Module MA2321: Analysis in Several Real Variables
Correspondences with MA2223
Michaelmas Term 2015

D. R. Wilkins

January 2016

Metric Space Correspondences

Many results included in MA2321 are particular cases of general results from the theories of metric spaces and topological spaces. As a result there are many correspondences relating material taught in the modules MA2321 and MA2223 in Michaelmas Term 2015. The purpose of these notes is to document some of these correspondences.

First we discuss some terminology to be used in what follows.

The following definition of a *metric space* is given in the MA2223 notes.

> A *metric* on a set X is a function d that assigns a real number to each pair of elements of X in such a way that the following properties hold.
>
> 1. *Non-negativity*: $d(x, y) \geq 0$ with equality if and only if $x = y$.
> 2. *Symmetry*: $d(x, y) = d(y, x)$ for all $x, y \in X$.
> 3. *Triangle Inequality*: $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The *Euclidean norm* $|.|$ on k-dimensional Euclidean space $\mathbb{R}^k$ determines a metric (or distance function) d_2 on $\mathbb{R}^k$ so that

$$d_2(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_k)$. Corollary 6.2 from MA2321 ensures that this distance function on $\mathbb{R}^k$ satisfies the Triangle Inequality, and the properties of non-negativity and symmetry set out in the definition of metrics above are clearly satisfied by the distance function d_2 determined by the Euclidean norm on k-dimensional Euclidean space $\mathbb{R}^k$, and thus $\mathbb{R}^k$ is a metric space. This fact is noted in the MA2223 notes immediately after the definition of metric spaces.

We say that a lemma, proposition, theorem or corollary in the MA2321 notes can be *generalized directly to metric spaces* if only uses those properties of the Euclidean distance function d_2 that are required in the definition of metrics on metric spaces, so that the proof of the theorem can be converted to a more general proof in the theory of metric spaces through systematic replacement of expressions of the form "$|\mathbf{x} - \mathbf{y}|$" in the Euclidean space (MA2321) context by expressions of the form "$d(x, y)$" in the metric space (MA2223) context.

We now discuss many of the correspondences between the content of MA2321 and MA2223.

MA2223 Theorem 1.1 Schwarz's Inequality (Proposition 6.1 of MA2321) is a special case of Hölder's Inequality when $p = q = 2$. Similarly Minkowski's Inequality is a special case of Similarly Corollary 6.2 of MA2321. is a special case of Minkowski's Inequality.

MA2223 Definitions of Open balls and Open Sets The general definition of open balls in metric spaces in MA2223 generalizes the definition of an open ball $B_X(\mathbf{p}, r)$ of radius r in a subset X of a Euclidean space centred on a point $\mathbf{p}$ of X that is given in subsection 6.5 of the MA2321 notes. Similarly the definition of an open set in a metric space X generalizes the definition of a subset V of X that is "open in X" in accordance with the definition adopted in subsection 6.5 of the MA2321 notes.

MA2223 Theorem 1.2 Parts 1,2 and 3 of this theorem generalize directly to metric spaces Proposition 6.15 of MA2321. The proof of Proposition 6.15 of MA2321 generalizes directly to metric spaces. Part 4 of MA2223 Theorem 1.2 generalizes the result of Lemma 6.13 to metric spaces. The proof of that lemma also generalizes directly to metric spaces. Part 5 of MA2223 Theorem 1.2 is not included in MA2321, though it should be an easy exercise for anyone who has mastered the course content.

MA2223 Theorem 1.3 No corresponding result on uniqueness of limits of convergent sequences in Euclidean spaces has been proved in MA2321, though this is a straightforward result.

MA2223 Definitions of Closed Set This definition correponds to the definition in subsection 6.6 of the MA2321 notes.

MA2223 Theorem 1.4 This result and its proof directly generalize Proposition 6.17 and Lemma 6.18 of the MA2321 notes.

MA2223 Definition of Continuity This definition of course generalizes the "epsilon-delta" definition of continuity for functions defined on subsets of Euclidean spaces given in subsection 6.3 of the MA2321 notes. The following comment, representing the definition in terms of open balls corresponds to the comments before Proposition 6.19 of MA2223.

MA2223 Theorem 1.5 This theorem directly generalizes Proposition 3.4 and Lemma 6.5 of MA2321.

MA2223 Theorem 1.6 This theorem directly generalizes Lemma 3.8 and Lemma 6.6 of MA2321.

MA2223 Theorem 1.7 This theorem directly generalizes Proposition 6.15 of MA2321. Lemma 6.6 of MA2321.

MA2223 Definition of Lipschitz Continuity This definition is not introduced in the sections of the MA2321 notes (sections 1 to 7) that were covered in lectures. However Lipschitz conditions (and thus Lipschitz continuity) are defined in Subsection 8.2 of the "Additional Notes" for MA2321. Lipschitz continuity is relevant for an understanding of the proof of the Inverse Function Theorem: part of the full proof of the Inverse Function Theorem requires the result that if the derivative of a function defined around a point in Euclidean space is continuously differentiable, and if the derivative at that point is invertible, then any inverse function will be Lipschitz continuous around the corresponding point in the codomain of the given function.

MA2223 Theorem 1.8 For a multidimensional generalization of this theorem, see Corollary 8.7 of the "Additional Notes" for MA2321.

MA2223 Theorem 1.9 No corresponding result was included in MA2321.

MA2223 Theorem 1.10 This theorem directly generalizes Proposition 5.11 of MA2321.

MA2223 Theorem 1.11 This theorem directly generalizes the more elementary part of Theorem 2.7 of MA2321. The remaining part of Theorem 2.7 of MA2321 ensures that the space $\mathbb{R}$ of real numbers (with the usual distance function) is a complete metric space.

MA2223 Theorem 1.12 This theorem directly generalizes Lemma 2.6 of MA2321.

MA2223 Theorem 1.13 This theorem in effect generalizes part of the proof of Theorem 2.7 of MA2321.

MA2223 Theorem 1.14 Part 1 of this theorem is Theorem 2.3 of MA2321. Part 2 of this theorem is Theorem 2.5, and moreover the proof presented in the MA2223 notes is in essence the second proof in the MA2321 notes: the natural numbers j belonging to the set Q_j defined in the second proof of Theorem 2.5 of MA2321 are those natural numbers j for which x_j is a "peak point", as that term is defined in the proof of Theorem 1.14 of MA2223. Part 3 of Theorem 1.14 of MA2223 is the more significant part of Theorem 2.7 of MA2321.

MA2223 Theorem 1.15 . Part 1 of this theorem is a consequence of Lemma 6.4 of MA2321.

MA2223 Theorem 1.16 . This result is not explicitly covered in MA2321, but the essentials, for subsets of $\mathbb{R}^n$, follow from Theorem 6.4 and Lemma 6.18 of the MA2321 notes.

MA2223 Theorem 1.17 . This general theorem covers the particular case, directly relevant to the proof of the Inverse Function Theorem, that is manifested in the MA2321 "Additional notes" as Lemma 8.12 of the "Additional Notes", that in turn is required for the given proof of the Inverse Function Theorem.

This theorem is also known as the "Contraction Mapping Theorem". This result is stated and proved as Theorem 4.13 in the lecture notes for Course 221, as taught by DRW in Michaelmas Term 2007 (and also in lecture notes for other modules covering similar ground). These lecture notes are available at the following URL:

`http://www.maths.tcd.ie/~dwilkins/Courses/221/221FirstSemester2007.pdf`

MA2223 Theorem 1.18 . This theorem on the existence, for short periods of time, of solutions of ordinary differential equations determined by Lipschitz continuous equations is often referred to as *Picard's Existence Theorem* or the *Picard-Lindelöf Theorem*(or just plain *Picard's Theorem*). The result is stated and proved as Theorem 4.14 of the lecture notes for Course 221 as taught by DRW in Michaelmas Term 2007 (available at the URL cited above in the note for MA2223 Theorem 1.17). Modern differential geometry is built on significant theorems of real analysis. These include the Inverse Function Theorem. They also include the result that, where systems of ordinary differential equations are specified through functions that are smooth, solutions for sufficiently short periods of time exist, are unique, and depend smoothly on initial conditions. Proofs of smooth dependence of solutions on initial conditions can be quite challenging.

MA2223 Theorem 1.19 . The construction of the completion of a metric space is carried through in Subsection 4.6 of the lecture notes for Course 221 as taught by DRW in Michaelmas Term 2007 (available at the URL cited above in the note for MA2223 Theorem 1.17).

Topological Space Correspondences

The MA2321 notes were designed to contain proofs of MA2223 Theorem 1.2 statements 1 to 4, and also MA2223 Theorem 1.7, in the particular case where the space in question is a subset of a Euclidean space. This ensures that, once the definition of topological spaces is given, it is apparent that subsets of Euclidean spaces are topological spaces with all the necessary properties, and that functions between subsets of Euclidean spaces are continuous according to the "epsilon-delta" definition if and only if they are continuous according to the topological space definition.

Compactness, and the multidimensional Heine-Borel Theorem are important. Section 9 of the "Additional Notes" for MA2321 are intended to show a route whereby the multidimensional Heine-Borel Theorem can be deduced from the multidimensional Bolzano-Weierstrass Theorem. One could go in the opposite direction: a metric space is compact if and only if every sequence in that space has a convergent subsequence. Thus where a proof makes use of the Heine-Borel Theorem, one should expect to be able to construct an alternative proof using the Bolzano-Weierstrass Theorem, and vice versa.

In particular the Extreme Value Theorem (Theorem 6.21) and the theorem that continuous functions defined on closed bounded subsets of

Euclidean spaces are uniformly continuous (Theorem 6.22) were proved in Section 6 of the MA2321 notes using the multidimensional Bolzano-Weierstrass Theorem (Theorem 6.20). Alternatively one could use the multidimensional Heine-Borel Theorem (Theorem 2.20 of MA2223, Theorem 9.10 of MA2321), to deduce these results as in Theorem 2.19 and Theorem 2.22 of MA2223.