma 7.5). Let ε_0 be a strictly positive number chosen such that $\varepsilon_0 < 1$ and $2M\varepsilon_0 < \varepsilon$. It then follows from the continuity and differentiability of φ at $\mathbf{p}$ that there exists some strictly positive real number δ satisfying $(L+1)\delta < \eta$ with the property that

$$\mathbf{p}+\mathbf{h} \in D \quad \text{and} \quad |\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\mathbf{h}| \leq \varepsilon_0 |\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$.

Let $\mathbf{h} \in \mathbb{R}^n$ satisfy $|\mathbf{h}| < \delta$, and let

$$\mathbf{k} = \varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) = \varphi(\mathbf{p}+\mathbf{h}) - \mathbf{q}.$$

Then

$$\begin{aligned}
&\psi(\varphi(\mathbf{p}+\mathbf{h})) - \psi(\varphi(\mathbf{p})) - (D\psi)_{\varphi(\mathbf{p})}(D\varphi)_{\mathbf{p}}\,\mathbf{h} \\
&\qquad = (\psi(\mathbf{q}+\mathbf{k}) - \psi(\mathbf{q}) - (D\psi)_{\mathbf{q}}\,\mathbf{k}) \\
&\qquad\qquad + (D\psi)_{\mathbf{q}}\,(\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,\mathbf{h})\,.
\end{aligned}$$

Also, on applying the Triangle Inequality satisfied by the Euclidean norm (see Corollary 6.2), we find that

$$\begin{aligned}
|\mathbf{k}| &= |\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p})| \\
&\leq |(D\varphi)_{\mathbf{p}}\mathbf{h}| + |\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\mathbf{h}| \\
&\leq L|\mathbf{h}| + \varepsilon_0|\mathbf{h}| \\
&\leq (L+1)|\mathbf{h}| < (L+1)\delta < \eta.
\end{aligned}$$

It follows that

$$\begin{aligned}
&|\psi(\varphi(\mathbf{p}+\mathbf{h})) - \psi(\varphi(\mathbf{p})) - (D\psi)_{\varphi(\mathbf{p})}(D\varphi)_{\mathbf{p}}\,\mathbf{h}| \\
&\qquad \leq |\psi(\mathbf{q}+\mathbf{k}) - \psi(\mathbf{q}) - (D\psi)_{\mathbf{q}}\,\mathbf{k}| \\
&\qquad\qquad + |(D\psi)_{\mathbf{q}}\,(\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,\mathbf{h})| \\
&\qquad \leq |\psi(\mathbf{q}+\mathbf{k}) - \psi(\mathbf{q}) - (D\psi)_{\mathbf{q}}\,\mathbf{k}| \\
&\qquad\qquad + M\,|\varphi(\mathbf{p}+\mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_{\mathbf{p}}\,\mathbf{h}| \\
&\qquad \leq \frac{1}{2(L+1)}\varepsilon|\mathbf{k}| + M\varepsilon_0|\mathbf{h}| \\
&\qquad \leq \tfrac{1}{2}\varepsilon|\mathbf{h}| + \tfrac{1}{2}\varepsilon|\mathbf{h}| = \varepsilon|\mathbf{h}|
\end{aligned}$$

whenever $|\mathbf{h}| < \delta$. It follows that the composition function $\psi \circ \varphi \colon \mathbb{R}^n \to \mathbb{R}^l$ is differentiable, and its derivative at the point $\mathbf{p}$ is $(D\psi)_{\varphi(\mathbf{p})}(D\varphi)_{\mathbf{p}}$, as required. ∎

Example Consider the function $\varphi \colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(x,y) = \begin{cases} x^2 y^3 \sin \dfrac{1}{x} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Now one can verify from the definition of differentiability that the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by

$$h(t) = \begin{cases} t^2 \sin \dfrac{1}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0 \end{cases}$$

is differentiable everywhere on $\mathbb{R}$, though its derivative $h'\colon \mathbb{R} \to \mathbb{R}$ is not continuous at 0. Also the functions $(x, y) \mapsto x$ and $(x, y) \mapsto y$ are differentiable everywhere on $\mathbb{R}$ (by Lemma 7.4). Now $\varphi(x, y) = y^3 h(x)$. Using Theorem 7.8 and Theorem 7.9, we conclude that φ is differentiable everywhere on $\mathbb{R}^2$.

Let $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ denote the standard basis of $\mathbb{R}^n$, where

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, \ldots, 1).$$

Let us denote by $f^i\colon V \to \mathbb{R}$ the ith component of the map $\varphi\colon V \to \mathbb{R}^m$, where V is an open subset of $\mathbb{R}^n$. Thus

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

for all $\mathbf{x} \in V$. The jth partial derivative of f_i at $\mathbf{p} \in V$ is then given by

$$\left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{p}} = \lim_{t \to 0} \frac{f_i(\mathbf{p} + t\mathbf{e}_j) - f_i(\mathbf{p})}{t}.$$

We see therefore that if φ is differentiable at $\mathbf{p}$ then

$$(D\varphi)_{\mathbf{p}} \mathbf{e}_j = \left(\frac{\partial f_1}{\partial x_j}, \frac{\partial f_2}{\partial x_j}, \ldots, \frac{\partial f_m}{\partial x_j} \right).$$

Thus the linear transformation $(D\varphi)_{\mathbf{p}}\colon \mathbb{R}^n \to \mathbb{R}^m$ is represented by the $m \times n$ matrix

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

This matrix is known as the *Jacobian matrix* of φ at $\mathbf{p}$.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Note that this function is not continuous at $(0,0)$. (Indeed $f(t,t) = 1/(4t^2)$ if $t \neq 0$ so that $f(t,t) \to +\infty$ as $t \to 0$, yet $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$, thus showing that

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

cannot possibly exist.) Because f is not continuous at $(0,0)$ we conclude from Lemma 7.6 that f cannot be differentiable at $(0,0)$. However it is easy to show that the partial derivatives

$$\frac{\partial f(x,y)}{\partial x} \text{ and } \frac{\partial f(x,y)}{\partial y}$$

exist everywhere on $\mathbb{R}^2$, even at $(0,0)$. Indeed

$$\left.\frac{\partial f(x,y)}{\partial x}\right|_{(x,y)=(0,0)} = 0, \qquad \left.\frac{\partial f(x,y)}{\partial y}\right|_{(x,y)=(0,0)} = 0$$

on account of the fact that $f(x,0) = f(0,y) = 0$ for all $x, y \in \mathbb{R}$.

Example Consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Given real numbers b and c, let $u_{b,c}\colon \mathbb{R} \to \mathbb{R}$ be defined so that $u_{b,c}(t) = g(bt, ct)$ for all $t \in \mathbb{R}$. If $b = 0$ or $c = 0$ then $u_{b,c}(t) = 0$ for all $t \in \mathbb{R}$, and the function $u_{b,c}$ is thus a smooth function of t. If $b \neq 0$ and $c \neq 0$ then

$$u_{b,c}(t) = \frac{bc^2t^3}{b^2t^2 + c^4t^4} = \frac{bc^2t}{b^2 + c^2t^2},$$

and therefore $u_{b,c}(t)$ is a smooth function of t. Moreover

$$\left.\frac{du_{b,c}(t)}{dt}\right|_{t=0} = \begin{cases} \dfrac{c^2}{b} & \text{if } b \neq 0; \\ 0 & \text{if } b = 0. \end{cases}$$

The restriction of the function g to any line passing through the origin determines a smooth function of distance along the line. The restriction of the function g to any other line in the plane also determines a smooth function of distance. It follows that, when restricted to any straight line in $\mathbb{R}^2$, the value of the function g is a smooth function of distance along that line.

However $g(x, y) = \frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x > 0$ and $y = \pm\sqrt{x}$, and similarly $g(x, y) = -\frac{1}{2}$ for all $(x, y) \in \mathbb{R}^2$ satisfying $x < 0$ and $y = \pm\sqrt{-x}$. It follows that every open disk about the origin $(0, 0)$ contains some points at which the function g takes the value $\frac{1}{2}$, and other points at which the function takes the value $-\frac{1}{2}$, and indeed the function g will take on all real values between $-\frac{1}{2}$ and $\frac{1}{2}$ on any open disk about the origin, no matter how small the disk. It follows that the function $g\colon \mathbb{R}^2 \to \mathbf{R}$ is not continuous at zero, even though the partial derivatives of the function g with respect to x and y exist at each point of $\mathbb{R}^2$.

Remark These last two examples exhibits an important point. They show that *even if all the partial derivatives of a function exist at some point, this does not necessarily imply that the function is differentiable at that point.* However Theorem 7.11 below shows that if the first order partial derivatives of the components of a function exist *and are continuous* throughout some neighbourhood of a given point then the function is differentiable at that point.

Proposition 7.10 *Let M and δ_0 be positive real numbers, and let*

$$V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -\delta_0 < x_j < \delta_0 \text{ for } j = 1, 2, \ldots, n\}.$$

let $f\colon V \to \mathbb{R}$ be a real-valued function defined over V. Suppose that the partial derivatives of the function f with respect to $x_1, x_2, \ldots, x_n$ exist throughout V, and satisfy

$$\left| \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_j} \right| \leq M$$

whenever $-\delta_0 < x_j \leq \delta_0$ for $j = 1, 2, \ldots, n$. Then

$$|f(\mathbf{v}) - f(\mathbf{u})| \leq \sqrt{n}\, M |\mathbf{v} - \mathbf{u}|$$

for all $\mathbf{u}, \mathbf{v} \in V$.

Proof Let points $\mathbf{w}_k$ for $k = 0, 1, 2, \ldots, n$ be defined so that

$$\mathbf{w}_k = (w_{k,1}, w_{k,2}, \ldots, w_{k,n}),$$

where

$$w_{k,j} = \begin{cases} u_j & \text{if } j > k; \\ v_j & \text{if } j \leq k. \end{cases}$$

Then $\mathbf{w}_0 = \mathbf{u}$ and $\mathbf{w}_n = \mathbf{v}$. Moreover $\mathbf{w}_k$ and $\mathbf{w}_{k-1}$ differ only in the kth coordinate for $k = 1, 2, \ldots, n$, and indeed $w_{k-1,k} = u_k$, $w_{k,k} = v_k$ and $w_{k,j} = w_{k-1,j}$ for $j \neq k$. Let $q_k\colon [0,1] \to \mathbb{R}$ be defined such that

$$q_k(t) = f((1-t)\mathbf{w}_{k-1} + t\mathbf{w}_k)$$

for all $t \in [0,1]$. Then $q_k(0) = f(\mathbf{w}_{k-1})$ and $q_k(1) = f(\mathbf{w}_k)$, and therefore

$$f(\mathbf{v}) - f(\mathbf{u}) = \sum_{k=1}^{n} (f(\mathbf{w}_k) - f(\mathbf{w}_{k-1})) = \sum_{k=1}^{n} (q_k(1) - q_k(0)).$$

Now

$$q_k'(t) = \frac{dq_k(t)}{dt} = (v_k - u_k)(\partial_k f)((1-t)\mathbf{w}_{k-1} + t\mathbf{w}_k)$$

for all $t \in [0,1]$, where $\partial_k f$ denotes the partial derivative of the function f with respect to x_k. Moreover $|(\partial_k f)(\mathbf{x})| \leq M$ for all $\mathbf{x} \in V$. It follows that $|q_k'(t)| \leq M|v_k - u_k|$ for all $t \in [0,1]$. Applying the Mean Value Function (Theorem 4.6) to the function q on the interval $[0,1]$, we see that

$$|q_k(1) - q_k(0)| \leq M|v_k - u_k|$$

for $k = 1, 2, \ldots, n$. It follows that

$$|f(\mathbf{v}) - f(\mathbf{u})| \leq \sum_{k=1}^{n} |q_k(1) - q_k(0)| \leq M \sum_{k=1}^{n} |v_k - u_k|.$$

Now

$$\sum_{k=1}^{n} |v_k - u_k| \leq \sqrt{n}|\mathbf{v} - \mathbf{u}|.$$

Indeed let $\mathbf{s} \in \mathbf{R}^n$ be defined such that $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ where $s_j = +1$ if $v_j \geq u_j$ and $s_j = -1$ if $v_j < u_j$. Then

$$\sum_{k=1}^{n} |v_k - u_k| = \mathbf{s} \,.\, (\mathbf{v} - \mathbf{u}) \leq |\mathbf{s}||\mathbf{v} - \mathbf{u}| = \sqrt{n}|\mathbf{v} - \mathbf{u}|.$$

The result follows. ∎

Theorem 7.11 *Let V be an open subset of $\mathbb{R}^m$ and let $f\colon V \to \mathbb{R}$ be a function mapping V into $\mathbb{R}$. Suppose that the first order partial derivatives of the components of f exist and are continuous on V. Then f is differentiable at each point of V, and*

$$(Df)_{\mathbf{p}}\mathbf{h} = \sum_{j=1}^{n} h_j \left.\frac{\partial f}{\partial x_j}\right|_{\mathbf{x}=\mathbf{p}}$$

for all $\mathbf{p} \in V$ and $\mathbf{h} \in \mathbb{R}^n$, where $\mathbf{h} = (h_1, h_2, \ldots, h_n)$.

Proof Let $\mathbf{p} \in V$, and let $g\colon V \to \mathbb{R}$ be defined such that

$$g(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{n} a_j(x_j - p_j)$$

for all $\mathbf{x} \in V$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and

$$a_j = (\partial_j f)(\mathbf{p}) = \left.\frac{\partial f}{\partial x_j}\right|_{\mathbf{p}}$$

for $j = 1, 2, \ldots, n$. The partial derivatives $\partial_j g$ of the function g are then determined by those of f so that

$$(\partial_j g)(\mathbf{x}) = (\partial_j f)(\mathbf{x}) - a_j$$

for $j = 1, 2, \ldots, n$. It follows that $(\partial_j g)(\mathbf{p}) = 0$ for $j = 1, 2, \ldots, n$. It follows from the continuity of the partial derivatives of f that, given any positive real number ε, there exists some positive real number δ such that $(x_1, x_2, \ldots, x_n) \in V$ and, for each integer k between 1 and n,

$$|(\partial_k g)(x_1, x_2, \ldots, x_n)| \leq \frac{\varepsilon}{\sqrt{n}}$$

whenever $p_j - \delta < x_j < p_j + \delta$ for $j = 1, 2, \ldots, n$. It then follows from Proposition 7.10 that

$$|g(\mathbf{p} + \mathbf{h}) - g(\mathbf{p})| \leq \varepsilon|\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$. But then

$$\left| f(\mathbf{p} + \mathbf{h}) - f(\mathbf{p}) - \sum_{j=1}^{n} h_j(\partial_j f)(\mathbf{p}) \right| \leq \varepsilon|\mathbf{h}|$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$, where $\mathbf{h} = (h_1, h_2, \ldots, h_n)$. It follows from Lemma 7.5 that the function f is differentiable at $\mathbf{p}$. Moreover the Cartesian components of the derivative of f at $\mathbf{p}$ are equal to the partial derivatives of f at that point, as required. ∎

We can generalize this result immediately to functions $u: V \to \mathbb{R}^m$ which map some open subset V of $\mathbb{R}^n$ into $\mathbb{R}^m$. Let u_i denote the ith component of u for $i = 1, 2, \ldots, m$. One sees easily from the definition of differentiability that u is differentiable at a point of V if and only if each u_i is differentiable at that point. We can therefore deduce immediately the following corollary.

Corollary 7.12 *Let V be an open subset of $\mathbb{R}^n$ and let $u: V \to \mathbb{R}^m$ be a function mapping V into $\mathbb{R}^m$. Suppose that the Jacobian matrix*

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_m}{\partial x_1} & \dfrac{\partial u_m}{\partial x_2} & \cdots & \dfrac{\partial u_m}{\partial x_n} \end{pmatrix}$$

exists at every point of V and that the entries of the Jacobian matrix are continuous functions on V. Then φ is differentiable at every point of V, and the derivative of φ at each point is represented by the Jacobian matrix.

We now summarize the main conclusions regarding differentiability of functions of several real variables. They are as follows.

(i) A function $\varphi: V \to \mathbb{R}^m$ defined on an open subset V of $\mathbb{R}^n$ is said to be *differentiable* at a point $\mathbf{p}$ of V if and only if there exists a linear transformation $(D\varphi)_\mathbf{p}: \mathbb{R}^n \to \mathbb{R}^m$ with the property that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{1}{|\mathbf{h}|} \left(\varphi(\mathbf{p} + \mathbf{h}) - \varphi(\mathbf{p}) - (D\varphi)_\mathbf{p} \, \mathbf{h} \right) = \mathbf{0}.$$

The linear transformation $(D\varphi)_\mathbf{p}$ (if it exists) is unique and is known as the *derivative* (or *total derivative*) of φ at $\mathbf{p}$.

(ii) If the function $\varphi: V \to \mathbb{R}^m$ is differentiable at a point $\mathbf{p}$ of V then the derivative $(D\varphi)_\mathbf{p}$ of φ at $\mathbf{p}$ is represented by the Jacobian matrix of the function φ at $\mathbf{p}$ whose entries are the first order partial derivatives of the components of φ.

(iii) There exist functions $\varphi: V \to \mathbb{R}^m$ whose first order partial derivatives are well-defined at a particular point of V but which are not differentiable at that point. Indeed there exist such functions whose first order partial derivatives exist throughout their domain, though the functions themselves are not even continuous. *Thus in order to show that a function is differentiable at a particular point, it is not sufficient to show that the first order partial derivatives of the function exist at that point.*

(iv) However if the first order partial derivatives of the components of a function $\varphi: V \to \mathbb{R}^m$ exist *and are continuous* throughout some neighbourhood of a given point then the function is differentiable at that point. (However the converse does not hold: there exist functions which are differentiable whose first order partial derivatives are not continuous.)

(v) Linear transformations are everywhere differentiable.

(vi) A function $\varphi: V \to \mathbb{R}^m$ is differentiable if and only if its components are differentiable functions on V (where V is an open set in $\mathbb{R}^n$).

(vii) Given two differentiable functions from V to $\mathbb{R}$, where V is an open set in $\mathbb{R}^n$, the sum, difference and product of these functions are also differentiable.

(viii) (The Chain Rule). The composition of two differentiable functions is differentiable, and the derivative of the composition of the functions at any point is the composition of the derivatives of the functions.

7.4 Second Order Partial Derivatives

Let V be an open subset of $\mathbb{R}^n$ and let $f: V \to \mathbb{R}$ be a real-valued function on V. We consider the second order partial derivatives of the function f defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

We shall show that if the partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad \frac{\partial f}{\partial x_j}, \quad \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ and } \frac{\partial^2 f}{\partial x_j \partial x_i}$$

all exist and are continuous then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

First though we give a counterexample which demonstrates that there exist functions f for which

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Example Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0); \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

For convenience of notation, let us write

$$\begin{aligned} f_x(x, y) &= \frac{\partial f(x, y)}{\partial x}, \\ f_y(x, y) &= \frac{\partial f(x, y)}{\partial y}, \\ f_{xy}(x, y) &= \frac{\partial^2 f(x, y)}{\partial x \partial y}, \\ f_{yx}(x, y) &= \frac{\partial^2 f(x, y)}{\partial y \partial x}. \end{aligned}$$

If $(x, y) \neq (0, 0)$ then

$$\begin{aligned} f_x &= \frac{yx^2 - y^3 + 2x^2 y}{x^2 + y^2} - \frac{2x^2 y(x^2 - y^2)}{(x^2 + y^2)^2} \\ &= \frac{3x^2 y(x^2 + y^2) - y^3(x^2 + y^2) - 2x^4 y + 2x^2 y^3}{(x^2 + y^2)^2} \\ &= \frac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2}. \end{aligned}$$

Similarly

$$f_y = -\frac{y^4 x + 4y^2 x^3 - x^5}{(y^2 + x^2)^2}.$$

Thus if $(x, y) \neq (0, 0)$ then

$$f_{xy} = f_{yx} = \frac{x^6 + 9x^4 y^2 - 9x^2 y^4 - y^6}{(x^2 + y^2)^3}.$$

Note that

$$\lim_{(x,y)\to(0,0)} f_x(x, y) = 0, \qquad \lim_{(x,y)\to(0,0)} f_y(x, y) = 0.$$

Indeed if $(x, y) \neq (0, 0)$ then

$$|f_x| \leq \frac{6r^5}{r^4} = 6r,$$

where $r = \sqrt{x^2 + y^2}$, and similarly $|f_y| \leq 6r$. However

$$\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$$

does not exist. Indeed

$$\begin{aligned} \lim_{x\to 0} f_{xy}(x,0) = \lim_{x\to 0} f_{yx}(x,0) &= \lim_{x\to 0} \frac{x^6}{x^6} = 1, \\ \lim_{y\to 0} f_{xy}(0,y) = \lim_{y\to 0} f_{yx}(0,y) &= \lim_{y\to 0} \frac{-y^6}{y^6} = -1. \end{aligned}$$

Next we show that f_x, f_y, f_{xy} and f_{yx} all exist at $(0,0)$, and thus exist everywhere on $\mathbb{R}^2$. Now $f(x,0) = 0$ for all x, hence $f_x(0,0) = 0$. Also $f(0,y) = 0$ for all y, hence $f_y(0,0) = 0$. Thus

$$f_y(x,0) = x, \qquad f_x(0,y) = -y$$

for all $x, y \in \mathbb{R}$. We conclude that

$$\begin{aligned} f_{xy}(0,0) &= \frac{d(f_y(x,0))}{dx} = 1, \\ f_{yx}(0,0) &= \frac{d(f_x(0,y))}{dy} = -1, \end{aligned}$$

Thus

$$\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$$

at $(0,0)$.

Observe that in this example the functions f_{xy} and f_{yx} are continuous throughout $\mathbb{R}^2 \setminus \{(0,0\}$ and are equal to one another there. Although the functions f_{xy} and f_{yx} are well-defined at $(0,0)$, they are not continuous at $(0,0)$ and $f_{xy}(0,0) \neq f_{yx}(0,0)$.

We now prove that the continuity of the first and second order partial derivatives of a function f of two variables x and y is sufficient to ensure that

$$\frac{\partial^2 f}{\partial x \partial y}.$$

Theorem 7.13 *Let V be an open set in $\mathbb{R}^2$ and let $f\colon V \to \mathbb{R}$ be a real-valued function on V. Suppose that the partial derivatives*

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x}$$

exist and are continuous on V. Then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Proof For convenience, we shall denote the values of

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \text{ and } \frac{\partial^2 f}{\partial y \partial x}$$

at a point (x, y) of V by $f_x(x, y)$, $f_y(x, y)$, $f_{xy}(x, y)$ and $f_{yx}(x, y)$ respectively.

Let (a, b) be a point of V. The set V is an open set in $\mathbb{R}^n$ and therefore there exists some positive real number R such that $(a + h, b + k) \in V$ for all $(h, k) \in \mathbb{R}^2$ satisfying $\sqrt{h^2 + k^2} < R$.

Let us define a differentiable function u by

$$u(t) = f(t, b + k) - f(t, b)$$

We apply the Mean Value Theorem to the function u on the closed interval $[a, a + h]$ to conclude that there exists θ_1, where $0 < \theta_1 < 1$, such that

$$u(a + h) - u(a) = hu'(a + \theta_1 h).$$

But

$$u(a + h) - u(a) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$$

and

$$u'(a + \theta_1 h) = f_x(a + \theta_1 h, b + k) - f_x(a + \theta_1 h, b).$$

Moreover, on applying the Mean Value Theorem to the function that sends $y \in [b, b + k]$ to $f_x(a + \theta_1 h, y)$, we see that there exists θ_2, where $0 < \theta_2 < 1$, such that

$$f_x(a + \theta_1 h, b + k) - f_x(a + \theta_1 h, b) = k f_{yx}(a + \theta_1 h, b + \theta_2 k)$$

Thus

$$\begin{aligned} & f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b) \\ & \qquad = \quad hk f_{yx}(a + \theta_1 h, b + \theta_2 k) = hk \left. \frac{\partial^2 f}{\partial y \partial x} \right|_{(x,y)=(a+\theta_1 h, b+\theta_2 k)}. \end{aligned}$$

Now let $\varepsilon > 0$ be given. Then there exists some positive real number δ_1, where $\delta_1 \leq R$, such that

$$|f_{yx}(x, y) - f_{yx}(a, b)| < \tfrac{1}{2}\varepsilon$$

whenever $(x - a)^2 + (y - b)^2 < \delta_1^2$, by the continuity of the function f_{yx}. Thus

$$\left| \frac{f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)}{hk} - f_{yx}(a, b) \right| < \tfrac{1}{2}\varepsilon$$

for all $(h,k) \in \mathbb{R}^2$ for which $hk \neq 0$ and $\sqrt{h^2+k^2} < \delta_1$.

A corresponding result holds with the roles of x and y interchanged, and therefore there exists some positive real number δ_2, where $\delta_2 \leq R$, such that

$$\left|\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk} - f_{xy}(a,b)\right| < \tfrac{1}{2}\varepsilon$$

for all $(h,k) \in \mathbb{R}^2$ for which $hk \neq 0$ and $\sqrt{h^2+k^2} < \delta_2$.

Take δ to be the minimum of δ_1 and δ_2. If $hk \neq 0$ and $\sqrt{h^2+k^2} < \delta^2$ then

$$\begin{aligned}\left|\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk} - f_{yx}(a,b)\right| &< \tfrac{1}{2}\varepsilon,\\ \left|\frac{f(a+h,b+k)-f(a+h,b)-f(a,b+k)+f(a,b)}{hk} - f_{xy}(a,b)\right| &< \tfrac{1}{2}\varepsilon.\end{aligned}$$

Using the triangle inequality we conclude that

$$|f_{yx}(a,b) - f_{xy}(a,b)| < \varepsilon.$$

But this inequality has to hold for all $\varepsilon > 0$. Therefore we must have

$$f_{yx}(a,b) = f_{xy}(a,b).$$

We conclude therefore that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

at each point (a,b) of V, as required. ∎

Remark It is actually possible to prove a somewhat stronger theorem which states that, if $f\colon V \to \mathbb{R}$ is a real-valued function defined on a open subset V of $\mathbb{R}^2$ and if the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \text{ and } \frac{\partial^2 f}{\partial x \partial y}$$

exist and are continuous at some point (a,b) of V then

$$\frac{\partial^2 f}{\partial y \partial x}$$

exists at (a,b) and

$$\left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(a,b)} = \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(a,b)}.$$

Corollary 7.14 *Let V be an open set in $\mathbb{R}^n$ and let $f: V \to \mathbb{R}$ be a real-valued function on V. Suppose that the partial derivatives*

$$\frac{\partial f}{\partial x_i} \text{ and } \frac{\partial^2 f}{\partial x_i \partial x_j}$$

exist and are continuous on V for all integers i and j between 1 and n. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

for all integers i and j between 1 and n.

7.5 Maxima and Minima

Let $f: V \to \mathbb{R}$ be a real-valued function defined over some open subset V of $\mathbb{R}^n$ whose first and second order partial derivatives exist and are continuous throughout V. Suppose that f has a local minimum at some point $\mathbf{p}$ of V, where $\mathbf{p} = (p_1, p_2, \ldots, a_n)$. Now for each integer i between 1 and n the map

$$t \mapsto f(p_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$$

has a local minimum at $t = a_i$, hence the derivative of this map vanishes there. Thus if f has a local minimum at $\mathbf{p}$ then

$$\left. \frac{\partial f}{\partial x_i} \right|_{\mathbf{x}=\mathbf{p}} = 0.$$

The following lemma applies Taylor's Theorem (for functions of a single real variable) the local behaviour of real-valued functions of several real variables that are twice continuously differentiable throughout an open neighbourhood of some given point.

Lemma 7.15 *Let f be a continuous real-valued function defined throughout an open ball in $\mathbb{R}^n$ of radius R about some point $\mathbf{p}$. Suppose that the partial derivatives of f of orders one and two exist and are continuous throughout this open ball. Then*

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \sum_{k=1}^{n} h_k \left. \frac{\partial f}{\partial x_k} \right|_{\mathbf{p}} + \tfrac{1}{2} \sum_{j,k=1}^{n} h_j h_k \left. \frac{\partial^2 f}{\partial x_j \, \partial x_k} \right|_{\mathbf{p}+\theta\mathbf{h}}$$

for all $\mathbf{h} \in \mathbb{R}^n$ satisfying $|\mathbf{h}| < \delta$.

Proof Let $\mathbf{h}$ satisfy $|\mathbf{h}| < R$, and let

$$q(t) = f(\mathbf{p} + t\mathbf{h})$$

for all $t \in [0, 1]$. It follows from the Chain Rule for functions of several variables Theorem 7.9

$$q'(t) = \sum_{j=1}^{n} h_k (\partial_k f)(\mathbf{p} + t\mathbf{h})$$

and

$$q''(t) = \sum_{j,k=1}^{n} h_j h_k (\partial_j \partial_k f)(\mathbf{p} + t\mathbf{h}),$$

where

$$(\partial_j f)(x_1, x_2, \ldots, x_n) = \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_j}$$

and

$$(\partial_j \partial_k f)(x_1, x_2, \ldots, x_n) = \frac{\partial^2 f(x_1, x_2, \ldots, x_n)}{\partial x_j \, \partial x_k}.$$

It follows from Taylor's Theorem for functions of a single real variable (Theorem 4.21) that if the function f has continuous partial derivatives of orders one and two then

$$q(1) = q(0) + q'(0) + \tfrac{1}{2} q''(\theta)$$

for some real number θ satisfying $0 < \theta < 1$. It follows that

$$\begin{aligned} f(\mathbf{p} + \mathbf{h}) &= f(\mathbf{p}) + \sum_{k=1}^{n} h_k (\partial_k f)(\mathbf{p}) + \tfrac{1}{2} \sum_{j,k=1}^{n} h_j h_k (\partial_j \partial_k f)(\mathbf{p}) \\ &= f(\mathbf{p}) + \sum_{k=1}^{n} h_k \left. \frac{\partial f}{\partial x_k} \right|_{\mathbf{p}} + \tfrac{1}{2} \sum_{j,k=1}^{n} h_j h_k \left. \frac{\partial^2 f}{\partial x_j \, \partial k} \right|_{\mathbf{p} + \theta \mathbf{h}}, \end{aligned}$$

as required. ∎

Let f be a real-valued function of several variables whose first second order partial derivatives exist and are continuous throughout some open neigbourhood of a given point $\mathbf{p}$, and let $R > 0$ be chosen such that the function f is defined throughout the open ball of radius R about the point $\mathbf{p}$. It follows from Lemma 7.15 that if

$$\left. \frac{\partial f}{\partial x_j} \right|_{\mathbf{p}} = 0$$

for $j = 1, 2, \ldots, n$, and if $|\mathbf{h}| < R$ then

$$f(\mathbf{p}+\mathbf{h}) = f(\mathbf{p}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{p}+\theta\mathbf{h}}$$

for some θ satisfying $0 < \theta < 1$. Let us denote by $(H_{i,j}(\mathbf{p}))$ the *Hessian matrix* at the point $\mathbf{p}$, defined by

$$H_{i,j}(\mathbf{p}) = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{p}}.$$

If the partial derivatives of f of second order exist and are continuous then $H_{i,j}(\mathbf{p}) = Hji(\mathbf{p})$ for all i and j, by Corollary 7.14. Thus the Hessian matrix is symmetric.

We now recall some facts concerning symmetric matrices.

Let $(c_{i,j})$ be a symmetric $n \times n$ matrix.

The matrix $(c_{i,j})$ is said to be *positive semi-definite* if $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j} h_i h_j \geq 0$ for all $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

The matrix $(c_{i,j})$ is said to be *positive definite* if $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j} h_i h_j > 0$ for all non-zero $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

The matrix $(c_{i,j})$ is said to be *negative semi-definite* if $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j} h_i h_j \leq 0$ for all $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

The matrix $(c_{i,j})$ is said to be *negative definite* if $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j} h_i h_j < 0$ for all non-zero $(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n$.

The matrix $(c_{i,j})$ is said to be *indefinite* if it is neither positive semi-definite nor negative semi-definite.

Lemma 7.16 *Let $(c_{i,j})$ be a positive definite symmetric $n \times n$ matrix. Then there exists some $\varepsilon > 0$ with the following property: if all of the components of a symmetric $n \times n$ matrix $(b_{i,j})$ satisfy the inequality $|b_{i,j} - c_{i,j}| < \varepsilon$ then the matrix $(b_{i,j})$ is positive definite.*

Proof Let S^{n-1} be the unit $n-1$-sphere in $\mathbb{R}^n$ defined by

$$S^{n-1} = \{(h_1, h_2, \ldots, h_n) \in \mathbb{R}^n : h_1^2 + h_2^2 + \cdots + h_n^2 = 1\}.$$

Observe that a symmetric $n \times n$ matrix $(b_{i,j})$ is positive definite if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}h_ih_j > 0$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$. Now the matrix $(c_{i,j})$ is positive definite, by assumption. Therefore

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}h_ih_j > 0$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$. But S^{n-1} is a closed bounded set in $\mathbb{R}^n$, it therefore follows from Theorem 6.21 that there exists some $(k_1, k_2, \ldots, k_n) \in S^{n-1}$ with the property that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}h_ih_j \geq \sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}k_ik_j$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$. Thus there exists a strictly positive constant $A > 0$ with the property that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}h_ih_j \geq A$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$. Set $\varepsilon = A/n^2$. If $(b_{i,j})$ is a symmetric $n \times n$ matrix all of whose components satisfy $|b_{i,j} - c_{i,j}| < \varepsilon$ then

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}(b_{i,j} - c_{i,j})h_ih_j\right| < \varepsilon n^2 = A,$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$, hence

$$\sum_{i=1}^{n}\sum_{j=1}^{n} b_{i,j}h_ih_j > \sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}h_ih_j - A \geq 0$$

for all $(h_1, h_2, \ldots, h_n) \in S^{n-1}$. Thus the matrix $(b_{i,j})$ is positive-definite, as required. ∎

Using the fact that a symmetric $n \times n$ matrix $(c_{i,j})$ is negative definite if and only if the matrix $(-c_{i,j})$ is positive-definite, we see that if $(c_{i,j})$ is a negative-definite matrix then there exists some $\varepsilon > 0$ with the following property: if all of the components of a symmetric $n \times n$ matrix $(b_{i,j})$ satisfy the inequality $|b_{i,j} - c_{i,j}| < \varepsilon$ then the matrix $(b_{i,j})$ is negative definite.

Let $f: V \to \mathbb{R}$ be a real-valued function whose partial derivatives of first and second order exist and are continuous throughout some open set V in $\mathbb{R}^n$. Let $\mathbf{p}$ be a point of V. We have already observed that if the function f has a local maximum or a local minimum at $\mathbf{p}$ then

$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{p}} = 0 \qquad (i = 1, 2, \ldots, n).$$

We now apply Taylor's theorem to study the behaviour of the function f around a point $\mathbf{p}$ at which the first order partial derivatives vanish. We consider the Hessian matrix $(H_{i,j}(\mathbf{p})$ defined by

$$H_{i,j}(\mathbf{p}) = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{p}}.$$

Lemma 7.17 *Let $f: V \to \mathbb{R}$ be a real-valued function whose partial derivatives of first and second order exist and are continuous throughout some open set V in $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V at which*

$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{p}} = 0 \qquad (i = 1, 2, \ldots, n).$$

If f has a local minimum at a point $\mathbf{p}$ of V then the Hessian matrix $(H_{i,j}(\mathbf{p}))$ at $\mathbf{p}$ is positive semi-definite.

Proof The first order partial derivatives of f vanish at $\mathbf{p}$. It therefore follows from Taylor's Theorem that, for any $\mathbf{h} \in \mathbb{R}^n$ which is sufficiently close to $\mathbf{0}$, there exists some θ satisfying $0 < \theta < 1$ (where θ depends on $\mathbf{h}$) such that

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j H_{i,j}(\mathbf{p} + \theta \mathbf{h}),$$

where

$$H_{i,j}(\mathbf{p} + \theta \mathbf{h}) = \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\mathbf{x}=\mathbf{p}+\theta \mathbf{h}}$$

(see Lemma 7.15). Suppose that the Hessian matrix $H_{i,j}(\mathbf{p})$ is not positive semi-definite. Then there exists some $\mathbf{k} \in \mathbb{R}^n$, where $|\mathbf{k}| = 1$ with the property that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_i k_j H_{i,j}(\mathbf{p}) < 0.$$

It follows from the continuity of the second order partial derivatives of f that there exists some $\delta > 0$ such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j H_{i,j}(\mathbf{x}) < 0$$

for all $\mathbf{x} \in V$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. Choose any λ such that $0 < \lambda < \delta$ and set $\mathbf{h} = \lambda\mathbf{k}$. Then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} h_i h_j H_{i,j}(\mathbf{p} + \theta\mathbf{h}) < 0$$

for all $\theta \in (0, 1)$. We conclude from Taylor's theorem that $f(\mathbf{p} + \lambda\mathbf{k}) < f(\mathbf{p})$ for all λ satisfying $0 < \lambda < \delta$ (see Lemma 7.15). We have thus shown that if the Hessian matrix at $\mathbf{p}$ is not positive semi-definite then $\mathbf{p}$ is not a local minimum. Thus the Hessian matrix of f is positive semi-definite at every local minimum of f, as required. ∎

Let $f\colon V \to \mathbb{R}$ be a real-valued function whose partial derivatives of first and second order exist and are continuous throughout some open set V in $\mathbb{R}^n$, and let $\mathbf{p}$ be a point at which the first order partial derivatives of f vanish. The above lemma shows that if the function f has a local minimum at $\mathbf{h}$ then the Hessian matrix of f is positive semi-definite at $\mathbf{p}$. However the fact that the Hessian matrix of f is positive semi-definite at $\mathbf{p}$ is not sufficient to ensure that f is has a local minimum at $\mathbf{p}$, as the following example shows.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 - y^3$. Then the first order partial derivatives of f vanish at $(0, 0)$. The Hessian matrix of f at $(0, 0)$ is the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

and this matrix is positive semi-definite. However $(0, 0)$ is not a local minimum of f since $f(0, y) < f(0, 0)$ for all $y > 0$.

The following theorem shows that if the Hessian of the function f is positive definite at a point at which the first order partial derivatives of f vanish then f has a local minimum at that point.

Theorem 7.18 *Let $f\colon V \to \mathbb{R}$ be a real-valued function whose partial derivatives of first and second order exist and are continuous throughout some open set V in $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V at which*

$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{p}} = 0 \qquad (i = 1, 2, \ldots, n).$$

Suppose that the Hessian matrix $H_{i,j}(\mathbf{p})$ at $\mathbf{p}$ is positive definite. Then f has a local minimum at $\mathbf{p}$.

Proof The first order partial derivatives of f vanish at $\mathbf{p}$. It therefore follows from Taylor's Theorem that, for any $\mathbf{h} \in \mathbb{R}^n$ which is sufficiently close to $\mathbf{0}$, there exists some θ satisfying $0 < \theta < 1$ (where θ depends on $\mathbf{h}$) such that

$$f(\mathbf{p} + \mathbf{h}) = f(\mathbf{p}) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j H_{i,j}(\mathbf{p} + \theta\mathbf{h}),$$

where

$$H_{i,j}(\mathbf{p} + \theta\mathbf{h}) = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\mathbf{x}=\mathbf{p}+\theta\mathbf{h}}$$

(see Lemma 7.15). Suppose that the Hessian matrix $(H_{i,j}(\mathbf{p}))$ is positive definite. It follows from Lemma 7.16 that there exists some $\varepsilon > 0$ such that if $|H_{i,j}(\mathbf{x}) - H_{i,j}(\mathbf{p})| < \varepsilon$ for all i and j then $(H_{i,j}(\mathbf{x}))$ is positive definite. But it follows from the continuity of the second order partial derivatives of f that there exists some $\delta > 0$ such that $|H_{i,j}(\mathbf{x}) - H_{i,j}(\mathbf{p})| < \varepsilon$ whenever $|\mathbf{x} - \mathbf{p}| < \delta$. Thus if $|\mathbf{h}| < \delta$ then $(H_{i,j}(\mathbf{p} + \theta\mathbf{h}))$ is positive definite for all $\theta \in (0, 1)$ so that $f(\mathbf{p} + \mathbf{h}) > f(\mathbf{p})$. Thus $\mathbf{p}$ is a local minimum of f. ∎

A symmetric $n \times n$ matrix C is positive definite if and only if all its eigenvalues are strictly positive. In particular if $n = 2$ and if λ_1 and λ_2 are the eigenvalues a symmetric 2×2 matrix C, then

$$\lambda_1 + \lambda_2 = \operatorname{trace} C, \qquad \lambda_1 \lambda_2 = \det C.$$

Thus a symmetric 2×2 matrix C is positive definite if and only if its trace and determinant are both positive.

Example Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 4x^2 + 3y^2 - 2xy - x^3 - x^2 y - y^3.$$

Now

$$\left. \frac{\partial f(x, y)}{\partial x} \right|_{(x,y)=(0,0)} = (0, 0), \qquad \left. \frac{\partial f(x, y)}{\partial y} \right|_{(x,y)=(0,0)} = (0, 0).$$

The Hessian matrix of f at $(0, 0)$ is

$$\begin{pmatrix} 8 & -2 \\ -2 & 6 \end{pmatrix}.$$

The trace and determinant of this matrix are 14 and 44 respectively. Hence this matrix is positive definite. We conclude from Theorem 7.18 that the function f has a local minimum at $(0, 0)$.

8 Curvilinear Coordinates and the Inverse Function Theorem

8.1 Higher Order Derivatives and Smoothness

Definition Let V be an open set in $\mathbb{R}^n$. A function $\varphi: V \to \mathbb{R}^m$ is *continuously differentiable* if the function sending each point $\mathbf{x}$ of V to the derivative $(D\varphi)$ of φ at the point $\mathbf{x}$ is a continuous function from V to $L(\mathbb{R}^n, \mathbb{R}^m)$.

Lemma 8.1 *Let V be an open set in $\mathbb{R}^n$. A function $\varphi: V \to \mathbb{R}^m$ is continuously differentiable if and only if the first order partial derivatives of the components of φ exist and are continuous throughout V.*

Proof The result follows directly from Theorem 7.11. ∎

A function of several real variables is said to be "C^1" if and only if it is continuously differentiable.

The process of differentiation can be repeated. Let $\varphi: V \to \mathbb{R}^m$ be a differentiable function defined over an open set V in $\mathbb{R}^m$. Suppose that the function φ is differentiable at each point $\mathbf{p}$. Then the derivative of φ can itself be regarded as a function on V taking values in the real vector space $L(\mathbb{R}^n, \mathbb{R}^m)$ of linear transformations between the real vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. Moreover $L(\mathbb{R}^n, \mathbb{R}^m)$ can itself be regarded as a Euclidean space whose Euclidean norm is the Hilbert-Schmidt norm on $L(\mathbb{R}^n, \mathbb{R}^m)$. It follows that the definition of differentiability can be applied to derivative of a differentiable function of several real variables to obtain the second derivative of a twice-differentiable function. Continuing the process, one can obtain the kth derivative of a k-times differentiable function for any positive integer k. A more detailed analysis of this process shows that if φ is a k-times differentiable function, and if the Cartesian components of φ are $f_1, f_2, \ldots, f_m$, so that

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$$

for all $\mathbf{x} \in V$, then the kth derivative of φ at each point of V is represented by the multilinear transformation that maps each k-tuple $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(k)})$ of vectors in $\mathbb{R}^n$ to the vector in $\mathbb{R}^m$ whose ith component is

$$\sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_k=1}^{n} \frac{\partial^k f_i}{\partial x_{j_1}\, \partial x_{j_2} \cdots \partial x_{j_k}} v_{j_1}^{(1)} v_{j_2}^{(2)} \cdots v_{j_k}^{(k)},$$

where $v_j^{(s)}$ denotes the jth component of the vector $\mathbf{v}^{(s)}$ for $j = 1, 2, \ldots, n$ and $s = 1, 2, \ldots, k$. The kth derivative of the function φ is thus represented by a function from the open set V to some real vector space of multilinear transformations. Such a function is said to be a (Cartesian) *tensor field* on V. Such tensor fields are ubiquitous in differential geometry and theoretical physics.

We can formally define the concept of functions of several variables being *differentiable* of order k by recursion on k.

Definition Let V be an open set in $\mathbb{R}^n$. A function $\varphi\colon V \to \mathbb{R}^m$ is *k-times differentiable*, where $k > 1$, if it is differentiable and the $D\varphi\colon V \to L(\mathbb{R}^n, \mathbb{R}^m)$ that maps each point $\mathbf{x}$ of V to the derivative of φ at that point is a $(k-1)$-times differentiable function on V.

Definition Let V be an open set in $\mathbb{R}^n$. A function $\varphi\colon V \to \mathbb{R}^m$ is *k-times continuously differentiable*, where $k > 1$, if the function $D\varphi\colon V \to L(\mathbb{R}^n, \mathbb{R}^m)$ that maps each point $\mathbf{x}$ of V to the derivative of φ at that point is a $(k-1)$-times continuously differentiable function on V.

A function of several real variables is said to be "C^k" for some positive integer k if and only if it is k-times continuously differentiable.

Definition A function $\varphi\colon V \to \mathbb{R}^m$ is said to be *smooth* (or C^∞) if it is k-times differentiable for all positive integers k.

If a function of several real variables is $(k+1)$-times differentiable, then the components of its kth order derivative must be continuous functions, because differentiability implies continuity (see Lemma 7.6). It follows that a function of several real variables is smooth if and only if it is C^k for all positive integers k.

Lemma 8.2 *Let V be an open set in $\mathbb{R}^n$. A function $\varphi\colon V \to \mathbb{R}^m$ is k-times continuously differentiable (or C^k) if and only if the partial derivatives of the components of φ of all orders up to and including k exist and are continuous throughout V.*

Proof The result can be proved by induction on k. The result is true for $k = 1$ by Lemma 8.1. Suppose as our induction hypothesis that $k > 1$ and that continuously differentiable vector-valued functions on V are C^{k-1} if and only if their partial derivatives of orders up to and including $k-1$ exist and are continuous throughout V. Now a vector-valued function is continuously differentiable if and only if its components are continuously differentiable. Moreover a vector-valued function is C^{k-1} if and only if its components are all C^{k-1}. It follows that the function φ is C^k if and only if the components of its derivative are C^{k-1}. These components are the first-order partial derivatives of φ. The induction hypothesis ensures that these first order partial derivatives of φ are C^{k-1} if and only if their partial derivatives of orders less than or equal to $k-1$ exist and are continuous throughout V. It follows that the function φ itself is C^k if and only if its partial derivatives of orders less than or equal to k exist and are continuous throughout V, as required. ∎

Lemma 8.3 *Let V be an open set in $\mathbb{R}^n$, and let $f: V \to \mathbb{R}$ and $g: V \to \mathbb{R}$ be real-functions on V, and let $f+g$, $f-g$ and $f\,.\,g$ denote the sum, difference and product of these functions, where*

$$(f+g)(\mathbf{x}) = f(\mathbf{x})+g(\mathbf{x}), \quad (f-g)(\mathbf{x}) = f(\mathbf{x})-g(\mathbf{x}), \quad (f\,.\,g)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$$

for all $\mathbf{x} \in V$. Suppose that the functions f and g are C^k for some positive integer k. Then so are the functions $f+g$, $f-g$ and $f\,.\,g$.

Proof The result can be proved by induction on k. It follows from Theorem 7.8 that the result is true when $k=1$.

A real-valued function on V is C^k for some positive integer k if and only if all the partial derivatives of its components of degree less than or equal to k exist and are continuous throughout the open set V. It follows from this that a real-valued function f on V is C^k if and only if its first order partial derivatives $\partial_i f$ are C^{k-1}, where $\partial_i f = \dfrac{\partial f}{\partial x_i}$ for $i = 1, 2, \ldots, n$.

Thus suppose as our induction hypothesis that $k > 1$ and that all sums, differences and products of C^{k-1} functions are known to be C^{k-1}. Let f and g be C^k functions. Then

$$\partial_i(f+g) = \partial_i f + \partial_i g, \quad \partial_i(f-g) = \partial_i f - \partial_i g,$$

$$\partial_i(f\,.\,g) = f\,.\,(\partial_i g) + (\partial_i f)\,.\,g$$

for $i = 1, 2, \ldots, n$. Now the functions f, g, $\partial_i f$ and $\partial_i g$ are all C^{k-1}. The induction hypothesis then ensures that $\partial_i(f+g)$, $\partial_i(f-g)$ and $\partial_i(f\,.\,g)$ are all C^{k-1} for $i = 1, 2, \ldots, n$, and therefore the functions $f+g$, $f-g$ and $f\,.\,g$ are C^k.

The required result therefore follows by induction on the degree k of the derivatives required to be continuous. ∎

Lemma 8.4 *Let V and W be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and let $\varphi: V \to \mathbb{R}^m$ and $\psi: W \to \mathbb{R}^l$ be functions mapping V and W into $\mathbb{R}^m$ and $\mathbb{R}^l$ respectively, where $\varphi(V) \subset W$. Suppose that the functions $\varphi: V \to \mathbb{R}^m$ and $\psi: W \to \mathbb{R}^l$ are C^k. Then the composition function $\psi \circ \varphi: V \to \mathbb{R}^l$ is also C^k.*

Proof We prove the result by induction on k. The Chain Rule for functions of several real variables (Theorem 7.9) ensures that the result is true for $k = 1$.

We have shown that sums, differences and products of C^k functions are C^k (see Lemma 8.3). We suppose as our induction hypothesis that all compositions of C^{k-1} functions of several real variables are C^{k-1} for some positive integer k, and show that this implies that all compositions of C^k functions of several real variables are C^k.

Let $\varphi: V \to \mathbb{R}^m$ and $\psi: W \to \mathbb{R}^l$ be C^k functions, where V is an open set in $\mathbb{R}^n$, W is an open set in $\mathbb{R}^m$ and $\varphi(V) = W$. Let the components of φ be $f_1, f_2, \ldots, f_n$ and let the components of ψ be $g_1, g_2, \ldots, g_m$, where $f_1, f_2, \ldots, f_n$ are real-valued functions on V, $g_1, g_2, \ldots, g_m$ are real-valued functions on W,

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x}))$$

for all $\mathbf{x} \in V$ and

$$\psi(\mathbf{y}) = (g_1(\mathbf{y}), g_2(\mathbf{y}), \ldots, f_m(\mathbf{y}))$$

for all $\mathbf{y} \in W$. It then follows from the Chain Rule (Theorem 7.9) that

$$\frac{\partial}{\partial x_i}\Big(g_j(\varphi(x_1, x_2, \ldots, x_n))\Big) = \sum_{s=1}^{m} \left(\frac{\partial g_j}{\partial u_s} \circ \varphi\right) \frac{\partial f_s}{\partial x_i}.$$

Now the functions $\dfrac{\partial g_j}{\partial u_s} \circ \varphi$ are compositions of C^{k-1} functions. The induction hypothesis therefore ensures that these functions are C^{k-1}. This then ensures that the functions $\dfrac{\partial}{\partial x_i}\Big(g_j(\varphi(x_1, x_2, \ldots, x_n))\Big)$ are expressible as sums of products of C^{k-1} functions, and must therefore themselves be C^{k-1} functions (see Lemma 8.3). We have thus shown that the first order partial derivatives of the components of the composition function $\psi \circ \varphi$ are C^{k-1} functions. It follows that $\psi \circ \varphi$ must itself be a C^k function.

The required result therefore follows by induction on the degree k of the derivatives required to be continuous. ∎

It follows from Lemma 8.3 and Lemma 8.4 that functions that are constructed from smooth vector-valued functions defined over open sets in Euclidean spaces by means of the operations of additions, subtraction, multiplication and composition of functions must themselves be smooth functions over the open sets over which they are defined.

We now prove a lemma that guarantees the smoothness of matrix-valued functions obtained from smooth matrix-valued functions through the operation of matrix inversion. The lemma applies to functions $F: V \to \mathrm{GL}(m, \mathbb{R})$

defined over an open subset V of a Euclidean space $\mathbb{R}^n$ and taking values in the set $\mathrm{GL}(m, \mathbb{R})$ of invertible $m \times m$ matrices. The value $F(\mathbf{x})$ of such a function at a point $\mathbf{x}$ of V is thus an invertible $m \times m$ matrix, and thus the function $F: V \to \mathrm{GL}(m, \mathbb{R})$ determines a corresponding function $G: V \to \mathrm{GL}(m, \mathbb{R})$, where $G(\mathbf{x}) = F(\mathbf{x})^{-1}$ for all $\mathbf{x} \in V$. The coefficients of the matrices $F(\mathbf{x})$ and $G(\mathbf{x})$ are then functions of $\mathbf{x}$ as $\mathbf{x}$ varies over the open set V. Now the function F is C^k if and only if, for all i and j between 1 and m, the coefficient of the matrix $F(\mathbf{x})$ in the ith row and jth column is a C^k function of $\mathbf{x}$ throughout the open set V. We prove that if the function F is C^k for some positive integer k then the function G is also C^k. It follows that if the function F is smooth, then the function G is smooth.

Lemma 8.5 *Let m be a positive integer, let $M_m(\mathbb{R})$ denote the real vector space consisting of all $m \times m$ matrices with real coefficients, and let $\mathrm{GL}(m, \mathbb{R})$ be the open set in $M_m(\mathbb{R})$ whose elements are the invertible $m \times m$ matrices with real coefficients. Let V be an open set in $\mathbb{R}^n$ let $F: V \to \mathrm{GL}(m, \mathbb{R})$ be a function mapping V into $\mathrm{GL}(m, \mathbb{R})$, and let $G: V \to \mathrm{GL}(m, \mathbb{R})$ be defined such that $G(\mathbf{x}) = F(\mathbf{x})^{-1}$ for all $\mathbf{x} \in V$. Suppose that the function F is C^k. Then the function G is C^k.*

Proof For each $\mathbf{x} \in V$, the matrices $F(\mathbf{x})$ and $G(\mathbf{x})$ satisfy $F(\mathbf{x})G(\mathbf{x}) = I$, where I is the identity matrix. On differentiating this identity with respect to the ith coordinate function x_i on V, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, we find that

$$\frac{\partial F(\mathbf{x})}{\partial x_i} G(\mathbf{x}) + F(\mathbf{x}) \frac{\partial G(\mathbf{x})}{\partial x_i} = 0,$$

and therefore

$$\frac{\partial G(\mathbf{x})}{\partial x_i} = -F(\mathbf{x})^{-1} \frac{\partial F(\mathbf{x})}{\partial x_i} G(\mathbf{x}) = -G(\mathbf{x}) \frac{\partial F(\mathbf{x})}{\partial x_i} G(\mathbf{x}).$$

(In the above equation $F(\mathbf{x})$, $G(\mathbf{x})$ and their inverses and partial derivatives are $m \times m$ matrices that are multiplied using the standard operation of matrix multiplication.) Now sums and products of C^k real-valued functions are themselves C^k (see Lemma 8.3). It follows that if matrices are multiplied together, where the coefficients of those matrices are C^k real-valued functions defined over the open set V, the coefficients of the resultant matrix will also be C^k real-valued functions defined over V.

The equation above ensures that if the matrix-valued function F is C^k (so that the functions determining the coefficients of the matrix are real-valued C^k functions on V), then the first order partial derivatives of the function G are continuous, and therefore the function G itself is C^1, where

$G(\mathbf{x}) = F(\mathbf{x})^{-1}$ for all $\mathbf{x} \in V$. Moreover if G is C^j, where $1 \leq j < k$ then the coefficients of the first order partial derivatives of G are expressible as a sums of products of C^j real-valued functions and thus are themselves C^j functions. Thus the matrix-valued function G itself is C^{j+1}. Repeated applications of this result ensure that G is a C^k function as required. ∎

8.2 Lipschitz Conditions satisfied locally by Continuously Differentiable Functions

Let $\varphi: X \to \mathbb{R}^m$ be a function defined over a subset X of $\mathbb{R}^n$. The function V is said to satisfy a *Lipschitz condition* with *Lipschitz constant* M on X if the inequality

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \leq M|\mathbf{x} - \mathbf{x}'|,$$

satisfied for all points $\mathbf{x}$ and $\mathbf{x}$ of X. A function on X that satisfies such a Lipschitz condition is said to be *Lipschitz continuous* on X.

A standard theorem (often referred to as *Picard's Theorem*) in the theory of ordinary diffential equations guaranteeing the existence and uniqueness of solutions of initial value problems is only applicable when the function determining the differential equation satisfies an appropriate Lipschitz condition.

We use the result of Proposition 7.10 to show that continuously differentiable functions satisfy Lipschitz conditions with arbitrarily small Lipschitz constants in the neighbourhood around points where their derivative is zero.

Proposition 8.6 *Let $\varphi: V \to \mathbb{R}^m$ be a continuously differentiable function defined over an open set V in $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V at which $(D\varphi)_{\mathbf{p}} = 0$. Then, given any positive real number λ, there exists some positive real number δ such that*

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \leq \lambda|\mathbf{x} - \mathbf{x}'|$$

for all points $\mathbf{x}$ and $\mathbf{x}'$ of V that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{x}' - \mathbf{p}| < \delta$.

Proof Let $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and let

$$\varphi(\mathbf{x}) = (f_1(\mathbf{x}),\, f_2(\mathbf{x}), \ldots\, f_m(\mathbf{x}))$$

for all $\mathbf{x} \in V$. Then the derivative $(D\varphi)_{\mathbf{x}}$ of φ at a point $\mathbf{x}$ of V is represented by value at $\mathbf{x}$ of the Jacobian matrix whose coefficients are the partial derivatives $\dfrac{\partial f_i}{\partial x_j}$ for $i, j = 1, 2, \ldots, n$. Now the first order partial derivatives of the functions $f_1, f_2, \ldots, f_m$ are continuous, because φ is a continuously differentiable function. It follows that there exists some positive real number δ such that

$$\left|\frac{\partial f_i}{\partial x_j}\right| < \frac{\lambda}{\sqrt{mn}}$$

at all points $(x_1, x_2, \ldots, x_n)$ that satisfy $|x_j - p_j| < \delta$ for $i = 1, 2, \ldots, n$. It then follows from Proposition 7.10 that if the points $\mathbf{x}$ and $\mathbf{x}'$ of V satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{x}' - \mathbf{p}| < \delta$ then

$$|f_i(\mathbf{x}') - f_i(\mathbf{x})| \leq \frac{\lambda}{\sqrt{m}} |\mathbf{x}' - \mathbf{x}|$$

for $i = 1, 2, \ldots, m$. But then

$$|\varphi(\mathbf{x}') - \varphi(\mathbf{x})|^2 = \sum_{i=1}^{m} |f_i(\mathbf{x}') - f_i(\mathbf{x})|^2 \leq \lambda^2 |\mathbf{x}' - \mathbf{x}|^2.$$

and therefore $|\varphi(\mathbf{x}') - \varphi(\mathbf{x})| \leq \lambda |\mathbf{x}' - \mathbf{x}|$, as required. ∎

We shall apply Proposition 8.6 in order to prove a result that yields a Lipschitz condition satisfied by continuously differentiable functions. The statement of the result will make reference to the *operator norm* of a linear transformation. We therefore proceed by giving the definition of the operator norm of a linear transformation between (finite-dimensional) Euclidean spaces.

Definition Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation between Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ The *operator norm* $\|T\|_{\mathrm{op}}$ of the linear transformation T is defined so that

$$\|T\|_{\mathrm{op}} = \sup\{|T\mathbf{v}| : \mathbf{v} \in \mathbb{R}^n \text{ and } |\mathbf{v}| = 1\}.$$

Let T, T_1 and T_2 be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ and let c be a real number. Let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^\ltimes$, and let $\hat{\mathbf{v}} = |\mathbf{v}|^{-1}\mathbf{v}$. Then

$$|T\mathbf{v}| = ||\mathbf{v}|\,(T\hat{\mathbf{v}})| = |\mathbf{v}|\,|T\hat{\mathbf{v}}| \leq \|T\|_{\mathrm{op}} |\mathbf{v}|.$$

Also $|T\mathbf{v}| = 0$ when $\mathbf{v} = \mathbf{0}$. It follows that $|T\mathbf{v}| \leq \|T\|_{\mathrm{op}}|\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^n$. Also

$$|(T_1 + T_2)\mathbf{v}| = |T_1\mathbf{v} + T_2\mathbf{v}| \leq |T_1\mathbf{v}| + |T_2\mathbf{v}| \leq (\|T_1\|_{\mathrm{op}} + \|T_2\|_{\mathbf{op}})|\mathbf{v}|$$

for all $\mathbf{v} \in \mathbb{R}^n$. It follows that $\|T_1 + T_2\|_{\mathrm{op}} \leq \|T_1\|_{\mathrm{op}} + \|T_2\|_{\mathbf{op}}$. Also $|(cT)\mathbf{v}| = |c|\,|T\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^n$, and therefore $\|cT\|_{\mathrm{op}} = |c|\,\|T\|_{\mathrm{op}}$. The linear transformation T satisfies $\|T\|_{\mathrm{op}} = 0$ if and only if $T = 0$.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S: \mathbb{R}^m \to \mathbb{R}^l$ be linear transformations. Then

$$|ST\mathbf{v}| \leq \|S\|_{\mathrm{op}} |T\mathbf{v}| \leq \|S\|_{\mathrm{op}}\,\|T\|_{\mathrm{op}} |\mathbf{v}|,$$

and therefore $\|ST\|_{\mathrm{op}} \leq \|S\|_{\mathrm{op}}\,\|T\|_{\mathrm{op}}$.

It was shown in Lemma 7.1 that $|T\mathbf{v}| \leq \|T\|_{\mathrm{HS}}|\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^n$, where $\|T\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm of the linear operator T. It follows that $\|T\|_{\mathrm{op}} \leq \|T\|_{\mathrm{HS}}$.

Corollary 8.7 *Let $\varphi \colon V \to \mathbb{R}^m$ be a continuously differentiable function defined over an open set V in $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V. Let M be a positive real number satisfying $M > \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}}$, where*

$$\|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}} = \sup\{|(D\varphi)_{\mathbf{p}}\mathbf{v}| : \mathbf{v} \in \mathbb{R}^n \textit{ and } |\mathbf{v}| = 1\}.$$

Then there exists a positive real number δ such that

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \leq M|\mathbf{x} - \mathbf{x}'|$$

for all points $\mathbf{x}$ and $\mathbf{x}'$ of V that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{x}' - \mathbf{p}| < \delta$.

Proof Let $T = (D\varphi)_{\mathbf{p}}$, and let

$$M_0 = \|(D\varphi)_{\mathbf{p}}\|_{\mathrm{op}} = \sup\{|T\mathbf{v}| : \mathbf{v} \in \mathbb{R}^n \text{ and } |\mathbf{v}| = 1\},$$

and let $\lambda = M - M_0$. Let $\varphi \colon V \to \mathbb{R}^m$ be defined such that

$$\psi(\mathbf{x}) = \varphi(\mathbf{x}) - T\mathbf{x}$$

for all $\mathbf{x} \in V$. Then $(D\psi)_{\mathbf{p}} = (D\varphi)_{\mathbf{p}} - T = 0$. It follows from Proposition 8.6 that there exists a positive real number δ such that

$$|\psi(\mathbf{x}) - \psi(\mathbf{x}')| \leq \lambda|\mathbf{x} - \mathbf{x}'|$$

for all points $\mathbf{x}$ and $\mathbf{x}'$ of V that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{x}' - \mathbf{p}| < \delta$. Then

$$\begin{aligned} |\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| &= |\psi(\mathbf{x}) - \psi(\mathbf{x}') + T(\mathbf{x} - \mathbf{x}')| \\ &\leq |\psi(\mathbf{x}) - \psi(\mathbf{x}')| + |T(\mathbf{x} - \mathbf{x}')| \\ &\leq \lambda|\mathbf{x} - \mathbf{x}'| + M_0|\mathbf{x} - \mathbf{x}'| = M|\mathbf{x} - \mathbf{x}'| \end{aligned}$$

for all points $\mathbf{x}$ and $\mathbf{x}'$ of V that satisfy $|\mathbf{x} - \mathbf{p}| < \delta$ and $|\mathbf{x}' - \mathbf{p}| < \delta$, as required. ∎

Corollary 8.7 ensures that continuously differentiable functions of several real variables are *locally Lipschitz continuous*. This means that they satisfy a Lipschitz condition in some sufficiently small neighbourhood of any given point. This in turn ensures that standard theorems concerning the existence and uniqueness of ordinary differential equations can be applied to systems of ordinary differential equations specified in terms of continuously differentiable functions.

8.3 Local Invertibility of Differentiable Functions

Definition Let $\varphi: V \to \mathbb{R}^n$ be a continuous function defined over an open set V in $\mathbb{R}^n$ and mapping that open set into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V. A *local inverse* of the map $\varphi: V \to \mathbb{R}^n$ around the point $\mathbf{p}$ is a continuous function $\mu: W \to V$ defined over an open set W in $\mathbb{R}^n$ that satisfies the following conditions:

(i) $\mu(W)$ is an open set in $\mathbb{R}^n$ contained in V, and $\mathbf{p} \in \mu(W)$;

(ii) $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$.

If there exists a function $\mu: W \to V$ satisfying these conditions, then the function φ is said to be *locally invertible* around the point $\mathbf{p}$.

Lemma 8.8 *Let $\varphi: V \to \mathbb{R}^n$ be a continuous function defined over an open set V in $\mathbb{R}^n$ and mapping that open set into $\mathbb{R}^n$, let $\mathbf{p}$ be a point of V. and let $\mu: W \to V$ be a local inverse for the map ϕ around the point $\mathbf{p}$. Then $\varphi(\mathbf{x}) \in W$ and $\mu(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mu(W)$.*

Proof The definition of local inverses ensures that $\mu(W)$ is an open subset of V, $\mathbf{p} \in \mu(W)$ and $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$. Let $\mathbf{x} \in \mu(W)$. Then $\mathbf{x} = \mu(\mathbf{y})$ for some $\mathbf{y} \in W$. But then $\varphi(\mathbf{x}) = \varphi(\mu(\mathbf{y})) = \mathbf{y}$, and therefore $\varphi(\mathbf{x}) \in W$. Moreover $\mu(\varphi(\mathbf{x})) = \mu(\mathbf{y}) = \mathbf{x}$, as required. ∎

Let $\varphi: V \to \mathbb{R}^n$ be a continuous function defined over an open set V in $\mathbb{R}^n$ and mapping that open set into $\mathbb{R}^n$, let $\mathbf{p}$ be a point of V. and let $\mu: W \to V$ be a local inverse for the map ϕ around the point $\mathbf{p}$. Then the function from the open set $\mu(W)$ to the open set W that sends each point $\mathbf{x}$ of $\mu(W)$ to $\varphi(x)$ is invertible, and its inverse is the continuous function from W to $\varphi(W)$ that sends each point $\mathbf{y}$ of W to $\mu(\mathbf{y})$. A function between sets is *bijective* if it has a well-defined inverse. A continuous bijective function whose inverse is also continuous is said to be a *homeomorphism*. We see therefore that the restriction of the map φ to the image $\mu(W)$ of the local inverse $\mu: W \to V$ determines a homeomorphism from the open set $\mu(W)$ to the open set W.

Example The function $\varphi: \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$ defined such that

$$\varphi(u, v) = (e^u \cos v, e^u \sin v)$$

for all $u, v \in \mathbb{R}^2$ is locally invertible, though it is not bijective. Indeed, given $(u_0, v_0) \in \mathbb{R}$, let

$$W = \{(r \cos(v_0 + \theta), r \sin(v_0 + \theta)) : r, \theta \in \mathbb{R},\ \ r > 0 \text{ and } -\pi < \theta < \pi\},$$

and let

$$\mu(r\,\cos(v_0+\theta), r\,\sin(v_0+\theta)) = (\log r, v_0+\theta)$$

whenever $r > 0$ and $-\pi < \theta < 1$. Then W is an open set in $\mathbb{R}^2$,

$$\mu(W) = \{(u,v) \in \mathbb{R}^2 : v_0 - \pi < v < v_0 + \pi\},$$

and $\mu(\varphi(u,v)) = (u,v)$ for all $(u,v) \in \mu(W)$. Note that the smoothness of the logarithm and inverse trigonometrical functions guarantees that the local inverse $\mu: W \to \mathbb{R}^2$ is itself smooth.

A smooth function may have a continuous inverse, but that inverse is not guaranteed to be differentiable, as the following example demonstrates.

Example Let $f: \mathbb{R} \to \mathbb{R}$ be defined so that $f(x) = x^3$ for all real numbers x. The function f is smooth and has a continuous inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$, where $f^{-1}(x) = \sqrt[3]{x}$ when $x \geq 0$ and $f^{-1}(x) = -\sqrt[3]{-x}$ when $x < 0$. This inverse function is not differentiable at zero.

Lemma 8.9 *Let $\varphi: V \to \mathbb{R}^n$ be a continuously differentiable function defined over an open set V in $\mathbb{R}^n$. Suppose that φ is locally invertible around some point $\mathbf{p}$ of V. Suppose also that a local inverse to φ around $\mathbf{p}$ is differentiable at the point $\varphi(\mathbf{p})$. Then the derivative $(D\varphi)_{\mathbf{p}}: \mathbb{R}^n \to \mathbb{R}^n$ of φ at the point $\mathbf{p}$ is an invertible linear operator on $\mathbb{R}^n$. Thus if*

$$\varphi(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n),$$

for all $(x_1, x_2, \ldots, x_n) \in V$, where $y_1, y_2, \ldots, y_n$ are differentiable functions of $x_1, x_2, \ldots, x_n$, and if φ has a differentiable local inverse around the point $\mathbf{p}$, then the Jacobian matrix

$$\begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} & \cdots & \dfrac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial y_n}{\partial x_1} & \dfrac{\partial y_n}{\partial x_2} & \cdots & \dfrac{\partial y_n}{\partial x_n} \end{pmatrix}$$

is invertible at the point $\mathbf{p}$.

Proof Let $\mu: W \to V$ be a local inverse of φ around $\mathbf{p}$, where W is an open set in $\mathbb{R}^n$, $\mathbf{p} \in \mu(W)$, $\mu(W) \subset V$ and $\mu(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mu(W)$. Suppose that $\mu: W \to V$ is differentiable at $\varphi(\mathbf{p})$. The identity $\mu(\varphi(\mathbf{x})) = \mathbf{x}$ holds throughout the open neighbourhood $\mu(W)$ of point $\mathbf{p}$. Applying the Chain Rule (Theorem 7.9), we find that $(D\mu)_{\varphi(\mathbf{p})}(D\varphi)_{\mathbf{p}}$ is the identity operator on $\mathbb{R}^n$. It follows that the linear operators $(D\mu)_{\varphi(\mathbf{p})}$ and $(D\varphi)_{\mathbf{p}}$ on $\mathbb{R}^n$ are inverses of one another, and therefore $(D\varphi)_{\mathbf{p}}$ is an invertible linear operator on $\mathbb{R}^n$. The result follows. ∎

Lemma 8.10 *Let $\varphi: V \to \mathbb{R}^n$ be a continuously differentiable function defined over an open set V in $\mathbb{R}^n$ that is locally invertible around some point of V and let $\mu: W \to \mathbb{R}^n$ be a local inverse for φ. Suppose that $\varphi: V \to \mathbb{R}^n$ is continuously differentiable and that the local inverse $\mu: W \to \mathbb{R}^n$ is Lipschitz continuous throughout W. Then $\mu: W \to \mathbb{R}^n$ is continuously differentiable throughout W.*

Proof The function $\mu: W \to \mathbb{R}^n$ is Lipschitz continuous, and therefore there exists a positive constant C such that

$$|\mu(\mathbf{y}) - \mu(\mathbf{y}')| \leq C\,|\mathbf{y} - \mathbf{y}'|$$

for all $\mathbf{q}, \mathbf{y} \in W$. Let $\mathbf{q} \in W$, let $\mathbf{p} = \mu(\mathbf{q})$, and let S be the derivative of φ at $\mathbf{p}$. Then

$$S\mathbf{v} = \lim_{t\to 0} \frac{1}{t}(\varphi(\mathbf{p} + t\mathbf{v}) - \varphi(\mathbf{p}))$$

for all $\mathbf{v} \in \mathbb{R}^n$ (see Lemma 7.7). If $|t|$ is sufficiently small then $\mathbf{p}+t\mathbf{v} \in \mu(W)$. It then follows from Lemma 8.8 that

$$t\mathbf{v} = \mu(\varphi(\mathbf{p} + t\mathbf{v})) - \mu(\varphi(\mathbf{p})),$$

and therefore

$$|t||\mathbf{v}| \leq C\,|\varphi(\mathbf{p} + t\mathbf{v}) - \varphi(\mathbf{p})|.$$

It follows that

$$|S\mathbf{v}| = \lim_{t\to 0} \frac{1}{|t|}|\varphi(\mathbf{p} + t\mathbf{v}) - \varphi(\mathbf{p})| \geq \frac{1}{C}|\mathbf{v}|$$

for all $\mathbf{v} \in \mathbb{R}^n$, and therefore $S\mathbf{v} \neq \mathbf{0}$ for all non-zero vectors $\mathbf{v}$. It follows from basic linear algebra that the linear operator S on $\mathbb{R}^n$ is invertible. Moreover $|S^{-1}\mathbf{v}| \leq C|\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^n$.

Now

$$\lim_{\mathbf{x}\to\mathbf{p}} \frac{1}{|\mathbf{x} - \mathbf{p}|}\,|\varphi(\mathbf{x}) - \varphi(\mathbf{p}) - S(\mathbf{x} - \mathbf{p})| = 0,$$

because the function φ is differentiable at $\mathbf{p}$. Now $\mu(\mathbf{y}) \neq \mathbf{p}$ when $\mathbf{y} \neq \mathbf{q}$, because $\mathbf{q} = \varphi(\mathbf{p})$ and $\mathbf{y} = \varphi(\mu(\mathbf{y}))$. Also the continuity of μ ensures that $\mu(\mathbf{y})$ tends to $\mathbf{p}$ as $\mathbf{y}$ tends to $\mathbf{q}$. It follows that

$$\lim_{\mathbf{y}\to\mathbf{q}} \frac{1}{|\mu(\mathbf{y}) - \mathbf{p}|}\,|\mathbf{y} - \mathbf{q} - S(\mu(\mathbf{y}) - \mathbf{p})| = 0.$$

Now

$$|S^{-1}(\mathbf{y}-\mathbf{q}) - (\mu(\mathbf{y}) - \mathbf{p})| \leq C|\mathbf{y}-\mathbf{q} - S(\mu(\mathbf{y}) - \mathbf{p})|$$

for all $\mathbf{y} \in W$. Also

$$\frac{1}{|\mathbf{y}-\mathbf{q}|} \leq \frac{C}{|\mathbf{p} - \mu(\mathbf{y})|}$$

for all $\mathbf{y} \in W$ satisfying $\mathbf{y} \neq \mathbf{q}$. It follows that

$$\frac{1}{|\mathbf{y}-\mathbf{q}|}|\mu(\mathbf{y}) - \mathbf{p} - S^{-1}(\mathbf{y}-\mathbf{q})| \leq \frac{C^2}{|\mu(\mathbf{y}) - \mathbf{p}|}\,|\mathbf{y}-\mathbf{q} - S(\mu(\mathbf{y}) - \mathbf{p})|.$$

It follows that

$$\lim_{\mathbf{y}\to\mathbf{q}} \frac{1}{|\mathbf{y}-\mathbf{q}|}|\mu(\mathbf{y}) - \mathbf{p} - S^{-1}(\mathbf{y}-\mathbf{q})| = 0,$$

and therefore the function μ is differentiable at $\mathbf{q}$ with derivative S^{-1}. Thus $(D\mu)_{\mathbf{q}} = (D\varphi)_{\mathbf{p}}^{-1}$ for all $\mathbf{q} \in W$. It follows from this that $(D\mu)_{\mathbf{q}}$ depends continuously on $\mathbf{q}$, and thus the function μ is continuously differentiable on W, as required. ∎

Lemma 8.11 *Let $\varphi: V \to \mathbb{R}^n$ be a continuously differentiable function defined over an open set V in $\mathbb{R}^n$ that is locally invertible around some point of V and let $\mu: W \to \mathbb{R}^n$ be a local inverse for φ. Suppose that $\varphi: V \to \mathbb{R}^n$ is C^k and that the local inverse $\mu: W \to \mathbb{R}^n$ is differentiable throughout W. Then $\mu: W \to \mathbb{R}^n$ is C^k throughout W.*

Proof The functions φ and μ are differentiable, and $\mu(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mu(W)$. The Chain Rule (Theorem 7.9) then ensures that $(D\mu)_{\varphi(\mathbf{x})} \circ (D\varphi)_{\mathbf{x}}$ is the identity operator. Let $F(\mathbf{x})$ denote the Jacobian matrix representing the derivative $(D\varphi)_{\mathbf{x}}$ of φ at each point $\mathbf{x}$ of $\mu(W)$, and let $G(\mathbf{x})$ denote the Jacobian matrix representing the derivative $(D\mu)_{\varphi(\mathbf{x})}$ of μ at $\varphi(\mathbf{x})$. Then the Chain Rule ensures that $G(\mathbf{x})F(\mathbf{x})$ is the identity matrix. It follows that $F(\mathbf{x})$ and $G(\mathbf{x})$ are invertible matrices and $G(\mathbf{x}) = F(\mathbf{x})^{-1}$ for all $\mathbf{x} \in \mu(W)$. Now the function φ is C^k on V and therefore the matrix-valued function $F: \mu(W) \to \mathrm{GL}(n, \mathbb{R})$ is is C^k on $\mu(W)$. It follows from Lemma 8.5 that the matrix-valued function $G: \mu(W) \to \mathrm{GL}(n, \mathbb{R})$ is also C^k on $\mu(W)$.

Now the $(D\mu)_{\mathbf{y}}$ is represented by the matrix $G(\mu(\mathbf{y}))$ for all $\mathbf{y} \in W$. It follows from the continuity of μ and G that the derivative $D\mu$ of μ is continuous on W. It follows that μ is C^1. Moreover if $\mu: W \to V$ is C^j for any integer j satisfying $1 \leq j < k$ then $G \circ \mu$ is a composition of C^j functions and is therefore C^j (Lemma 8.4). But the coefficients of the matrix $G(\mu(\mathbf{y}))$ are the first order partial derivatives of the components of μ at $\mathbf{y}$ at each point $\mathbf{y}$ of W. It follows therefore that the first order partial derivatives of μ are C^j and therefore the function μ itself is C^{j+1}. It follows by repeated application of this process that the function μ is C^k on W, as required. ∎

8.4 The Inverse Function Theorem

The *Inverse Function Theorem* ensures that, for a C^k function of several real variables, mapping an open set in one Euclidean space into a Euclidean space of the same dimension, the invertibility of the derivative of the function at a given point is sufficient to ensure the local invertibility of that function around the given point, and moreover ensures that the inverse function is also locally a C^k function.

The proof uses the method of successive approximations, using a convergence criterion for infinite sequences of points in Euclidean space that we establish in the following lemma.

Lemma 8.12 *Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ be an infinite sequence of points in n-dimensional Euclidean space $\mathbb{R}^n$, and let λ be a real number satisfying $0 < \lambda < 1$. Suppose that*

$$|\mathbf{x}_{j+1} - \mathbf{x}_j| \leq \lambda |\mathbf{x}_j - \mathbf{x}_{j-1}|$$

for all integers j satisfying $j > 1$. Then the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is convergent.

Proof We show that an infinite sequence of points in Euclidean space satisfying the stated criterion is a Cauchy sequence and is therefore convergent. Now the infinite sequence satisfies

$$|\mathbf{x}_{j+1} - \mathbf{x}_j| \leq C\lambda^j$$

for all positive integers j, where $C = |\mathbf{x}_2 - \mathbf{x}_1|/\lambda$. Let j and k be positive integers satisfying $j < k$. Then

$$\begin{aligned} |\mathbf{x}_k - \mathbf{x}_j| &= \left| \sum_{s=j}^{k-1} (\mathbf{x}_{s+1} - \mathbf{x}_s) \right| \leq \sum_{s=j}^{k-1} |\mathbf{x}_{s+1} - \mathbf{x}_s| \\ &\leq C \sum_{s=j}^{k-1} \lambda^s = C\lambda^j \frac{1 - \lambda^{k-j}}{1 - \lambda} < \frac{C\lambda^j}{1 - \lambda}. \end{aligned}$$

We now show that the infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is a Cauchy sequence. Let some positive real number ε be given. Then a positive integer N can be chosen large enough to ensure that $C\lambda^N < (1 - \lambda)\varepsilon$. Then

$|\mathbf{x}_k - \mathbf{x}_j| < \varepsilon$ whenever $j \geq N$ and $k \geq N$. Therefore the given infinite sequence is a Cauchy sequence. Now all Cauchy sequences in $\mathbb{R}^n$ are convergent (see Lemma 6.4). Therefore the given infinite sequence is convergent, as required. ∎

Theorem 8.13 (Inverse Function Theorem) *Let $\varphi: V \to \mathbb{R}^n$ be a continuously differentiable function defined over an open set V in n-dimensional Euclidean space $\mathbb{R}^n$ and mapping V into $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of V. Suppose that $k \geq 1$ and that the derivative $(D\varphi)_{\mathbf{p}}: \mathbb{R}^n \to \mathbb{R}^n$ of the map φ at the point $\mathbf{p}$ is an invertible linear transformation. Then there exists an open set W in $\mathbb{R}^n$ and a continuously differentiable function $\mu: W \to V$ that satisfies the following conditions:—*

(i) $\mu(W)$ is an open set in $\mathbb{R}^n$ contained in V, and $\mathbf{p} \in \mu(W)$;

(ii) $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$.

Moreover if the function $\varphi: V \to \mathbb{R}^n$ is C^k for some positive integer k, then so is the function $\mu: W \to V$.

Proof We may assume, without loss of generality, that $\mathbf{p} = \mathbf{0}$ and $\varphi(\mathbf{p}) = \mathbf{0}$. Indeed the result in the general case can then be deduced by applying the result in this special case to the function that sends $\mathbf{z}$ to $\varphi(\mathbf{p}+\mathbf{z}) - \varphi(\mathbf{p})$ for all $\mathbf{z} \in \mathbb{R}^n$ for which $\mathbf{p} + \mathbf{z} \in V$.

Now $(D\varphi)_{\mathbf{0}}: \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, by assumption. Let $T = (D\varphi)_{\mathbf{0}}^{-1}$, and let $\psi: V \to \mathbb{R}^n$ be defined such that

$$\psi(\mathbf{x}) = \mathbf{x} - T(\varphi(\mathbf{x}))$$

for all $\mathbf{x} \in V$. Now the derivative of any linear transformation at any point is equal to that linear transformation (see Lemma 7.4). It follows from the Chain Rule that the derivative of the composition function $T \circ \varphi$ at any point $\mathbf{x}$ of V is equal to $T(D\varphi)_{\mathbf{x}}$. It follows that $(D\psi)_{\mathbf{x}} = I - T(D\varphi)_{\mathbf{x}}$ for all $\mathbf{x} \in V$, where I denotes the identity operator on $\mathbb{R}^n$. In particular $(D\psi)_{\mathbf{0}} = I - T(D\varphi)_{\mathbf{0}} = 0$. It then follows from Proposition 8.6 that there exists a positive real number δ such that

$$|\psi(\mathbf{x}') - \psi(\mathbf{x})| \leq \tfrac{1}{2}|\mathbf{x}' - \mathbf{x}|$$

whenever $|\mathbf{x}| < \delta$ and $|\mathbf{x}'| < \delta$.

Now $\psi(\mathbf{0}) = \mathbf{0}$. It follows from the inequality just proved that $|\psi(\mathbf{x})| \leq \frac{1}{2}|\mathbf{x}|$ whenever $|\mathbf{x}| < \delta$.

Let W be the open set in $\mathbb{R}^n$ defined so that

$$W = \{\mathbf{y} \in \mathbb{R}^n : |T(\mathbf{y})| < \tfrac{1}{2}\delta\},$$

and let $\mu_0, \mu_1, \mu_2, \ldots$ be the infinite sequence of functions from W to $\mathbb{R}^n$ defined so that $\mu_0(\mathbf{y}) = 0$ for all $\mathbf{y} \in W$ and

$$\mu_j(\mathbf{y}) = \mu_{j-1}(\mathbf{y}) + T(\mathbf{y} - \varphi(\mu_{j-1}(\mathbf{y})))$$

for all positive integers j. We shall prove that there is a well-defined function $\mu: W \to \mathbb{R}^n$ defined such that $\mu(\mathbf{y}) = \lim_{j\to+\infty} \mu_j(\mathbf{y})$ and that this function μ is a local inverse for φ defined on the open set W that satisfies the required properties.

Let $\mathbf{y} \in W$ and let $\mathbf{x}_j = \mu_j(\mathbf{y})$ for all non-negative integers j. Then $\mathbf{x}_0 = \mathbf{0}$ and

$$\begin{aligned} \mathbf{x}_j &= \mathbf{x}_{j-1} + T(\mathbf{y} - \varphi(\mathbf{x}_{j-1})) \\ &= \psi(\mathbf{x}_{j-1}) + T\mathbf{y} \end{aligned}$$

for all positive integers j. Now we have already shown that $|\psi(\mathbf{x})| \leq \frac{1}{2}|\mathbf{x}|$ whenever $|\mathbf{x}| < \delta$. Also the definition of the open set W ensures that $|T\mathbf{y}| < \frac{1}{2}\delta$. It follows that if $|\mathbf{x}_{j-1}| < \delta$ then

$$|\mathbf{x}_j| \leq |\psi(\mathbf{x}_{j-1})| + |T\mathbf{y}| \leq \tfrac{1}{2}|\mathbf{x}_{j-1}| + |T\mathbf{y}| < \tfrac{1}{2}\delta + |T\mathbf{y}| < \delta.$$

It follows by induction on j that $|\mathbf{x}_j| < \frac{1}{2}\delta + |T\mathbf{y}|$ for all non-negative integers j. Also

$$\begin{aligned} \mathbf{x}_{j+1} - \mathbf{x}_j &= \mathbf{x}_j - \mathbf{x}_{j-1} - T(\varphi(\mathbf{x}_j) - \varphi(\mathbf{x}_{j-1})) \\ &= \psi(\mathbf{x}_j) - \psi(\mathbf{x}_{j-1}) \end{aligned}$$

for all positive integers j. But $|\mathbf{x}_j| < \delta$ and $|\mathbf{x}_{j-1}| < \delta$ and therefore

$$|\mathbf{x}_{j+1} - \mathbf{x}_j| = |\psi(\mathbf{x}_j) - \psi(\mathbf{x}_{j-1})| \leq \tfrac{1}{2}|\mathbf{x}_j - \mathbf{x}_{j-1}|$$

for all positive integers j. It then follows from Lemma 8.12 that the infinite sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ is convergent. Now $\mathbf{x}_j = \mu_j(\mathbf{y})$ for all non-negative integers j, where $\mathbf{y}$ is an arbitrary element of the open set W. The convergence result just obtained therefore guarantees that there is a well-defined function $\mu: W \to \mathbb{R}^n$ which satisfies

$$\mu(\mathbf{y}) = \lim_{j\to+\infty} \mu_j(\mathbf{y})$$

for all $\mathbf{y} \in W$. Moreover $|\mu_j(\mathbf{y})| < \frac{1}{2}\delta + |T\mathbf{y}|$ for all positive integers j and for all $\mathbf{y} \in W$, and therefore

$$|\mu(\mathbf{y})| \leq \tfrac{1}{2}\delta + |T\mathbf{y}| < \delta$$

for all $\mathbf{y} \in W$.

Next we prove that $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$. Now

$$\begin{aligned} \mu(\mathbf{y}) &= \lim_{j \to +\infty} \mu_j(\mathbf{y}) = \lim_{j \to +\infty} (\mu_{j-1}(\mathbf{y}) + T(\mathbf{y} - \varphi(\mu_{j-1}(\mathbf{y})))) \\ &= \mu(\mathbf{y}) + T(\mathbf{y} - \varphi(\mu(\mathbf{y}))) \end{aligned}$$

It follows that $T(\mathbf{y} - \varphi(\mu(\mathbf{y}))) = \mathbf{0}$, and therefore

$$\mathbf{y} - \varphi(\mu(\mathbf{y})) = (D\varphi)_{\mathbf{0}}(T(\mathbf{y} - \varphi(\mu(\mathbf{y})))) = (D\varphi)_{\mathbf{0}}(\mathbf{0}) = \mathbf{0}.$$

Thus $\mathbf{y} = \varphi(\mu(\mathbf{y}))$ for all $\mathbf{y} \in W$. Also $\mu_j(\mathbf{0}) = \mathbf{0}$ for all non-negative integers j, and therefore $\mu(\mathbf{0}) = \mathbf{0}$.

Next we show that if $\mathbf{x} \in \mathbb{R}^n$ satisfies $|\mathbf{x}| < \delta$ and if $\varphi(x) \in W$ then $\mathbf{x} = \mu(\varphi(\mathbf{x}))$. Now $\mathbf{x} = \psi(\mathbf{x}) + T\varphi(\mathbf{x})$ for all $\mathbf{x} \in V$. Also

$$|T\varphi(\mathbf{x})| \leq \|T\|_{\mathrm{op}}\, |\varphi(\mathbf{x})|$$

for all $\mathbf{x} \in V$, where

$$\|T\|_{\mathrm{op}} = \sup\{|T\mathbf{v}| : \mathbf{v} \in \mathbb{R}^n \text{ and } |\mathbf{v}| = 1\}.$$

It follows that

$$\begin{aligned} |\mathbf{x} - \mathbf{x}'| &= |\psi(\mathbf{x}) - \psi(\mathbf{x}') + T(\varphi(\mathbf{x}) - \varphi(\mathbf{x}'))| \\ &\leq |\psi(\mathbf{x}) - \psi(\mathbf{x}')| + |T^{-1}(\varphi(\mathbf{x}) - \varphi(\mathbf{x}'))| \\ &\leq \tfrac{1}{2}|\mathbf{x} - \mathbf{x}'| + \|T\|_{\mathrm{op}}\, |\varphi(\mathbf{x}) - \varphi(\mathbf{x}')| \end{aligned}$$

whenever $|\mathbf{x}| < \delta$ and $|\mathbf{x}'| < \delta$. Subtracting $\frac{1}{2}|\mathbf{x} - \mathbf{x}'|$ from both sides of the above inequality, and then multiplying by two, we find that

$$|\mathbf{x} - \mathbf{x}'| \leq 2\|T\|_{\mathrm{op}}\, |\varphi(\mathbf{x}) - \varphi(\mathbf{x}')|.$$

whenever $|\mathbf{x}| < \delta$ and $|\mathbf{x}'| < \delta$. Substituting $\mathbf{x}' = \mu(\mathbf{y})$, we find that

$$|\mathbf{x} - \mu(\mathbf{y})| \leq 2\|T\|_{\mathrm{op}}\, |\varphi(\mathbf{x}) - \mathbf{y}|$$

for all $\mathbf{x} \in V$ satisfying $|\mathbf{x}| < \delta$ and for all $\mathbf{y} \in W$. It follows that if $\mathbf{x} \in V$ satisfies $|\mathbf{x}| < \delta$ and if $\varphi(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{y} \in W$ then $\mathbf{x} = \mu(\mathbf{y})$. The inequality also ensures that

$$|\mu(\mathbf{y}) - \mu(\mathbf{y}')| \leq 2\|T\|_{\mathrm{op}}\, |\mathbf{y} - \mathbf{y}'|$$

for all $\mathbf{y}, \mathbf{y}' \in W$. Thus the function $\mu: W \to V$ is Lipschitz continuous. It then follows from Lemma 8.10 that the function μ is continuously differentiable.

Next we prove that $\mu(W)$ is an open subset of V. Now $\mu(W) \subset \varphi^{-1}(W)$ because $\mathbf{y} = \varphi(\mu(\mathbf{y}))$ for all $\mathbf{y} \in W$. We have also proved that $|\mu(\mathbf{y})| < \delta$ for all $\mathbf{y} \in W$. It follows that

$$\mu(W) \subset \varphi^{-1}(W) \cap \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < \delta\}.$$

But we have also shown that if $\mathbf{x} \in V$ satisfies $|\mathbf{x}| < \delta$, and if $\varphi(\mathbf{x}) \in W$ then $\mathbf{x} = \mu(\varphi(\mathbf{x}))$, and therefore $\mathbf{x} \in \mu(W)$. It follows that

$$\mu(W) = \varphi^{-1}(W) \cap \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < \delta\}.$$

Now $\varphi^{-1}(W)$ is an open subset in V, because $\varphi: V \to \mathbb{R}^n$ is continuous and W is an open set in $\mathbb{R}^n$ (see Proposition 6.19). It follows that $\mu(W)$ is an intersection of two open sets, and is thus itself an open set. Moreover $\mathbf{0} \in \mu(W)$, because $\mu(\mathbf{0}) = \mathbf{0}$. We have now completed the proof that $\mu: W \to V$ satisfies properties (i) and (ii) in the statement of the theorem, and is thus a continuously differentiable local inverse for the map $\varphi: V \to \mathbb{R}^n$.

The result that this local inverse is C^k when φ is C^k then follows from Lemma 8.11. This completes the proof of the Inverse Function Theorem. ∎

Corollary 8.14 *Let $\varphi: V \to \mathbb{R}^n$ be a smooth function defined over an open set V in n-dimensional Euclidean space $\mathbb{R}^n$ and mapping V into $\mathbb{R}^n$. Then φ has a smooth local inverse around any point $\mathbf{p}$ at which the derivative $(D\varphi)_{\mathbf{p}}$ is invertible.*

Proof This result follows directly from the Inverse Function Theorem (Theorem 8.13), in view of the fact that a function $\varphi: V \to \mathbb{R}^n$ is smooth if and only if it is C^k for all positive integers k.

Definition Let V and W be open sets in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\varphi: V \to W$ be a function from V to W. The function φ is said to be a *diffeomorphism* if it has a well-defined inverse $\varphi^{-1}: W \to V$ and both the function $\varphi: V \to W$ and its inverse $\varphi^{-1}: W \to V$ are smooth functions.

Definition Let V be an open set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\varphi: V \to \mathbb{R}^n$ be a smooth function from V to $\mathbb{R}^n$. Let U be an open subset of V. We say that φ maps U *diffeomorphically* onto an open set of $\mathbb{R}^n$ if $\varphi(U)$ is an open set in $\mathbb{R}^n$ and the restriction of the function φ to U is a diffeomorphism from U to $\varphi(U)$.

Corollary 8.15 *Let V be an open set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $\varphi: V \to \mathbb{R}^n$ be a smooth function from V to $\mathbb{R}^n$, and let $\mathbf{p} \in V$. Suppose that the derivative $(D\varphi)_{\mathbf{p}}$ of φ is invertible at the point $\mathbf{p}$. Then there exists an open subset U of V, where $\mathbf{p} \in U$, that is mapped diffeomorphically by φ onto an open set in $\mathbb{R}^n$.*

Proof The derivative $(D\varphi)_{\mathbf{p}}$ of φ is invertible at the point $\mathbf{p}$. It follows from the Inverse Function Theorem (Theorem 8.13) that there exists an open set W in $\mathbb{R}^n$ and a smooth map $\mu: W \to V$ such that $\mu(W)$ is an open subset of V, $\mathbf{p} \in \mu(W)$ and $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$. Let $U = \mu(W)$. Then $\varphi(U) = W$, because $\varphi(\mu(\mathbf{y})) = \mathbf{y}$ for all $\mathbf{y} \in W$. Moreover if $\mathbf{x} \in U$ then $\mathbf{x} = \mu(\mathbf{y})$ for some point $\mathbf{y}$ of W. But then

$$\mu(\varphi(\mathbf{x})) = \mu(\varphi(\mu(\mathbf{y}))) = \mu(\mathbf{y}) = \mathbf{x}.$$

Thus $\mu(\varphi(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in U$. It follows that φ maps the open set U diffeomorphically onto W, and the inverse of this diffeomorphism from U to W is the smooth map μ. The result follows. ∎

8.5 Smooth Curvilinear Coordinate Systems

Definition Let U be an open set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $u_1, u_2, \ldots, u_n$ be smooth real-valued functions defined throughout U, and let

$$\tilde{U} = \{(u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x})) : \mathbf{x} \in U\}.$$

Then the smooth real-valued functions $u_1, u_2, \ldots, u_n$ are said to constitute a *smooth curvilinear coordinate system* on U if $\tilde{U}$ is an open set in $\mathbb{R}^n$ and there exist smooth real-valued functions $\xi_1, \xi_2, \ldots, \xi_n$ defined over $\tilde{U}$ such that

$$x_i = \xi_i(u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2, \ldots, x_n), \ldots, u_n(x_1, x_2, \ldots, x_n))$$

for all $\mathbf{x} \in U$.

Let U be an open set in n-dimensional Euclidean space $\mathbb{R}^n$, and let $u_1, u_2, \ldots, u_n$ be smooth real-valued functions defined throughout U, and let

$$\tilde{U} = \{(u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x})) : \mathbf{x} \in U\}.$$

Let $\varphi: U \to \tilde{U}$ be defined so that

$$\varphi(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x})).$$

Then the smooth real-valued functions $u_1, u_2, \ldots, u_n$ constitute a smooth curvilinear coordinate system on U if and only if $\varphi: U \to \tilde{U}$ is a diffeomorphism.

Suppose that $u_1, u_2, \ldots, u_n$ constitute a smooth curvilinear coordinate system on the open set U. Let the open set $\tilde{U}$ and the diffeomorphism $\varphi U \to \tilde{U}$ be defined as described above. A differentiable function $f: U$ determines a corresponding differentiable function $f \circ \varphi^{-1}: \tilde{U} \to U$. The partial derivatives of the function f with respect to the curvilinear coordinates $u_1, u_2, \ldots, u_n$ are then defined so that

$$\begin{aligned} \left.\frac{\partial f}{\partial u_j}\right|_{\mathbf{p}} &= (\partial_j(f \circ \varphi^{-1}))(\varphi(\mathbf{p})) \\ &= \left.\frac{\partial f(\xi_1(y_1, \ldots, y_n), \ldots, \xi_n(y_1, \ldots, y_n))}{\partial y_j}\right|_{(y_1, \ldots, y_n) = \varphi(\mathbf{p})} \end{aligned}$$

for all $\mathbf{p} \in U$, where $\partial_j(f \circ \varphi^{-1})$ denotes the partial derivative of $f \circ \varphi^{-1}$ with respect to the jth Cartesian coordinate on the open set $\tilde{U}$. The Chain Rule (Theorem 7.9) ensures that

$$(Df)_{\mathbf{p}} = D(f \circ \varphi^{-1})_{\varphi(\mathbf{p})} (D\varphi)_{\mathbf{p}}.$$

It follows that

$$\begin{aligned} \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{p}} &= (\partial_i f)(\mathbf{p}) = \sum_{j=1}^{n} (\partial_j(f \circ \varphi^{-1}))(\varphi(\mathbf{p}))\, (\partial_i u_j)(\mathbf{p}) \\ &= \sum_{j=1}^{n} (\partial_j(f \circ \varphi^{-1}))(\varphi(\mathbf{p}))\, (\partial_i u_j)(\mathbf{p}) \\ &= \sum_{j=1}^{n} \left.\frac{\partial f}{\partial u_j}\right|_{\mathbf{p}} \left.\frac{\partial u_j}{\partial x_i}\right|_{\mathbf{p}}. \end{aligned}$$

This establishes the Chain Rule

$$\frac{\partial f}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial x_i}$$

used to compute the partial derivatives of a smooth real-valued function f on the domain U of a smooth curvilinear coordinate system $u_1, u_2, \ldots, u_n$.

We can apply the Chain Rule when the functions to be differentiated are the Cartesian coordinate functions on U itself. We find that

$$\sum_{j=1}^{n} \frac{\partial x_k}{\partial u_j} \frac{\partial u_j}{\partial x_i} = \begin{cases} 1 & \text{if } i = k; \\ 0 & \text{if } i \neq k. \end{cases}$$

It follows that the Jacobian matrices associated with the change of coordinates satisfy

$$\begin{pmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\ \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_2}{\partial u_2} & \cdots & \dfrac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial u_1} & \dfrac{\partial x_n}{\partial u_2} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{pmatrix}^{-1}.$$

Let $v_1, v_2, ldots, v_n$ be another smooth local coordinate system defined over an open set V, where $U \cap V$ is non-empty. Then

$$\sum_{j=1}^{n} \frac{\partial f}{\partial u_j} \frac{\partial u_j}{\partial x_i} = \frac{\partial f}{\partial x_i} = \sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial x_i} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_j} \frac{\partial u_j}{\partial x_i}$$

throughout $U \cap V$ for $i = 1, 2, \ldots, n$. It then follows from the invertibility of the Jacobian matrix of partial derivatives of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ that

$$\frac{\partial f}{\partial u_j} = \sum_{j=1}^{n} \frac{\partial f}{\partial v_k} \frac{\partial v_k}{\partial u_j}$$

throughout $U \cap V$ for $j = 1, 2, \ldots, n$.

The fact that compositions of smooth functions are smooth ensures that the smooth curvilinear coordinates $v_1, v_2, \ldots, v_n$ can be expressed as smooth functions of $u_1, u_2, \ldots, u_n$ and vice versa throughout the open set $U \cap V$ where the domains of the smooth curvilinear coordinate systems overlap.

Proposition 8.16 *Let $u_1, u_2, \ldots, u_n$ be smooth real-valued functions defined throughout some open neighbourhood of a point $\mathbf{p}$ of n-dimensional Euclidean space $\mathbb{R}^n$. Suppose that the Jacobian matrix*

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{pmatrix}$$

of partial derivatives of $u_1, u_2, \ldots, u_n$ with respect to $x_1, x_2, \ldots, x_n$ is invertible at the point $\mathbf{p}$. Then there exists an open set U containing the point $\mathbf{p}$

such that the restrictions of the functions $u_1, u_2, \ldots, u_n$ to the open set U constitute a smooth curvilinear coordinate system over the open set U.

Proof This result is essentially a restatement of the Inverse Function Theorem (Theorem 8.13), and follows directly from Corollary 8.15 and the definition of smooth curvilinear coordinate systems. ∎

8.6 The Implicit Function Theorem

Theorem 8.17 *Let $\mathbf{p}$ be a point of $\mathbb{R}^n$, where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ and let $u_1, u_2, \ldots, u_m$ be a smooth real-valued functions defined over an open neighbourhood V of the point $\mathbf{p}$ in $\mathbb{R}^n$, where $m < n$, and let*

$$M = \{\mathbf{x} \in V : u_j(\mathbf{x}) = 0 \text{ for } j = 1, 2, \ldots, m\}.$$

Suppose that $u_1, u_2, \ldots, u_n$ are zero at $\mathbf{p}$ and that the matrix

$$\begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_m} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_m}{\partial x_1} & \dfrac{\partial u_m}{\partial x_2} & \cdots & \dfrac{\partial u_m}{\partial x_m} \end{pmatrix}$$

is invertible at the point $\mathbf{p}$. Then there exists an open neighbourhood U of $\mathbf{p}$ and a smooth functions $f_1, f_2, \ldots, f_m$ of $n - m$ real variables, defined around $(p_{m+1}, \ldots, p_n)$ in $\mathbb{R}^{n-m}$, such that

$$M \cap U = \{(x_1, x_2, \ldots, x_n) \in U : x_j = f_j(x_{m+1}, \ldots, x_n) \text{ for } j = 1, 2, \ldots, m\}.$$

Proof Let $u_j = x_j$ for $j = m + 1, \ldots, n$, and let

$$J_0 = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_m} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_m}{\partial x_1} & \dfrac{\partial u_m}{\partial x_2} & \cdots & \dfrac{\partial u_m}{\partial x_m} \end{pmatrix}, \quad J = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{pmatrix}.$$

(The matrix J_0 is thus the leading $m \times m$ minor of the $n \times n$ matrix J.) Now

$$J = \begin{pmatrix} \dfrac{\partial u_1}{\partial x_1} & \dots & \dfrac{\partial u_1}{\partial x_m} & \dfrac{\partial u_1}{\partial x_{m+1}} & \dfrac{\partial u_1}{\partial x_{m+2}} & \dots & \dfrac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \\ \dfrac{\partial u_m}{\partial x_1} & \dots & \dfrac{\partial u_m}{\partial x_m} & \dfrac{\partial u_m}{\partial x_{m+1}} & \dfrac{\partial u_m}{\partial x_{m+2}} & \dots & \dfrac{\partial u_m}{\partial x_n} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

It follows from basic properties of determinants that $\det J = \det J_0$, and therefore $\det J$ is non-zero at the point $\mathbf{p}$. It follows that matrix J whose coefficients are the first order partial derivatives of $u_1, u_2, \dots, u_n$ with respect to $x_1, x_2, \dots, x_n$ is invertible at the point p. It then follows from Proposition 8.16 that $u_1, u_2, \dots, u_n$ is a smooth curvilinear coordinate system defined over some open set U that contains the point $\mathbf{p}$ and satisfies $u_j = x_j$ for $j > m$. It then follows that there exist smooth real-valued functions ξ_1, ξ_2, ξ_n such that

$$x_j = \xi_j(u_1, u_2, \dots, u_m, x_{m+1}, \dots, x_n)$$

for $j = 1, 2, \dots, n$. Let

$$f_j(x_{m+1}, \dots, x_n) = \xi_j(0, 0, \dots, 0, x_{m+1}, \dots, x_n)$$

for $j = 1, 2, \dots, m$. Then

$$M \cap U = \{(x_1, x_2, \dots, x_n) \in U : x_j = f_j(x_{m+1}, \dots, x_n) \text{ for } j = 1, 2, \dots, m\},$$

as required. ∎

Corollary 8.18 *Let $u: V \to \mathbb{R}$ be a smooth real-valued function defined over an open subset V of $\mathbb{R}^n$. Suppose that $\dfrac{\partial u}{\partial x_n} \neq 0$ at some point $\mathbf{p}$ of V, where $p = (p_1, p_2, \dots, p_n)$. Then there exist an open neighbourhood U of $\mathbf{p}$ and a smooth real-valued function f, defined throughout some open neighbourhood of $(p_1, p_2, \dots, p_{n-1})$ in $\mathbb{R}^{n-1}$, such that*

$$\{\mathbf{x} \in U : u(\mathbf{x}) = 0\} = \{(x_1, x_2, \dots, x_n) \in U : x_n = f(x_1, x_2, \dots, x_{n-1})\}.$$

Proof This result comes directly on applying the Implicit Function Theorem (Theorem 8.17), after reordering Cartesian coordinates so that x_n precedes $x_1, x_2, \dots, x_{n-1}$. ∎

8.7 Submanifolds of Euclidean Spaces

A function is said to be *injective* (or *one-to-one*) if distinct points of the domain get mapped to distinct points of the codomain.

Let M be a subset of n-dimensional Euclidean space $\mathbb{R}^n$, and let Let $\alpha: U \to M$ be a smooth function mapping some open subset U of a Euclidean space $\mathbb{R}^k$ into M, where $0 < k < n$. The function α is injective if and only if $\alpha(\mathbf{u}) \neq \alpha(\mathbf{u}')$ for all $\mathbf{u}, \mathbf{u}' \in U$ satisfying $\mathbf{u} \neq \mathbf{u}'$. If $\alpha: U \to M$ is injective, then there is a well-defined function $\rho: \alpha(U) \to U$ defined such that $\rho(\alpha(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in U$.

The range $\alpha(U)$ of the map α is open in M if and only if, given any point $\mathbf{p}$ of $\alpha(U)$, there exists some $\delta > 0$ such that all points of M that lie within a distance δ of the point $\mathbf{p}$ belong to $\alpha(U)$.

The derivative $(D\alpha)_{\mathbf{u}}$ of α at each point $\mathbf{u}$ has a *rank* which is by definition the dimension of the image $(D\alpha)_{\mathbf{u}}(\mathbb{R}^k)$ of the linear transformation $(D\alpha)_{\mathbf{u}}: \mathbb{R}^k \to \mathbb{R}^n$. The rank of $(D\alpha)_{\mathbf{u}}$ is some integer between 0 and k. We consider the local properties of the image of a smooth injective function $\alpha: U \to \mathbb{R}^n$ defined over an open subset U of $\mathbb{R}^k$ in the case where the rank of the derivative of α at each point of U has its maximum possible value, which is k.

Proposition 8.19 *Let k and n be positive integers satisfying $k < n$, let let U be an open set in $\mathbb{R}^k$, let $\alpha: U \to \mathbb{R}^n$ be a smooth injective function from U into $\mathbb{R}^n$. Suppose that the following conditions are satisfied:—*

(i) the function $\alpha: U \to \mathbb{R}^n$ is injective;

(ii) the inverse of α on the set $\alpha(U)$ is a continuous map from $\alpha(U)$ to U;

(iii) the derivative $(D\alpha)_{\mathbf{u}}$ of α at each point $\mathbf{u}$ of U has rank k.

Then, given any point $\mathbf{p}$ of $\alpha(U)$, there exists an open set W in $\mathbb{R}^n$, where $\mathbf{p} \in W$, and a smooth curvilinear coordinate system $w_1, w_2, \ldots, w_n$ defined over W such that

$$\alpha(U) \cap W = \{\mathbf{x} \in W : w_j(\mathbf{x}) = 0 \text{ for } k < j \leq n\}$$

and $w_j(\alpha(u_1, u_2, \ldots, u_k)) = u_j$ for all $(u_1, u_2, \ldots, u_k) \in U$.

Proof Let $\mathbf{p}$ be a point of $\alpha(U)$. Then there exists $\mathbf{u} \in U$ such that $\mathbf{p} = \alpha(\mathbf{u})$. We may assume, without loss of generality, that $\mathbf{p} = \alpha(\mathbf{0})$. Then $(D\alpha)_{\mathbf{0}}(\mathbb{R}^k)$ is a vector subspace of $\mathbb{R}^n$ of dimension k. Let

$$\mathbf{v}_j = \left.\frac{\partial \alpha(u_1, u_2, \ldots, u_k)}{\partial u_j}\right|_{(u_1, u_2, \ldots, u_k) = \mathbf{0}}$$

for $j = 1, 2, \ldots, k$. Then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent vectors in $(D\alpha)_{\mathbf{p}}(\mathbb{R}^k)$ that span this vector space. It follows from standard linear algebra that there exist vectors $\mathbf{v}_j$ in $\mathbb{R}^n$ for $j = k+1, \ldots, n$ such that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ consistute a basis of the vector space $\mathbb{R}^n$. The smooth function $\alpha\colon U \to \mathbb{R}^n$ then extends to a smooth function $\beta\colon V \to \mathbb{R}^n$, where

$$V = \{(u_1, u_2, \ldots, u_k, 0, \ldots, 0) \in \mathbb{R}^n : (u_1, u_2, \ldots, u_k) \in U\}$$

and

$$\beta(u_1, u_2, \ldots, u_n) = \alpha(u_1, u_2, \ldots, u_k) + \sum_{j=k+1}^{n} u_j \mathbf{v}_j.$$

Let $\lambda\colon U \to V$ be defined so that

$$\lambda(u_1, u_2, \ldots, u_k) = (u_1, u_2, \ldots, u_k, 0, \ldots, 0)$$

for all $(u_1, u_2, \ldots, u_k) \in U$. Then $\alpha(\mathbf{u}) = \beta(\lambda(\mathbf{u}))$ for all $\mathbf{u} \in U$.

Let $\mathbf{e}_j$ be the unit vector in $\mathbb{R}^n$ whose jth component has the value 1 and whose other components are zero for $j = 1, 2, \ldots, n$. Then $(D\beta)_{\mathbf{0}}\,\mathbf{e}_j = \mathbf{v}_j$ for $j = 1, 2, \ldots, n$. It follows that $(D\beta)_{\mathbf{0}}$ is an invertible linear transformation whose inverse sends $\mathbf{v}_j$ to $\mathbf{e}_j$ for $j = 1, 2, \ldots, n$. It then follows from the Inverse Function Theorem (Theorem 8.13) that there exists a smooth local inverse $\mu\colon W_0 \to V$ for the map α around the point $\mathbf{p}$ defined over some open set W_0 in $\mathbb{R}^n$. Then $\mathbf{p} \in W_0$, $\mu\colon W_0 \to V$ is a smooth function from W_0 to V, $\mu(W_0)$ is an open subset of V and $\beta(\mu(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in W_0$.

Now $\lambda^{-1}(\mu(W_0))$ is an open set in $\mathbb{R}^k$ and $\mathbf{0} \in \lambda^{-1}(\mu(W_0))$. It follows that there exists some positive number η such that $\mathbf{u} \in \lambda^{-1}(\mu(W_0))$ for all $\mathbf{u} \in \mathbb{R}^k$ satisfying $|\mathbf{u}| < \eta$. The continuity of the inverse of α on $\alpha(U)$ then ensures the existence of a positive real number δ such that $|\mathbf{u}| < \eta$ for all $\mathbf{u} \in U$ satisfying $|\alpha(\mathbf{u}) - \mathbf{p}| < \delta$. Let

$$W = \{\mathbf{x} \in W_0 : |\mathbf{x} - \mathbf{p}| < \delta\}.$$

If $\mathbf{u} \in U$ and if $\alpha(\mathbf{u}) \in W$ then $|\alpha(\mathbf{u}) - \mathbf{p}| < \delta$, and therefore $|\mathbf{u}| < \eta$. But then $\mathbf{u} \in \lambda^{-1}(\mu(W_0))$, and therefore $\lambda(\mathbf{u}) = \mu(\mathbf{x})$ for some $\mathbf{x} \in W_0$. But then

$$\alpha(\mathbf{u}) = \beta(\lambda(\mathbf{u})) = \beta(\mu(\mathbf{x})) = \mathbf{x}.$$

It follows that

$$\mu(\alpha(\mathbf{u})) = \mu(\mathbf{x}) = \lambda(\mathbf{u}).$$

We have thus shown that $\mu(\alpha(\mathbf{u})) = \lambda(\mathbf{u})$ for all $\mathbf{u} \in U$ for which $\alpha(\mathbf{u}) \in W$.

Let the smooth real-valued functions $w_1, w_2, \ldots, w_n$ be defined throughout the open subset W of $\mathbb{R}^n$ so that

$$\mu(\mathbf{x}) = (w_1(\mathbf{x}), w_2(\mathbf{x}), \ldots, w_n(\mathbf{x}))$$

for all $\mathbf{x} \in W$. Then $w_1, w_2, \ldots, w_n$ is a smooth curvilinear coordinate system defined over the open set W. Let $\mathbf{u} \in U$. Suppose that $\alpha(\mathbf{u}) \in W$. Then $\mu(\alpha(\mathbf{u})) = \lambda(\mathbf{u})$. It follows that $w_j(\alpha(\mathbf{u})) = u_j$ for $j = 1, 2, \ldots, k$, and $w_j(\alpha(\mathbf{u})) = 0$ when $j > k$. It follows from this that

$$\alpha(U) \cap W = \{\mathbf{x} \in W : w_j(\mathbf{x}) = 0 \text{ for } k < j \leq n\},$$

as required. ∎

The proof of Proposition 8.19 involves some technicalities that depend on the requirement that the inverse of the map $\alpha\colon U \to \mathbb{R}^n$ in the statement of the proposition be continuous on $\alpha(U)$. The following example demonstrates that the conclusions of the proposition may fail to hold in situations where the other requirements of the proposition are satisfied by the continuity requirement (ii) in the statements of the proposition is not satisfied.

Example Let ν be an irrational number and let $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ be the smooth curve in $\mathbb{R}^3$ defined such that

$$\gamma(t) = ((2 + \cos 2\pi t)\, \cos 2\pi\nu t, (2 + \cos 2\pi t)\, \sin 2\pi\nu t \sin 2\pi t)$$

for all $t \in \mathbb{R}$. Then γ is a smooth curve which winds around the torus

$$\{(x, y, z) \in \mathbb{R}^3 : (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}.$$

Morover the velocity vector $\dfrac{d\gamma(t)}{dt}$ is everywhere non-zero. The map γ is injective. Indeed suppose that t_1 and t_2 are real numbers satisfying $\gamma(t_1) = \gamma(t_2)$. Then both $t_1 - t_2$ and $\nu(t_1 - t_2)$ are integers, and the fact that ν is irrational ensures that this can only happen when $t_1 = t_2$.

Now if p, p', q, q' are integers and if and if $p - \nu q = p' - \nu q'$ then $p = p'$ and $q = q'$. We use this fact to construct infinite sequences $p_1, p_2, p_3, \ldots$ and $q_1, q_2, q_3, \ldots$ of integers such that $p_n - q_n\nu > 0$ and

$$0 < p_{n+1} - \nu q_{n+1} < \tfrac{1}{2}(p_n - \nu q_n)$$

for all positive integers n. Choose integers p_1 and q_1 for which $0 < p_1 - \nu q_1 < 1$. Then suppose that integers $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ have been determined so as to satisfy the required inequalities. Then $0 < p_n - \nu q_n < 1$. We show

how to determine integers p_{n+1} and q_{n+1} for which $0 < p_{n+1} - \nu q_{n+1} < \frac{1}{2}(p_n - \nu q_n)$.

Let $p = p_n + 1$ and $q = q_n$. Then $p - \nu q > p_n - \nu q_n$ and $p - \nu q \neq k(p_n - \nu q_n)$ for all integers k. Let k_0 be the largest integer for which $k_0(p_n - \nu q_n) < p - \nu q$, and let $p' = p - k_0 p_n$ and $q' = q - k_0 q_n$. Then $p' - \nu q' < p_n - \nu q_n$. Then let $p_{n+1} = (1 - k_1)p_n - k_1 p'$ and $q_{n+1} = (1 - k_1)q_n - k_1 q'$, where k_1 be the largest positive integer for which $(p_n - \nu q_n) + k_1(p' - p_n - \nu(q' - q_n)) > 0$. Then $0 < p_{n+1} - \nu q_{n+1} < \frac{1}{2}(p_n - \nu q_n)$. The infinite sequences of integers constructed in this fashion have the property that

$$0 < p_n - q_n\omega < \frac{1}{2^{n-1}}(p_1 - q_1\omega)$$

for all positive integers n. It follows that if $u_n = p_n - q_n\omega$ then the real numbers $u_1, u_2, u_3, \ldots$ constitute a decreasing sequence of real numbers converging to zero. Moreover the real numbers u_j are all distinct, and each u_j is uniquely determined by the value of q_j. It follows that the integers $q_1, q_2, q_3, \ldots$ are distinct.

Now $\cos 2\pi q_n = 1$, $\sin 2\pi q_n = 0$ for all positive integers n. Also

$$\cos 2\pi\nu q_n = \cos 2\pi(p_n + u_n) = \cos 2\pi u_n$$

and similarly $\sin 2\pi\nu q_n = \sin 2\pi u_n$ for all positive integers n. It follows that

$$\gamma(q_n) = (3\cos 2\pi u_n, 3\sin 2\pi u_n, 0)$$

for all positive integers n, and therefore $\gamma(q_n) \to (3, 0, 0)$ as $n \to +\infty$. But the infinite sequence $q_1, q_2, q_3, \ldots$ of distinct integers is not convergent. It follows that the inverse of the function γ is not continuous on $\gamma(\mathbb{R})$. Also, given any open neighbourhood of $(3, 0, 0)$, no matter how small, the curve γ passes infinitely often through that open neighbourhood. It is not therefore possible to find a smooth curvilinear coordinate system around $(3, 0, 0)$ satisfying the requirements in the statement of Proposition 8.19.

Definition Let M be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then M is said to be a *smooth submanifold* of $\mathbb{R}^n$ of dimension k if and only if, given any point $\mathbf{p}$ of M, there exists an open set W, where $\mathbf{p} \in W$, and a smooth curvilinear coordinate system $w_1, w_2, \ldots, w_n$ defined over the open set W such that

$$M \cap W = \{\mathbf{x} \in W : w_j(\mathbf{x}) = 0 \text{ for } k < j \leq n\}$$

and $w_j(\alpha(u_1, u_2, \ldots, u_k)) = u_j$ for all $(u_1, u_2, \ldots, u_k) \in U$.

Let M be a k-dimensional smooth submanifold of $\mathbb{R}^n$, and let $\mathbf{p}$ be a point of M. Then there exists s smooth curvilinear coordinate system $w_1, w_2, \ldots, w_n$ defined over an open set W in $\mathbb{R}^n$ such that

$$M \cap W = \{\mathbf{x} \in W : w_j(\mathbf{x}) = 0 \text{ for } k < j \leq n\}.$$

Let U be the open set in $\mathbb{R}^k$ defined so that

$$U = \{(w_1(\mathbf{x}), w_2(\mathbf{x}), \ldots, w_k(\mathbf{x})) : \mathbf{x} \in M \cap W\},$$

and let $\alpha\colon U \to \mathbb{R}^n$ be the smooth map from U to W defined such that

$$w_j(\alpha(u_1, u_2, \ldots, u_k) = u_j \quad \text{for} \quad j = 1, 2, \ldots, k$$

and

$$w_j(\alpha(u_1, u_2, \ldots, u_k)) = u_j \quad \text{for} \quad j > k.$$

Then $\alpha(U) = M \cap W$, and therefore $\alpha(U)$ is open in M. Also the smooth map $\alpha\colon U \to \mathbb{R}^n$ is injective, and the components of its inverse on $\alpha(U)$ are the restrictions of the smooth real-valued functions $w_1, w_2, \ldots, w_k$ to $\alpha(U)$. It follows that the inverse of α is continuous on $\alpha(U)$. Finally the derivative $(D\alpha)_{\mathbf{q}}$ of α at any point $\mathbf{q}$ of $\mathbf{U}$ is represented by a matrix product JA where the components of the matrices $n \times n$ matrix J and the $n \times k$ matrix A satisfy

$$J_{i,j} = \left.\frac{\partial x_i}{\partial w_j}\right|_{\alpha(\mathbf{q})} \qquad \text{for } i, j = 1, 2, \ldots, n$$

and

$$A_{j,l} = \left.\frac{\partial w_j(\alpha(u_1, \ldots, u_k))}{\partial u_l}\right|_{\mathbf{q}} \qquad \text{for } j = 1, 2, \ldots, n \text{ and } l = 1, 2, \ldots, k.$$

Now $A_{i,m} = 1$ when $m = i$, and $A_{i,m} = 0$ when $m \neq i$. It follows that the matrix A has rank k. Also the matrix J is invertible. It follows that the derivative $(D\alpha)_{\mathbf{q}}$ of the function α has rank k at each point of U.

Corollary 8.20 *Let M be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then M is a k-dimensional smooth submanifold of $\mathbb{R}^n$ if and only if, given any point $\mathbf{p}$ of M, there exists a smooth map $\alpha\colon U \to M$, defined over some open set in $\mathbb{R}^k$, which satisfies the following conditions:*

(i) $\alpha(U)$ is open in M and $\mathbf{p} \in \alpha(U)$;

(ii) the function $\alpha\colon U \to M$ is injective;

(iii) the inverse of α on the set $\alpha(U)$ is a continuous map from $\alpha(U)$ to U;

(iv) the derivative $(D\alpha)_{\mathbf{u}}$ of α at each point $\mathbf{u}$ of U has rank k.

Proof The preceding remarks show that if M is a smooth k-dimensional submanifold of $\mathbb{R}^k$, so that, given any point $\mathbf{p}$ of M, there exists an open set W in $\mathbb{R}^n$, where $\mathbf{p} \in W$, and a smooth curvilinear coordinate system $w_1, w_2, \ldots, w_n$ on W such that

$$M \cap W = \{\mathbf{x} \in W : w_j(\mathbf{x}) = 0 \text{ for } k < j \leq n\},$$

if

$$U = \{(w_1(\mathbf{x}), w_2(\mathbf{x}), \ldots, w_k(\mathbf{x})) : \mathbf{x} \in M \cap W\},$$

and if $\alpha: U \to \mathbb{R}^n$ is the smooth map from U to W defined such that

$$w_j(\alpha(u_1, u_2, \ldots, u_k) = u_j \quad \text{for} \quad j = 1, 2, \ldots, k,$$

then conditions (i), (ii), (iii) and (iv) of the corollary are satisfied by the map α.

Conversely if, given any point $\mathbf{p}$ there exist a smooth map α satisfying conditions (i), (ii), (iii) and (iv) of the corollary, then Proposition 8.19 ensures that M is a smooth submanifold of $\mathbb{R}^n$, as required. ∎

9 Topologies, Compactness, and the Multidimensional Heine-Borel Theorem

9.1 Open Sets in Subsets of Euclidean Spaces

Let X be a subset of n-dimensional Euclidean space $\mathbb{R}^n$. A subset U of X is said to be *open in* X if, given any point $\mathbf{u}$ of U, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

Lemma 9.1 *Let X be a subset of $\mathbb{R}^n$, and let U be a subset of X. Then U is open in X if and only if there exists some open set V in $\mathbb{R}^n$ for which $U = V \cap X$.*

Proof First suppose that $U = V \cap X$ for some open set V in $\mathbb{R}^n$. Let $\mathbf{u} \in U$. Then the definition of open sets in $\mathbb{R}^n$ ensures that there exists some positive real number δ such that

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta\} \subset V.$$

Then

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

This shows that U is open in X.

Conversely suppose that the subset U of X is open in X. For each point $\mathbf{u}$ of U there exists some positive real number $\delta_{\mathbf{u}}$ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\} \subset U.$$

For each $\mathbf{u} \in U$, let $B(\mathbf{u}, \delta_{\mathbf{u}})$ denote the open ball in $\mathbb{R}^n$ of radius $\delta_{\mathbf{u}}$ about the point $\mathbf{u}$, so that

$$B(\mathbf{u}, \delta_{\mathbf{u}}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{u}| < \delta_{\mathbf{u}}\}$$

for all $\mathbf{u} \in U$, and let V be the union of all the open balls $B(\mathbf{u}, \delta_{\mathbf{u}})$ as $\mathbf{u}$ ranges over all the points of U. Then V is an open set in $\mathbb{R}^n$. Indeed every open ball in $\mathbb{R}^n$ is an open set (Lemma 6.13), and any union of open sets in $\mathbb{R}^n$ is itself an open set (Proposition 6.15). The set V is a union of open balls. It is therefore a union of open sets, and so must itself be an open set.

Now $B(\mathbf{u}, \delta_{\mathbf{u}}) \cap X \subset U$. for all $\mathbf{u} \in U$. Also every point of V belongs to $B(\mathbf{u}, \delta_{\mathbf{u}})$ for at least one point $\mathbf{u}$ of U. It follows that $V \cap X \subset U$. But $\mathbf{u} \in B(\mathbf{u}, \delta_{\mathbf{u}})$ and $B(\mathbf{u}, \delta_{\mathbf{u}}) \in V$ for all $\mathbf{u} \in U$, and therefore $U \subset V$, and thus $U \subset V \cap X$. It follows that $U = V \cap X$, as required. ∎

9.2 Topological Spaces

Definition A *topological space* X consists of a set X together with a collection of subsets, referred to as *open sets*, such that the following conditions are satisfied:—

(i) the empty set $\emptyset$ and the whole set X are open sets,

(ii) the union of any collection of open sets is itself an open set,

(iii) the intersection of any *finite* collection of open sets is itself an open set.

The collection consisting of all the open sets in a topological space X is referred to as a *topology* on the set X.

Remark If it is necessary to specify explicitly the topology on a topological space then one denotes by (X, τ) the topological space whose underlying set is X and whose topology is τ. However if no confusion will arise then it is customary to denote this topological space simply by X.

It follows from Proposition 6.15 that if X is a subset of n-dimensional Euclidean space then the collection of subsets of X that are open in X is a topology on X. We refer to this topology as the *usual topology* on X. A subset U of X is open with respect to the usual topology on X if and only if, given any point $\mathbf{u}$ of U, there exists some positive real number δ such that

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\} \subset U.$$

Definition A topological space X is said to be a *Hausdorff space* if and only if it satisfies the following *Hausdorff Axiom*:

- if x and y are distinct points of X then there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Any subset of a Euclidean space is a Hausdorff space. Indeed let X be a subset of a Euclidean space $\mathbb{R}^n$, and let $\mathbf{x}$ and $\mathbf{y}$ be distinct points of X. Let $\delta = \frac{1}{2}|\mathbf{x} - \mathbf{y}|$. Then the open balls of radius δ about the points $\mathbf{x}$ and $\mathbf{y}$ are open sets in X containing $\mathbf{x}$ and $\mathbf{y}$ respectively whose intersection is the empty set.

Let X be a topological space with topology τ, and let A be a subset of X. Let τ_A be the collection of all subsets of A that are of the form $V \cap A$ for $V \in \tau$. Then τ_A is a topology on the set A. (It is a straightforward exercise to verify that the topological space axioms are satisfied.) The topology τ_A on A is referred to as the *subspace topology* on A.

Any subset of a Hausdorff space is itself a Hausdorff space (with respect to the subspace topology).

Euclidean space $\mathbb{R}^n$ of dimension n is a topological space with the usual topology. It follows from Lemma 9.1 that the usual topology on any subset X of $\mathbb{R}^n$ is the subspace topology on that subset.

Definition A function $f: X \to Y$ from a topological space X to a topological space Y is said to be *continuous* if $f^{-1}(V)$ is an open set in X for every open set V in Y, where

$$f^{-1}(V) = \{x \in X : f(x) \in V\}.$$

A continuous function from X to Y is often referred to as a *map* from X to Y.

It follows from Proposition 6.19 that the definition of continuity for functions between topological spaces generalizes the standard definition of continuity for functions between subsets of Euclidean spaces.

It is an easy exercise to prove from the definition of continuity for functions between topological spaces that any composition of continuous functions is continuous.

Let $f: X \to Y$ be a continuous function between topological spaces X and Y. Then $f^{-1}(G)$ is closed in X for all closed sets G in Y. Indeed if G is a closed set in Y then the complement $Y \setminus G$ of Y in G is an open set in Y. The continuity of $f: X \to Y$ ensures that $f^{-1}(Y \setminus G)$ is closed in X. But it is straightforward to verify that $f^{-1}(Y \setminus G) = X - f^{-1}(G)$. It follows that $f^{-1}(G)$ is closed in X.

Definition Let X and Y be topological spaces. A function $h: X \to Y$ is said to be a *homeomorphism* if and only if the following conditions are satisfied:

- the function $h: X \to Y$ is both injective and surjective (so that the function $h: X \to Y$ has a well-defined inverse $h^{-1}: Y \to X$),
- the function $h: X \to Y$ and its inverse $h^{-1}: Y \to X$ are both continuous.

Two topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $h: X \to Y$ from X to Y.

If $h: X \to Y$ is a homeomorphism between topological spaces X and Y then h induces a one-to-one correspondence between the open sets of X and the open sets of Y. Thus the topological spaces X and Y can be regarded as being identical as topological spaces.

9.3 Compact Topological Spaces

Let X be a topological space, and let A be a subset of X. A collection of subsets of X in X is said to *cover* A if and only if every point of A belongs to at least one of these subsets. In particular, an *open cover* of X is collection of open sets in X that covers X.

If $\mathcal{U}$ and $\mathcal{V}$ are open covers of some topological space X then $\mathcal{V}$ is said to be a *subcover* of $\mathcal{U}$ if and only if every open set belonging to $\mathcal{V}$ also belongs to $\mathcal{U}$.

Definition A topological space X is said to be *compact* if and only if every open cover of X possesses a finite subcover.

Lemma 9.2 *Let X be a topological space. A subset A of X is compact (with respect to the subspace topology on A) if and only if, given any collection $\mathcal{U}$ of open sets in X covering A, there exists a finite collection $V_1, V_2, \ldots, V_r$ of open sets belonging to $\mathcal{U}$ such that $A \subset V_1 \cup V_2 \cup \cdots \cup V_r$.*

Proof A subset B of A is open in A (with respect to the subspace topology on A) if and only if $B = A \cap V$ for some open set V in X. The desired result therefore follows directly from the definition of compactness. ∎

Lemma 9.3 *Let A be a closed subset of some compact topological space X. Then A is compact.*

Proof Let $\mathcal{U}$ be any collection of open sets in X covering A. On adjoining the open set $X \setminus A$ to $\mathcal{U}$, we obtain an open cover of X. This open cover of X possesses a finite subcover, since X is compact. Moreover A is covered by the open sets in the collection $\mathcal{U}$ that belong to this finite subcover. It follows from Lemma 9.2 that A is compact, as required. ∎

Lemma 9.4 *Let $f: X \to Y$ be a continuous function between topological spaces X and Y, and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof Let $\mathcal{V}$ be a collection of open sets in Y which covers $f(A)$. Then A is covered by the collection of all open sets of the form $f^{-1}(V)$ for some $V \in \mathcal{V}$. It follows from the compactness of A that there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to $\mathcal{V}$ such that

$$A \subset f^{-1}(V_1) \cup f^{-1}(V_2) \cup \cdots \cup f^{-1}(V_k).$$

But then $f(A) \subset V_1 \cup V_2 \cup \cdots \cup V_k$. This shows that $f(A)$ is compact. ∎

Lemma 9.5 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then f is bounded above and below on X.*

Proof For each positive integer m let

$$U_m = \{x \in X : -m < f(x) < m\}.$$

Then $U_m = f^{-1}((-m, m))$, where $(-m, m)$ is the open interval in $\mathbb{R}$ consisting of all real numbers t that satisfy $-m < t < m$. It follows from the definition of continuity for functions between topological space that U_m is open in X for all positive integers k. Now, given any point x of X, there exists some positive integer m such that $-m < f(x) < m$. It follows that the open sets $U_1, U_2, U_3, \ldots$ cover the compact space X. The definition of compactness ensures the existence of a finite subcover $U_{m_1}, U_{m_2}, \ldots, U_{m_k}$, where $m_1, m_2, \ldots, m_k$ are positive integers. Let M be the maximum of $m_1, m_2, \ldots, m_k$. Then $-M < f(x) < M$ for all $x \in X$. The result follows. ∎

Proposition 9.6 *Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a compact topological space X. Then there exist points u and v of X such that $f(u) \leq f(x) \leq f(v)$ for all $x \in X$.*

Proof Let $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x) : x \in X\}$. There must exist $v \in X$ satisfying $f(v) = M$, for if $f(x) < M$ for all $x \in X$ then the function $x \mapsto 1/(M - f(x))$ would be a continuous real-valued function on X that was not bounded above, contradicting Lemma 9.5. Similarly there must exist $u \in X$ satisfying $f(u) = m$, since otherwise the function $x \mapsto 1/(f(x) - m)$ would be a continuous function on X that was not bounded above, again contradicting Lemma 9.5. But then $f(u) \leq f(x) \leq f(v)$ for all $x \in X$, as required. ∎

9.4 Compact Subsets of Euclidean Spaces

Proposition 9.7 *Let A be a compact subset of n-dimensional Euclidean space $\mathbb{R}^n$. Then A is closed and bounded in $\mathbb{R}^n$.*

Proof The function that sends each $\mathbf{x} \in A$ to $|\mathbf{x}|$ is a continuous function on A. Every continuous function on a compact topological space is bounded (Lemma 9.5). It follows that there exists a real number M such that $|\mathbf{x}| < M$ for all $\mathbf{x}$ in A. Thus the set A is bounded.

Let $\mathbf{p}$ be a point of $\mathbb{R}^n$ that does not belong to A, and let $f(\mathbf{x}) = |\mathbf{x} - \mathbf{p})$. The function f is continuous on $\mathbb{R}^n$. It therefore follows from Proposition 9.6

that there is a point $\mathbf{q}$ of A such that $f(\mathbf{x}) \geq f(\mathbf{q})$ for all $\mathbf{x} \in A$, since A is compact. Now $f(\mathbf{q}) > 0$, since $\mathbf{q} \neq \mathbf{p}$. Let δ satisfy $0 < \delta \leq f(\mathbf{q})$. Then the open ball of radius δ about the point $\mathbf{p}$ is contained in the complement of A, since $f(\mathbf{x}) < f(\mathbf{q})$ for all points $\mathbf{x}$ of this open ball. It follows that the complement of A is an open set in $\mathbb{R}^n$, and thus A itself is closed in $\mathbb{R}^n$. ∎

We shall prove the converse of Proposition 9.7. The proof will make use of the following proposition.

Proposition 9.8 *Let X be a closed bounded set in n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L with the property that, given any point $\mathbf{u}$ of X, there exists a member V of the open cover $\mathcal{V}$ for which*

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta_L\} \subset V.$$

Proof Let

$$B_X(\mathbf{u}, \delta) = \{\mathbf{x} \in X : |\mathbf{x} - \mathbf{u}| < \delta\}$$

for all $\mathbf{u} \in X$ and for all positive real numbers δ. Suppose that there did not exist any positive real number δ_L with the stated property. Then, given any positive number δ, there would exist a point $\mathbf{u}$ of X for which the ball $B_X(\mathbf{u}, \delta)$ would not be wholly contained within any open set V belonging to the open cover $\mathcal{V}$. Then $B_X(\mathbf{u}, \delta) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$. There would therefore exist an infinite sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ of points of X with the property that, for all positive integers j, the open ball $B_X(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$. The sequence $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$ would be bounded, because the set X is bounded. It would then follow from the multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) that there would exist a convergent subsequence $\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3}, \ldots$ of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots$. Let $\mathbf{p}$ be the limit of this convergent subsequence. Then the point $\mathbf{p}$ would then belong to X, because X is closed (see Lemma 6.18). But then the point $\mathbf{p}$ would belong to an open set V belonging to the open cover $\mathcal{V}$. It would then follow from the definition of open sets that there would exist a positive real number δ for which $B_X(\mathbf{p}, 2\delta) \subset V$. Let $j = j_k$ for a positive integer k large enough to ensure that both $1/j < \delta$ and $\mathbf{u}_j \in B_X(\mathbf{p}, \delta)$. The Triangle Inequality would then ensure that every point of X within a distance $1/j$ of the point $\mathbf{u}_j$ would lie within a distance 2δ of the point $\mathbf{p}$, and therefore

$$B_X(\mathbf{u}_j, 1/j) \subset B_X(\mathbf{p}, 2\delta) \subset V.$$

But $B(\mathbf{u}_j, 1/j) \cap (X \setminus V) \neq \emptyset$ for all members V of the open cover $\mathcal{V}$, and therefore it would not be possible for this open set to be contained in the open set V. Thus the assumption that there is no positive number δ_L with the required property has led to a contradiction. Therefore there must exist some positive number δ_L with the property that, for all $\mathbf{u} \in X$ the open ball $B_X(\mathbf{u}, \delta_L)$ in X is contained wholly within at least one open set belonging to the open cover $\mathcal{V}$, as required. ∎

Definition Let X be a subset of n-dimensional Euclidean space, and let $\mathcal{V}$ be an open cover of X. A positive real number δ_L is said to be a *Lebesgue number* for the open cover $\mathcal{V}$ if, given any point $\mathbf{p}$ of X, there exists some member V of the open cover $\mathcal{V}$ for which

$$\{\mathbf{x} \in X : |\mathbf{x} - \mathbf{p}| < \delta_L\} \subset V.$$

Proposition 9.8 ensures that, given any open cover of a closed bounded subset of n-dimensional Euclidean space, there exists a positive real number that is a Lebesgue number for that open cover.

Definition The *diameter* $\mathrm{diam}(A)$ of a bounded subset A of n-dimensional Euclidean space is defined so that

$$\mathrm{diam}(A) = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in A\}.$$

It follows from this definition that $\mathrm{diam}(A)$ is the smallest real number K with the property that $|\mathbf{x} - \mathbf{y}| \leq K$ for all $\mathbf{x}, \mathbf{y} \in A$.

A *hypercube* in n-dimensional Euclidean space $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_i \leq x_i \leq u_i + l\},$$

where l is a positive constant that is the length of the edges of the hypercube and $(u_1, u_2, \ldots, u_n)$ is the point in $\mathbb{R}^n$ at which the Cartesian coordinates of points in the hypercube attain their minimum values. The diameter of a hypercube with edges of length l is $l\sqrt{n}$.

Lemma 9.9 *Let X be a bounded subset of n-dimensional Euclidean space, and let δ be a positive real number. Then there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that the* $\mathrm{diam}(A_i) < \delta$ *for $i = 1, 2, \ldots, s$ and*

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Proof The set X is bounded, and therefore there exists some positive real number M such that that if $(x_1, x_2, \ldots, x_n) \in X$ then $-M \leq x_j \leq M$ for

$j = 1, 2, \ldots, n$. Choose k large enough to ensure that $2M/k < \delta_L/\sqrt{n}$. Then the large hypercube

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : -M \leq x_j \leq M \text{ for } j = 1, 2, \ldots, n\}$$

can be subdivided into k^n hypercubes with edges of length l, where $l = 2M/k$. Each of the smaller hypercubes is a set of the form

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : u_j \leq x_j \leq u_j + l \text{ for } j = 1, 2, \ldots, n\},$$

where $(u_1, u_2, \ldots, u_n)$ is the corner of the hypercube at which the Cartesian coordinates have their minimum values. If $\mathbf{p}$ is a point belonging to such a small hypercube, then all points of the hypercube lie within a distance $l\sqrt{n}$ of the point $\mathbf{p}$. It follows that the small hypercube is wholly contained within the open ball $B_{\mathbb{R}^n}(\mathbf{p}, \delta_L)$ of radius δ about the point $\mathbf{p}$.

Now the number of small hypercubes resulting from the subdivision is finite. Let $H_1, H_2, \ldots, H_s$ be a listing of the small hypercubes that intersect the set X, and let $A_i = H_i \cap X$. Then $\operatorname{diam}(H_i) \leq \sqrt{n}l < \delta_L$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

as required. ∎

Theorem 9.10 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.*

Proof It follows from Proposition 9.15 that a compact subset of $\mathbb{R}^n$ is both closed and bounded. We must prove the converse.

Let X be a closed bounded subset of $\mathbb{R}^n$, and let $\mathcal{V}$ be an open cover of X. It follows from Proposition 9.8 that there exists a Lebesgue number δ_L for the open cover $\mathcal{V}$. It then follows from Lemma 9.9 that there exist subsets $A_1, A_2, \ldots, A_s$ of X such that $\operatorname{diam}(A_i) < \delta_L$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_s.$$

We may suppose that A_i is non-empty for $i = 1, 2, \ldots, s$ (because if $A_i = \emptyset$ then A_i could be deleted from the list). Choose $\mathbf{p}_i \in A_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset B_X(\mathbf{p}_i, \delta_L)$ for $i = 1, 2, \ldots, s$. The definition of the Lebesgue number δ_L then ensures that there exist members $V_1, V_2, \ldots, V_s$ of the open cover $\mathcal{V}$ such that $B_X(\mathbf{p}_i, \delta_L) \subset V_i$ for $i = 1, 2, \ldots, s$. Then $A_i \subset V_i$ for $i = 1, 2, \ldots, s$, and therefore

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_s.$$

Thus $V_1, V_2, \ldots, V_s$ constitute a finite subcover of the open cover $\mathcal{U}$. We have therefore proved that every closed bounded subset of n-dimensional Euclidean space is compact, as required. ∎

9.5 Compact Metric Spaces

Definition A *metric space* (X, d) consists of a set X together with a *distance function* $d: X \times X \to [0, +\infty)$ on X satisfying the following axioms:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) $d(x, y) = 0$ if and only if $x = y$.

The quantity $d(x, y)$ should be thought of as measuring the *distance* between the points x and y. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is referred to as the *Triangle Inequality.*

An n-dimensional Euclidean space $\mathbb{R}^n$ is a metric space with with respect to the *Euclidean distance function* d, defined by

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Any subset X of $\mathbb{R}^n$ may be regarded as a metric space whose distance function is the restriction to X of the Euclidean distance function on $\mathbb{R}^n$ defined above.

Definition Let (X, d) be a metric space. Given a point x of X and $r \geq 0$, the *open ball* $B_X(x, r)$ of *radius* r about x in X is defined by

$$B_X(x, r) = \{x' \in X : d(x', x) < r\}.$$

Definition Let (X, d) be a metric space. A subset V of X is said to be an *open set* if and only if the following condition is satisfied:

- given any point v of V there exists some $\delta > 0$ such that $B_X(v, \delta) \subset V$.

By convention, we regard the empty set $\emptyset$ as being an open subset of X. (The criterion given above is satisfied vacuously in this case.)

Let (X, d) be a metric space. Then $\emptyset$ and X itself are open subsets of X. Every union of open subsets in X is itself an open set in X. Also any finite intersection of open sets in X is an open set in X. (The proof of these results is a straightforward generalization of the proof of Proposition 6.15).

Lemma 9.11 *Any open ball in a metric space is an open set.*

Proof Let X be a metric space with distance function d let x be a point of X, and let r be a positive real number. If $y \in B_X(x,r)$ and if $z \in B_X(y,\delta)$, where $B_X(x,r)$ and $B_X(y,\delta)$ are the open balls of radius r and δ about the points x and y respectively, then

$$d(z,x) \leq d(z,y) + d(y,x) < d(y,x) + \delta.$$

But $d(y,x) < r$. It follows that if $0 < \delta < r - d(y,x)$ then $B_X(y,\delta) \subset B_X(x,r)$. ∎

Lemma 9.12 *All metric spaces are Hausdorff spaces.*

Proof Let X be a metric space with distance function d, and let x and y be points of X, where $x \neq y$. Let $\delta = \frac{1}{2}d(x,y)$. Then $x \in B_X(x,\delta)$ and $y \in B_X(y,\delta)$. Moreover $B_X(x,\delta) \cap B_X(y,\delta) = \emptyset$. Indeed were there to exist some point z in the intersection of $B_X(x,\delta) \cap B_X(y,\delta) = \emptyset$ then $d(x,y) \leq d(x,y) + d(y,z) < 2\delta$; but this contradicts the choice of δ. The balls $B_X(x,\delta)$ and $B_X(y,\delta)$ are open in X (Lemma 9.11). The result follows. ∎

The following definition of continuity for functions between metric spaces generalizes that for functions of a real or complex variable.

Proposition 9.13 *Let X and Y be metric spaces with distance functions d_X and d_Y respectively. Then one can prove that a function $f\colon X \to Y$ from X to Y is continuous (in accordance with the definition of continuity for functions between topological spaces) if and only if, given any point x of X and given any positive real number ε, there exists some positive real number δ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x' of X satisfying $d_X(x,x') < \delta$.*

The proof of Proposition 9.13 this result is a straightforward generalization of the proof of Proposition 6.19.

Lemma 9.14 *Let X be a metric space with distance function d, and let p be a point of X. Let $f_p\colon X \to \mathbb{R}$ be the function defined such that $f_p(x) = d(x,p)$ for all $x \in X$. Then the function f_p is continuous on X. Moroever $|f_p(x) - f_p(y)| \leq d(x,y)$ for all $x, y \in X$.*

Proof Let x and y be points of X. Then

$$f_p(x) = d(x,p) \leq d(x,y) + d(y,p) = f_p(y) + d(x,y)$$

and therefore $f_p(x) - f_p(y) \leq d(x,y)$. Interchanging x and y, we find that $f_p(y) - f_p(x) \leq d(x,y)$. It follows that $|f_p(x) - f_p(y)| \leq d(x,y)$ for all $x, y \in X$. The required result then follows on applying Lemma 9.14. ∎

Proposition 9.15 *Let A be a compact subset of a metric space X. Then A is closed in X.*

Proof Let p be a point of X that does not belong to A, and let $f_p(x) = d(x, p)$, where d is the distance function on X. It follows from Proposition 9.6 that there is a point a_0 of A such that $f_p(a) \geq f_p(a_0)$ for all $a \in A$, since A is compact. Now $f_p(a_0) > 0$, since $a_0 \neq p$. Let δ satisfy $0 < \delta \leq f(a_0)$. Then the open ball of radius δ about the point p is contained in the complement of A, since $f_p(x) < f_p(a_0)$ for all points x of this open ball. It follows that the complement of A is an open set in X, and thus A itself is closed in X. ∎

Let X be a metric space with distance function d. Given a closed subset A of X, we denote by $d(x, A)$ the greatest lower bound on the distances from x to the points of the set A. Thus

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Lemma 9.16 *Let X be a metric space with distance function d, let A be a closed set in X, and let $f_A: X \to \mathbb{R}$ be defined so that*

$$f_A(x) = d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Then the function f_A is continuous on X, and

$$A = \{x \in X : f_A(x) = 0\}.$$

Moreover $|f_A(x) - f_A(y)| \leq d(x, y)$ for all $x, y \in A$.

Proof Let x and y be points of X. Then $d(x, a) \geq f_A(x)$ and $d(y, a) \geq f_A(y)$ for all $x \in A$. Let some positive real number ε be given. Then there exist points p and q of A such that $d(x, p) < f_A(x) + \varepsilon$ and $d(y, q) < f_A(y) + \varepsilon$. Then

$$f_A(x) \leq d(x, q) \leq d(x, y) + d(y, q) \leq f_A(y) + d(x, y) + \varepsilon.$$

It follows from this that $f_A(x) - f_A(y) < d(x, y) + \varepsilon$ for all positive real numbers ε, and therefore $f_A(x) - f_A(y) \leq d(x, y)$. Similarly $f_A(y) - f_A(x) \leq d(x, y)$. Thus $|f_A(x) - f_A(y)| < d(x, y)$ for all $x, y \in X$. It follows from Lemma 9.14 that the function $f_A: X \to \mathbb{R}$ is continuous. If $x \in A$ then $0 \leq f_A(x) \leq d(x, x)$, and $d(x, x) = 0$, and therefore $f_A(x) = 0$. If $x \notin A$ then there exists some positive real number δ such that the open ball of radius δ about the point A is contained in the complement of A and therefore $f_A(x) \geq \delta > 0$. Therefore A point x of X belongs to the subset A if and only if $f_A(x) = 0$. The result follows. ∎

Definition Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists a non-negative real number K such that $d(x,y) \leq K$ for all $x, y \in A$. The smallest real number K with this property is referred to as the *diameter* of A, and is denoted by $\operatorname{diam} A$. (Note that $\operatorname{diam} A$ is the supremum of the values of $d(x,y)$ as x and y range over all points of A.)

Let X be a metric space with distance function d. A subset A of X is said to be *bounded* if there exists some non-negative real number K with the property that $d(x,y) \leq K$ for all $x, y \in A$.

Definition Let X be a metric space with distance function d. The *diameter* $\operatorname{diam}(A)$ of a bounded subset A of X is defined so that

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}.$$

Lemma 9.17 (Lebesgue Covering Lemma) *Let (X,d) be a compact metric space. Let $\mathcal{U}$ be an open cover of X. Then there exists a positive real number δ_L such that every subset of X whose diameter is less than δ_L is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.*

1st Proof The open cover $\mathcal{U}$ of X has a finite subcover, because X is compact. Therefore there exists a finite collection $V_1, V_2, \ldots, V_k$ of open sets belonging to the open cover $\mathcal{U}$ which covers X. Let $A_i = X \setminus V_i$ for $i = 1, 2, \ldots, k$, let

$$f_i(x) = d(x, A_i) = \inf\{d(x,a) : a \in A_i\}.$$

for $i = 1, 2, \ldots, k$, and let

$$F(x) = \sum_{i=1}^{k} f_i(x) = \sum_{i=1}^{k} d(x, A_i).$$

It follows from Lemma 9.16 that each function f_i is a continuous function on X. Therefore the function $F\colon X \to \mathbb{R}$ is a continuous real-valued function on X.

Given any point x of X there exists some integer i between 1 and k for which $x \in V_i$. Then $x \notin A_i$. It follows from Lemma 9.16 that $f_i(x) > 0$. Therefore $F(x) > 0$. Thus $F(x)$ is strictly positive for all $x \in X$. It follows from Proposition 9.6 that there exists some point u of X with the property that $F(x) \geq F(u)$ for all $x \in X$. Let δ_L be a positive real number for which $k\delta_L < F(u)$.

Let $g(x) = \text{maximum}(f_1(x), f_2(x), \ldots, f_k(x)$ for all $x \in X$. Then $k\delta_L < F(u) \leq F(x) \leq kg(x)$ for all $x \in X$. Therefore, given any point x in X, there exists some integer i between 1 and k for which $f_i(x) > \delta_L$. But then $d(x, A_i) > \delta_L$, and therefore the open ball $B_X(x, \delta_L)$ of radius δ_L about the point x is wholly contained in the open set V_i. Now any non-empty subset of X of diameter less than δ_L is contained within $B_X(x, \delta)$ for any $x \in L$. Therefore every subset of X of diameter less than δ_L is wholly contained within one of the open sets belonging to the open cover $\mathcal{U}$, as required. ∎

2nd Proof Every point of X is contained in at least one of the open sets belonging to the open cover $\mathcal{U}$. It follows from this that, for each point x of X, there exists some $\delta_x > 0$ such that the open ball $B(x, 2\delta_x)$ of radius $2\delta_x$ about the point x is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. But then the collection consisting of the open balls $B(x, \delta_x)$ of radius δ_x about the points x of X forms an open cover of the compact space X. Therefore there exists a finite set $x_1, x_2, \ldots, x_r$ of points of X such that

$$B(x_1, \delta_1) \cup B(x_2, \delta_2) \cup \cdots \cup B(x_r, \delta_r) = X,$$

where $\delta_i = \delta_{x_i}$ for $i = 1, 2, \ldots, r$. Let $\delta_L > 0$ be given by

$$\delta_L = \text{minimum}(\delta_1, \delta_2, \ldots, \delta_r).$$

Suppose that A is a subset of X whose diameter is less than δ_L. Let u be a point of A. Then u belongs to $B(x_i, \delta_i)$ for some integer i between 1 and r. But then it follows that $A \subset B(x_i, 2\delta_i)$, since, for each point v of A,

$$d(v, x_i) \leq d(v, u) + d(u, x_i) < \delta_L + \delta_i \leq 2\delta_i.$$

But $B(x_i, 2\delta_i)$ is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$. Thus A is contained wholly within one of the open sets belonging to $\mathcal{U}$, as required. ∎

Definition Let $\mathcal{U}$ be an open cover of a compact metric space X. A *Lebesgue number* for the open cover $\mathcal{U}$ is a positive real number δ_L such that every subset of X whose diameter is less than δ_L is contained wholly within one of the open sets belonging to the open cover $\mathcal{U}$.

The Lebesgue Covering Lemma thus states that there exists a Lebesgue number for every open cover of a compact metric space.

Definition Let X and Y be metric spaces with distance functions d_X and d_Y respectively, and let $f: X \to Y$ be a function from X to Y. The function f

is said to be *uniformly continuous* on X if and only if, given $\varepsilon > 0$, there exists some $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for all points x and x' of X satisfying $d_X(x, x') < \delta$. (The value of δ should be independent of both x and x'.)

Theorem 9.18 *Let X and Y be metric spaces. Suppose that X is compact. Then every continuous function from X to Y is uniformly continuous.*

Proof Let d_X and d_Y denote the distance functions for the metric spaces X and Y respectively. Let $f: X \to Y$ be a continuous function from X to Y. We must show that f is uniformly continuous.

Let $\varepsilon > 0$ be given. For each $y \in Y$, define

$$V_y = \{x \in X : d_Y(f(x), y) < \tfrac{1}{2}\varepsilon\}.$$

Note that $V_y = f^{-1}\left(B_Y(y, \frac{1}{2}\varepsilon)\right)$, where $B_Y(y, \frac{1}{2}\varepsilon)$ denotes the open ball of radius $\frac{1}{2}\varepsilon$ about y in Y. Now the open ball $B_Y(y, \frac{1}{2}\varepsilon)$ is an open set in Y, and f is continuous. Therefore V_y is open in X for all $y \in Y$. Note that $x \in V_{f(x)}$ for all $x \in X$.

Now $\{V_y : y \in Y\}$ is an open cover of the compact metric space X. It follows from the Lebesgue Lemma (Lemma 9.17) that there exists some $\delta > 0$ such that every subset of X whose diameter is less than δ is a subset of some set V_y. Let x and x' be points of X satisfying $d_X(x, x') < \delta$. The diameter of the set $\{x, x'\}$ is $d_X(x, x')$, which is less than δ. Therefore there exists some $y \in Y$ such that $x \in V_y$ and $x' \in V_y$. But then $d_Y(f(x), y) < \frac{1}{2}\varepsilon$ and $d_Y(f(x'), y) < \frac{1}{2}\varepsilon$, and hence

$$d_Y(f(x), f(x')) \leq d_Y(f(x), y) + d_Y(y, f(x')) < \varepsilon.$$

This shows that $f: X \to Y$ is uniformly continuous, as required. ∎

Definition A metric space X with distance function d is said to be *totally bounded* if and only if, given any positive real number δ, there exists a finite collection $A_1, A_2, \ldots, A_s$ of subsets of X such that $\operatorname{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k.$$

Lemma 9.9 ensures that every bounded subset of n-dimensional Euclidean space is totally bounded.

Lemma 9.19 *Let X be a metric space that is totally bounded. Suppose that every open cover of X has a Lebesgue number. Then X is compact.*

Proof Let $\mathcal{V}$ be an open cover of X. Then there exists a positive real number δ_L that is a Lebesgue number for this open cover. There then exists a finite collection $A_1, A_2, \ldots, A_k$ of subsets of X such that $\operatorname{diam}(A_i) < \delta$ for $i = 1, 2, \ldots, s$ and

$$X = A_1 \cup A_2 \cup \cdots \cup A_k,$$

because X is totally bounded. The definition of Lebesgue numbers then ensures that, for each integer i between 1 and k, there exists an open set V_i belonging to the open cover $\mathcal{V}$ such that $A_i \subset V_i$. Then

$$X \subset V_1 \cup V_2 \cup \cdots \cup V_k.$$

Thus the open cover $\mathcal{V}$ has a finite subcover. This proves that X is compact, as required. ∎

Remark The proof of Lemma 9.19 is an obvious generalization of part of the proof of the multidimensional Heine-Borel Theorem (Theorem 9.10) given above.

Definition A metric space X is said to be *sequentially compact* if every sequence of points in X has a convergent subsequence.

The multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20) and Lemma 6.18 together ensure that every closed bounded subset of a Euclidean space is sequentially compact.

Proposition 9.20 *Let X be a sequentially compact metric space. Then, given any open cover of X, there exists a Lebesgue number for that open cover.*

Proposition 9.8 is a special case of Proposition 9.20, and the proof of the latter proposition is an obvious generalization of that of the former.

Let X be a metric space with distance function d An infinite sequence $x_1, x_2, x_3, \ldots$ of points in X is said to be a *Cauchy sequence* if, given any positive real number ε, there exists some positive integer N such that $d(x_j, x_k) < \varepsilon$ whenever $j \geq N$ and $k \geq N$.

It can be shown that the three following conditions on a metric space are equivalent:—

(i) the metric space is compact;

(ii) the metric space is sequentially compact;

(iii) the metric space is complete and totally bounded;

(iv) the metric space is totally bounded and, given any open cover of the space, there is a Lebesgue number for that open cover.

9.6 Norms on a Finite-Dimensional Vector Space

Definition A *norm* $\|.\|$ on a real or complex vector space X is a function, associating to each element x of X a corresponding real number $\|x\|$, such that the following conditions are satisfied:—

(i) $\|x\| \geq 0$ for all $x \in X$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$,

(iii) $\|\lambda x\| = |\lambda| \, \|x\|$ for all $x \in X$ and for all scalars λ,

(iv) $\|x\| = 0$ if and only if $x = 0$.

A *normed vector space* $(X, \|.\|)$ consists of a a real or complex vector space X, together with a norm $\|.\|$ on X.

Any normed vector space $(X, \|.\|)$ is a metric space with distance function d defined so that $d(x, y) = \|x - y\|$ for all $x, y \in X$.

In addition to the Euclidean norm, the norms on $\mathbb{R}^n$ include the norms $\|.\|_1$ and $\|.\|_{\sup}$, where

$$\|(x_1, x_2, \ldots, x_n)\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

and

$$\|(x_1, x_2, \ldots, x_n)\|_{\sup} = \text{maximum}(|x_1|, |x_2|, \ldots, |x_n|).$$

Definition Let X and Y be normed vector spaces. A linear transformation $T: X \to Y$ is said to be *bounded* if there exists some non-negative real number C with the property that $\|Tx\| \leq C\|x\|$ for all $x \in X$. If T is bounded, then the smallest non-negative real number C with this property is referred to as the *operator norm* of T, and is denoted by $\|T\|$.

A linear transformation between normed vector spaces is continuous if and only if it is bounded.

Definition Let $\|.\|$ and $\|.\|_*$ be norms on a real vector space X. The norms $\|.\|$ and $\|.\|_*$ are said to be *equivalent* if and only if there exist constants c and C, where $0 < c \leq C$, such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in X$.

If two norms on a real vector space are equivalent to a third norm then they are equivalent to each other.

Suppose that norms $\|.\|$ and $\|.\|_*$ be equivalent norms on a real vector space X. Then there exist positive constants C and C_* such that $\|x\|_* \leq C\|x\|$ and $\|x\| \leq C_*\|x\|_*$ for all $x \in X$. Let V be a subset of X that is open with respect to the norm $\|.\|_*$, and let $p \in V$. Then there exists a positive real number δ small enough to ensure that

$$\{x \in X : \|x - p\|_* < C\delta\} \subset V.$$

Then

$$\{x \in X : \|x - p\| < \delta\} \subset V.$$

It follows that if V is open in the topology generated by the $\|.\|_*$ norm then it is also open in the topology generated by the $\|.\|$ norm. Conversely if V is open in the topology generated by the $\|.\|$ norm then it is also open in the topology generated by the $\|.\|_*$ norm. Thus if norms $\|.\|$ and $\|.\|_*$ are equivalent, then they generate the same topology on X.

We shall show that all norms on a finite-dimensional real vector space are equivalent.

Lemma 9.21 *Let $\|.\|$ be a norm on $\mathbb{R}^n$. Then the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous with respect to the topology generated by the Euclidean norm on $\mathbb{R}^n$.*

Proof Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the basis of $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \cdots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1).$$

Let $\mathbf{x}$ and $\mathbf{y}$ be points of $\mathbb{R}^n$, given by

$$\mathbf{x} = (x_1, x_2, \ldots, x_n), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_n).$$

Using Schwarz' Inequality, we see that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &= \left\| \sum_{j=1}^n (x_j - y_j)\mathbf{e}_j \right\| \leq \sum_{j=1}^n |x_j - y_j| \, \|\mathbf{e}_j\| \\ &\leq \left(\sum_{j=1}^n (x_j - y_j)^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\mathbf{e}_j\|^2 \right)^{\frac{1}{2}} = C|\mathbf{x} - \mathbf{y}|, \end{aligned}$$

where

$$C^2 = \|\mathbf{e}_1\|^2 + \|\mathbf{e}_2\|^2 + \cdots + \|\mathbf{e}_n\|^2$$

and $|\mathbf{x} - \mathbf{y}|$ denotes the Euclidean norm of $\mathbf{x} - \mathbf{y}$, defined so that

$$|\mathbf{x} - \mathbf{y}| = \left(\sum_{j=1}^{n} (x_j - y_j)^2 \right)^{\frac{1}{2}}.$$

Also $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x} - \mathbf{y}\|$, since

$$\|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|, \qquad \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x}\|.$$

We conclude therefore that

$$|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq C|\mathbf{x} - \mathbf{y}|,$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, and thus the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous on $\mathbb{R}^n$ with respect to the topology generated by the Euclidean norm on $\mathbb{R}^n$. ∎

Theorem 9.22 *Any two norms on $\mathbb{R}^n$ are equivalent.*

Proof Let $\|.\|$ be any norm on $\mathbb{R}^n$. We show that $\|.\|$ is equivalent to the Euclidean norm $|.|$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$, defined by

$$S^{n-1} = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| = 1\}.$$

Now S^{n-1} is a compact subset of $\mathbb{R}^n$, since it is both closed and bounded. Also the function $\mathbf{x} \mapsto \|\mathbf{x}\|$ is continuous (Lemma 9.21). Also it follows from the Extreme Value Theorem (Theorem 6.21) that any continuous real-valued function on a closed bounded subset of Euclidean space attains both its maximum and minimum values on that subset. Therefore there exist points $\mathbf{u}$ and $\mathbf{v}$ of S^{n-1} such that $\|\mathbf{u}\| \leq \|\mathbf{x}\| \leq \|\mathbf{v}\|$ for all $\mathbf{x} \in S^{n-1}$. Set $c = \|\mathbf{u}\|$ and $C = \|\mathbf{v}\|$. Then $0 < c \leq C$ (since it follows from the definition of norms that the norm of any non-zero element of $\mathbb{R}^n$ is necessarily non-zero).

If $\mathbf{x}$ is any non-zero element of $\mathbb{R}^n$ then $\lambda\mathbf{x} \in S^{n-1}$, where $\lambda = 1/|\mathbf{x}|$. But $\|\lambda\mathbf{x}\| = |\lambda|\,\|\mathbf{x}\|$ (see the the definition of norms). Therefore $c \leq |\lambda|\,\|\mathbf{x}\| \leq C$, and hence $c|\mathbf{x}| \leq \|\mathbf{x}\| \leq C|\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^n$, showing that the norm $\|.\|$ is equivalent to the Euclidean norm $|.|$ on $\mathbb{R}^n$. If two norms on a vector space are equivalent to a third norm, then they are equivalent to each other. It follows that any two norms on $\mathbb{R}^n$ are equivalent, as required. ∎

Let X be a finite-dimensional real vector space. Then X is isomorphic to $\mathbb{R}^n$, where n is the dimension of X. It follows immediately from Theorem 9.22 and that all norms on X are equivalent and therefore generate the same topology on X. This result does not generalize to infinite-dimensional vector spaces.