MA1S11—Calculus Portion
School of Mathematics, Trinity College
Michaelmas Term 2016
Appendix: The Wave Equation

David R. Wilkins

102.1. Partial Derivatives

Definition

Let $\Psi(x, y, z, t)$ be a function of four variables x, y, z and t. The *partial derivatives*

$$\frac{\partial \Psi}{\partial x}, \quad \frac{\partial \Psi}{\partial y}, \quad \frac{\partial \Psi}{\partial z}, \quad \frac{\partial \Psi}{\partial t}$$

of Ψ with respect to x, y, z and t respectively are defined as follows:

$$\frac{\partial \Psi}{\partial x} = \lim_{\Delta x \to 0} \frac{\Psi(x + \Delta x, y, z, t)}{\Delta x}, \quad \frac{\partial \Psi}{\partial y} = \lim_{\Delta y \to 0} \frac{\Psi(x, y + \Delta y, z, t)}{\Delta y},$$

$$\frac{\partial \Psi}{\partial z} = \lim_{\Delta z \to 0} \frac{\Psi(x, y, z + \Delta z, t)}{\Delta z}, \quad \frac{\partial \Psi}{\partial t} = \lim_{\Delta t \to 0} \frac{\Psi(x, y, z, t + \Delta t)}{\Delta t}.$$

102.2. The Wave Equation in Three Dimensions

Let c be a positive constant. The three-dimensional (classical) *wave equation* characterizing waves moving with speed c is a partial differential equation satisfied by functions $\Psi(x, y, z, t)$ that are of the form

$$\begin{aligned}\Psi(x, y, z, t) &= A\cos(\mathbf{k}\,.\,\mathbf{r} - c\|\mathbf{k}\|t) \\ &\qquad + B\sin(\mathbf{k}\,.\,\mathbf{r} - c\|\mathbf{k}\|t) \\ &= A\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t) \\ &\qquad + B\sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t),\end{aligned}$$

where A and B are constants, $\mathbf{r}$ is the three-dimensional vector with components x, y, z, so that $\mathbf{r} = (x, y, z)$, and $\mathbf{k}$ is a three-dimensional vector whose Cartesian components are k_x, k_y and k_z respectively, so that $\mathbf{k} = (k_x, k_y, k_z)$. The length $\|\mathbf{k}\|$ of the vector k is defined so that

$$\|k\|^2 = k_x^2 + k_y^2 + k_z^2.$$

We calculate partial derivatives by taking the derivative of the function with respect to one of the variables whilst holding the values of the other variables fixed. Thus

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\| t)\right) \\ = &\; -k_x \sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\| t) \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial}{\partial x}\left(\sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\| t)\right) \\ = &\; k_x \cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\| t) \end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right)\right) \\
&= \frac{\partial}{\partial x}\left(-k_x \sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right) \\
&= -k_x^2 \cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(\sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right)\right) \\
&= \frac{\partial}{\partial x}\left(k_x \cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\right) \\
&= -k_x^2 \sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t).
\end{aligned}$$

It follows that if

$$\begin{aligned}\Psi(x, y, z, t) &= A\cos(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t) \\ &\quad + B\sin(k_x x + k_y y + k_z z - c\|\mathbf{k}\|t)\end{aligned}$$

then

$$\frac{\partial^2\Psi(x, y, z, t)}{\partial x^2} = -k_x^2\Psi(x, y, z, t),$$

and similarly

$$\begin{aligned}\frac{\partial^2\Psi(x, y, z, t)}{\partial y^2} &= -k_y^2\Psi(x, y, z, t), \\ \frac{\partial^2\Psi(x, y, z, t)}{\partial z^2} &= -k_z^2\Psi(x, y, z, t), \\ \frac{\partial^2\Psi(x, y, z, t)}{\partial t^2} &= -c^2\|\mathbf{k}\|^2\Psi(x, y, z, t).\end{aligned}$$

But

$$k_x^2 + k_y^2 + k_z^2 = |\mathbf{k}|^2.$$

It follows that

$$\begin{aligned}
\frac{\partial^2 \Psi(x, y, z, t)}{\partial x^2} + \frac{\partial^2 \Psi(x, y, z, t)}{\partial y^2} + \frac{\partial^2 \Psi(x, y, z, t)}{\partial z^2} \\
&= -(k_x^2 + k_y^2 + k_z^2)\Psi(x, y, z, t) \\
&= -\|\mathbf{k}\|^2 \Psi(x, y, z, t) \\
&= \frac{1}{c^2} \frac{\partial^2 \Psi(x, y, z, t)}{\partial t^2}.
\end{aligned}$$

Let c be a positive constant. The (classical) three-dimensional *wave equation* characterizing wave motion with speed c is the partial differential equation

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2}.$$

Note that if Ψ_1 and Ψ_2 are functions of x, y, z and t that satisfy the wave equation, then the function $A_1 \Psi_1 + A_2 \Psi_2$ also satisfies the wave equation for all real constants A_1 and A_2. Therefore solutions of the wave equation can be superposed, and the resultant solutions may often exhibit the phenomenon of *interference* which is characteristic behaviour of waves interacting with one another.

102.3. The Wave Equation in One Dimension

We now restrict our attention to the one-dimensional wave equation, which describes waves travelling in directions parallel to the x-axis whose wave fronts are perpendicular to the x-axis. Such waves are represented by functions of the form $\Psi(x, t)$ that have no dependence on the values of the Cartesian coordinates y and z. Such waves satisfy the one-dimensional wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2}.$$

102. The Wave Equation (continued)

Let c be a positive constant, let f and g be twice-differentiable real-valued functions of a single real variable, and let

$$\Psi(x,t) = f(x-ct) + g(x+ct).$$

Then

$$\begin{aligned}
\frac{\partial \Psi(x,t)}{\partial x} &= f'(x-ct) + g'(x+ct), \\
\frac{\partial \Psi(x,t)}{\partial t} &= -cf'(x-ct) + cg'(x+ct), \\
\frac{\partial^2 \Psi(x,t)}{\partial x^2} &= f''(x-ct) + g''(x+ct), \\
\frac{\partial^2 \Psi(x,t)}{\partial x^t} &= c^2 f''(x-ct) + c^2 g''(x+ct).
\end{aligned}$$

It follows that

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2}.$$

Thus the function Ψ satisfies the wave equation.

One may regard $\Psi(x, t)$ as a superposition of two waves: one wave travelling with speed c in the positive x-direction with shape represented by the function f; the other wave travelling with speed c in the negative x-direction with shape represented by the function g. The shape of the resultant wave will be determined by the interference of these two waves travelling in opposite directions.

We now investigate waves of a fixed frequency. If we fix position, the variation in time should be represented by a sinusoidal wave function that is a superposition of a sine and a cosine function. Given constants c and ω, where $c > 0$, a solution $\Phi(x,t)$ of the wave equation with speed c and frequency ν should take the form

$$\Psi(x,t) = \psi_1(x)\cos(2\pi\nu t) + \psi_2(x)\sin(2\pi\nu t).$$

Differentiating, we find that

$$\frac{\partial^2\Psi(x,t)}{\partial t^2} = -4\pi^2\nu^2\Psi(x,t).$$

It follows from this that $\Psi(x,t)$ satisfies the wave equation if and only if the functions ψ_1 and ψ_2 satisfy the ordinary differential equations

$$\frac{d^2\psi_1(x)}{dx^2} = -\frac{4\pi^2\nu^2}{c^2}\psi_1(x) \quad \text{and} \quad \frac{d^2\psi_2(x)}{dx^2} = -\frac{4\pi^2\nu^2}{c^2}\psi_2(x).$$

Standard results in the theory of ordinary differential equations ensure that the functions ψ_1 and ψ_2 satisfy these equations if and only if there exist constants A_1, A_2, B_1 and B_2 such that

$$\psi_1(x) = A_1 \cos\left(\frac{2\pi\nu x}{c}\right) + B_1 \sin\left(\frac{2\pi\nu x}{c}\right).$$

and

$$\psi_2(x) = A_2 \cos\left(\frac{2\pi\nu x}{c}\right) + B_2 \sin\left(\frac{2\pi\nu x}{c}\right).$$

Thus if the function $\Psi(x,t)$ satisfies the wave equation with speed c, where $c > 0$, and if, for each fixed x, the function sending time t to $\Psi(x,t)$ is sinusoidal, representing an oscillation with frequency ν, then the wave equation ensures that, at any given fixed time, the shape of the wave in space is a superposition of sinusoidal waves, where these sinusoidal waves each represent a waveform with wavelength λ satisfying the equation $\nu\lambda = c$. The waveforms of these sinusoidal waves are then represented by functions of the form

$$\sin\left(\frac{2\pi(x - x_0)}{\lambda}\right),$$

where x_0 is a constant that determines the phase of the wave.