MA1S11—Calculus Portion
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Appendix: Complex Numbers

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101. Complex Numbers

101.1. The Algebra of Algebraic Couples

Sir William Rowan Hamilton developed a construction of the system of complex numbers in which complex numbers are identified with “algebraic couples”.

Hamilton gave a brief description of his approach at the 4th meeting of the *British Association for the Advancement of Science* in Edinburgh in 1834. In particular he used his approach in order to justify a formula for the logarithm of a complex number to a complex base advocated by his friend John T. Graves. A much fuller account of his approach to the fundamental principles of algebra was published in the *Transactions of the Royal Irish Academy* in 1837, with the following title:

> *Theory of Conjugate Functions, or Algebraic Couples; with a Preliminary and Elementary Essay on Algebra as the Science of Pure Time.*

Following Hamilton's approach fairly closely, we construct an algebra whose elements are (in Hamilton's terminology) *algebraic couples*. These algebraic couples can subsequently be identified with complex numbers.

Definition

An *algebraic couple* is an ordered pair (x, y) whose components x and y are real numbers.

Ordered pairs of real numbers, by themselves, are not particularly useful. But we can define natural operations of addition and subtraction of algebraic couples "componentwise" so that

$$(x, y) + (u, v) = (x + u, y + v)$$

and

$$(x, y) - (u, v) = (x - u, y - v).$$

We define the *zero couple* $\mathbf{O}$ so that

$$\mathbf{O} = (0, 0).$$

And, given any algebraic couple $\mathbf{V}$, we define the *negative* $-\mathbf{V}$ of $\mathbf{V}$ so that

$$-\mathbf{V} = (-x, -y) \quad \text{when} \quad \mathbf{V} = (x, y).$$

These operations of addition and subtraction on algebraic couples conform to the following basic "Laws" that are analogous to corresponding properties of the real number system.

(i) (*The Commutative Law for Addition*) $\mathbf{V} + \mathbf{U} = \mathbf{U} + \mathbf{V}$ for all algebraic couples $\mathbf{V}$ and $\mathbf{U}$;

(ii) (*The Associative Law for Addition*) $(\mathbf{U} + \mathbf{V}) + \mathbf{W} = \mathbf{U} + (\mathbf{V} + \mathbf{W})$ for all algebraic couples $\mathbf{V}$, $\mathbf{U}$ and $\mathbf{W}$;

(iii) (*Existence of a Zero Couple*) there exists a zero couple $\mathbf{O}$ that satisfies $\mathbf{V} + \mathbf{O} = \mathbf{V}$ for all algebraic couples $\mathbf{V}$;

(iv) (*Existence of Negatives of Couples*) given an algebraic couple $\mathbf{V}$ there exists an algebraic couple $-\mathbf{V}$ (the *negative* of $\mathbf{V}$) characterized by the property that $\mathbf{V} + (-\mathbf{V}) = \mathbf{O}$.

Algebraic couples, considered purely in relation to their behaviour under operations of addition and subtraction, are then indistinguishable from two-dimensional vectors. Also, like two-dimensional vectors, they can be represented by points of a plane, where that plane is provided with a standard Cartesian coordinate system.

The next stage is to introduce an appropriate definition of multiplication of algebraic couples, assigning to algebraic couples $\mathbf{U}$ and $\mathbf{V}$ an algebraic couple $\mathbf{U} \otimes \mathbf{V}$ that is to be regarded as the *product* of $\mathbf{U}$ and $\mathbf{V}$.

Now there is no law of logic that compels one to adopt one definition of multiplication of algebraic couples in preference to any other. Nevertheless there are design criteria that guide the selection of laws of multiplication of algebraic couples that provide useful tools for mathematical investigations.

Three basic requirements that guided Hamilton are the following:

- the system of algebraic couples, with the proposed law of multiplication, sould preserve, so far as is possible, the basic "Laws of Algebra" that identify fundamental properties of the real number system;
- the system of algebraic couples, with the proposed law of multiplication, should have fruitful applications when applied in investigating problems in geometry, physics or other sciences;
- the real number system should be naturally embedded within the system of algebraic couples.

The operations of addition and multiplication on the system of real numbers are related by the *Distributive Law*:

$$(x + y)z = xz + yz \text{ and } x(y + z) = xy + xz$$

for all real numbers x and z. This is a useful property that the system of algebraic couples should possess. Therefore we restrict the choice of multiplication operation $\otimes$ on algebraic couples by requiring that the Distributive Law be satisfied also in the system of algebraic couples so that

$$(\mathbf{U} + \mathbf{V}) \otimes \mathbf{W} = \mathbf{U} \otimes \mathbf{W} + \mathbf{V} \otimes \mathbf{W}$$

and

$$\mathbf{U} \otimes (\mathbf{V} + \mathbf{W}) = \mathbf{U} \otimes \mathbf{V} + \mathbf{U} \otimes \mathbf{W}$$

for all algebraic couples $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$.

Now the real number system possesses a number 1 with the property that $1 \times x = x \times 1 = x$ for all real numbers x. It would be useful to require that the algebra of algebraic couples possess an element $\mathbf{E}$ with the property that

$$\mathbf{E} \otimes \mathbf{Z} = \mathbf{Z} \otimes \mathbf{E} = \mathbf{Z}$$

for all algebraic couples $\mathbf{Z}$. Moreover this algebraic couple $\mathbf{E}$ should be the algebraic couple that corresponds to the number 1 under the natural embedding of the real numbers within the system of algebraic couples that the latter system should possess.

Now, given any real number x, let $\mathbf{M}_x$ denote the algebraic couple that corresponds to x under the natural embedding of the real number system $\mathbb{R}$ that the system of algebraic couples should possess. Then, for the algebraic systems to correspond, we require that

$$\mathbf{M}_{x+y} = \mathbf{M}_x + \mathbf{M}_y, \quad \mathbf{M}_{x-y} = \mathbf{M}_x - \mathbf{M}_y, \quad \mathbf{M}_{xy} = \mathbf{M}_x \otimes \mathbf{M}_y$$

for all real numbers x and y. It follows that

$$\begin{aligned}
\mathbf{M}_{2x} &= \mathbf{M}_{x+x} = \mathbf{M}_x + \mathbf{M}_x, \\
\mathbf{M}_{3x} &= \mathbf{M}_{x+x+x} = \mathbf{M}_x + \mathbf{M}_x + \mathbf{M}_x, \\
\mathbf{M}_{4x} &= \mathbf{M}_{x+x+x+x} = \mathbf{M}_x + \mathbf{M}_x + \mathbf{M}_x + \mathbf{M}_x, \\
&\vdots
\end{aligned}$$

Let us now choose an algebraic couple **E** that will correspond to the real number 1. We set

$$\mathbf{E} = (1, 0).$$

Experience shows that this is a *simple* and *natural* choice that makes the resultant algebraic system *easier to work with*. This choice is not compelled by any logical requirement.

We now investigate some of the logical consequences of the “laws of algebra” and other design choices that we have chosen to adopt as design requirements for the algebraic system that we are seeking to construct.

Now $\mathbf{M}_1 = \mathbf{E} = (1, 0)$. It follows that

$$\begin{aligned}
\mathbf{M}_2 &= \mathbf{M}_{1+1} = \mathbf{M}_1 + \mathbf{M}_1 = \mathbf{E} + \mathbf{E} = (2, 0), \\
\mathbf{M}_3 &= \mathbf{M}_{1+1+1} = \mathbf{M}_1 + \mathbf{M}_1 + \mathbf{M}_1 = \mathbf{E} + \mathbf{E} + \mathbf{E} = (3, 0), \\
\mathbf{M}_4 &= \mathbf{M}_{1+1+1+1} = \mathbf{M}_1 + \mathbf{M}_1 + \mathbf{M}_1 + \mathbf{M}_1 \\
&\qquad = \mathbf{E} + \mathbf{E} + \mathbf{E} + \mathbf{E} = (4, 0), \\
&\vdots
\end{aligned}$$

In general $\mathbf{M}_n = (n, 0)$ for all positive integers n.

We require that

$$\mathbf{M}_0 + \mathbf{M}_n = \mathbf{M}_{0+n} = \mathbf{M}_n$$

for all positive integers n. It follows that

$$\mathbf{M}_0 = \mathbf{O} = (0, 0).$$

Then

$$\mathbf{M}_{-n} + (n, 0) = \mathbf{M}_{-n} + \mathbf{M}_n = \mathbf{M}_{(-n)+n} = \mathbf{M}_0 = (0, 0),$$

and therefore $\mathbf{M}_{-n} = (-n, 0)$ for all positive integers n. We conclude therefore that $\mathbf{M}_n = (n, 0)$ for all integers n, whether they be positive, negative or zero.

101. Complex Numbers (continued)

Next let $\mathbf{Z}$ be an arbitrary algebraic couple. Then $\mathbf{Z} = (x, y)$, where x and y are real numbers. Now we have required the algebraic couple $\mathbf{E}$ to satisfy the identities

$$\mathbf{E} \otimes \mathbf{Z} = \mathbf{Z} \otimes \mathbf{E} = \mathbf{Z}.$$

Combining this result with the distributive law, we find that

$$\begin{aligned}
\mathbf{M}_2 \otimes \mathbf{Z} &= (\mathbf{E} + \mathbf{E}) \otimes \mathbf{Z} = \mathbf{E} \otimes \mathbf{Z} + \mathbf{E} \otimes \mathbf{Z} \\
&= \mathbf{Z} + \mathbf{Z} = (x, y) + (x, y) \\
&= (2x, 2y), \\
\mathbf{M}_3 \otimes \mathbf{Z} &= (\mathbf{E} + \mathbf{E} + \mathbf{E}) \otimes \mathbf{Z} = \mathbf{E} \otimes \mathbf{Z} + \mathbf{E} \otimes \mathbf{Z} + \mathbf{E} \otimes \mathbf{Z} \\
&= \mathbf{Z} + \mathbf{Z} = (x, y) + (x, y) + (x, y) \\
&= (3x, 3y), \\
&\vdots
\end{aligned}$$

It follows that $\mathbf{M}_n \otimes (x, y) = (nx, ny)$ for all positive integers n. Also

$$\mathbf{M}_0 \otimes (x, y) + \mathbf{M}_n \otimes (x, y) = \mathbf{M}_{0+n} \otimes (x, y) = \mathbf{M}_n \otimes (x, y),$$

and therefore $\mathbf{M}_0 \otimes (x, y) = (0, 0)$. It then follows that, for all positive integers n,

$$\mathbf{M}_{-n} \otimes (x, y) + \mathbf{M}_n \otimes (x, y) = \mathbf{M}_0 \otimes (x, y) = (0, 0),$$

and therefore

$$\mathbf{M}_{-n} \otimes (x, y) = (-nx, -ny)$$

for all positive integers n. We conclude that

$$\mathbf{M}_n \otimes (x, y) = (nx, ny)$$

for all algebraic couples (x, y) and for all integers n, whether they be positive, negative or zero.

What algebraic couple $\mathbf{M}_{\frac{p}{q}}$ should represent a rational number of the form $\dfrac{p}{q}$, where p and q are integers and $q > 0$? Now $q \times \dfrac{p}{q} = p$. Our design requirements therefore require that

$$\mathbf{M}_q \otimes M_{\frac{p}{q}} = \mathbf{M}_p = (p, 0).$$

But if $M_{\frac{p}{q}} = (u, v)$ then

$$\mathbf{M}_q \otimes M_{\frac{p}{q}} = (qu, qv).$$

It follows that $qu = p$ and $qv = 0$ and therefore $u = p/q$ and $v = 0$. It follows that

$$\mathbf{M}_{\frac{p}{q}} = \left(\frac{p}{q}, 0\right).$$

Now multiplication of real numbers satisfies the Associative Law. Therefore, if possible, it is natural to adopt the design requirement that multiplication of algebraic couples be associative. If we adopt this design requirement then it follows that

$$\mathbf{M}_r \otimes (x, y) = (rx, ry)$$

for all rational numbers r and for all real numbers x and y. Indeed suppose that $r = \dfrac{p}{q}$, where p and q are integers and $q > 0$. Then

$$\mathbf{M}_q \otimes (\mathbf{M}_{\frac{p}{q}} \otimes (x, y)) = (\mathbf{M}_q \otimes \mathbf{M}_{\frac{p}{q}}) \otimes (x, y) = \mathbf{M}_p \otimes (x, y) = (px, py).$$

It follows that

$$\mathbf{M}_{\frac{p}{q}} \otimes (x, y) = \left(\frac{p}{q}x, \frac{p}{q}y\right).$$

We next adopt the design requirement that the embedding of the real number system into the system of algebraic couples be *continuous*. Because real numbers can be approximated by rational numbers to any desired degree of precision, the continuity requirement ensures that

$$\mathbf{M}_t \otimes (x, y) = (tx, ty)$$

for all real numbers t. Our design requirements have therefore ensured that, if an algebraic system of algebraic couples can be constructed satisfying those design requirements, then it must satisfy the identity

$$\mathbf{M}_t \otimes (x, y) = (tx, ty)$$

for all real numbers t.

In order to simplify notation, it is natural now to define an operation of multiplication of algebraic couples by real numbers that conforms to the standard definition in vector algebra, setting

$$t(x, y) = (tx, ty)$$

for all real numbers x, y and t. Then

$$\mathbf{M}_t \otimes (x, y) = t(x, y)$$

for all real numbers x, y and t.

Earlier we introduced an algebraic couple $\mathbf{E}$, where $\mathbf{E} = (1, 0)$. Let us now define $\mathbf{I} = (0, 1)$. Let $\mathbf{Z}$ be an arbitrary algebraic couple, and let $\mathbf{Z} = (x, y)$, where x and y are real numbers. Then

$$\mathbf{Z} = (x, 0) + (0, y) = \mathbf{M}_x \otimes \mathbf{E} + \mathbf{M}_y \otimes \mathbf{I} = x\mathbf{E} + y\mathbf{I}.$$

Now, whether or not we adopt the Commutative Law for multiplication as a design requirement, the identity

$$(x, y) \otimes \mathbf{E} = (x, y)$$

ensures that

$$\begin{aligned} (x, y) \otimes \mathbf{M}_2 &= (x, y) \otimes (\mathbf{E} + \mathbf{E}) = (x, y) \otimes \mathbf{E} + (x, y) \otimes \mathbf{E} \\ &= (x, y) + (x, y) = (2x, 2y), \end{aligned}$$

etc.

Adapting the argument presented above, we find first that

$$(x, y) \otimes (n, 0) = (x, y) \otimes \mathbf{M}_n = (nx, ny)$$

for all positive integers n. It easily follows from this that the identity $(x, y) \otimes (n, 0) = (nx, ny)$ is valid for all real numbers x and y and for all integers n, positive, negative or zero. It then follows that $(x, y) \otimes (r, 0) = (rx, ry)$ is valid for all rational numbers r. The continuity of the embedding of the real numbers into the system of algebraic couples then ensures that

$$(x, y) \otimes (t, 0) = (tx, ty) = t(x, y)$$

for all real numbers t.

Now let $\mathbf{Z}$ and $\mathbf{W}$ be algebraic couples, and let $\mathbf{Z} = (x, y)$ and $\mathbf{W} = (u, v)$. Then the logical consequences of our design requirements ensure that

$$\begin{aligned}
(x, y) \otimes (u, v) &= \Big((x, 0) + (0, y)\Big) \otimes \Big((u, 0) + (0, v)\Big) \\
&= (x, 0) \otimes (u, 0) + (x, 0) \otimes (0, v) \\
&\qquad + (0, y) \otimes (u, 0) + (0, y) \otimes (0, v) \\
&= (xu, 0) + (0, xv) + (0, yu) + (0, y) \otimes (0, v) \\
&= (xu, xv + yu) + (0, y) \otimes (0, v)
\end{aligned}$$

Moreover, using the Associative Law for multiplication of algebraic couples (which we have adopted as a design requirement), we see that

$$
\begin{aligned}
(0, y) \otimes (0, v) &= (0, y) \otimes \Big((v, 0) \otimes (0, 1) \Big) \\
&= \Big((0, y) \otimes (v, 0) \Big) \otimes (0, 1) \\
&= (0, yv) \otimes (0, 1) \\
&= \Big((yv, 0) \otimes (0, 1) \Big) \otimes (0, 1) \\
&= (yv, 0) \otimes \Big((0, 1) \otimes (0, 1) \Big) \\
&= (yv, 0) \otimes \Big(I \otimes I \Big).
\end{aligned}
$$

It follows that

$$(x, y) \otimes (u, v) = (xu, xv + yu) + yv\,(I \otimes I).$$

It follows that the design requirements for the sought algebra of algebraic couples that we have so far adopted ensure that the operation $\otimes$ of multiplication of algebraic couples is completely determined by the algebraic couple $I \otimes I$, where $I = (0, 1)$. Let us follow Hamilton's example and notation by writing $I \otimes I = (\gamma_1, \gamma_2)$, where γ_1 and γ_2 are real numbers. We then find that

$$(x, y) \otimes (u, v) = (xu + \gamma_1 yv, xv + yu + \gamma_2 yv).$$

(Hamilton's derivation of this equation, with x, y, u and v replaced by a_1, a_2, a_1 and a_2 respectively, can be found in Section 4 of his paper on the *Theory of Conjugate Functions, or Algebraic Couples*, published in the *Transactions of the Royal Irish Academy* in 1837.)

Now the design requirements adopted so far are not sufficient to constrain the values of the constants γ_1 and γ_2. Accordingly Hamilton adopted a further requirement: it should always be possible to *divide* an algebraic couple by a non-zero algebraic couple. Hamilton showed that, for this to be the case, it was necessary and sufficient that

$$\gamma_1 + \frac{1}{4}\gamma_2^2 < 0.$$

(see equation (45) in Hamilton's 1837 paper cited above). And, as Hamilton pointed out (prior to equation (36):

> *"It is clear, however, that the simplicity and elegance of our future operations and results must mainly depend on our making a simple and suitable choice of thse two constants of multiplication. . .*

Having stated his requirement that division by non-zero algebraic couples be always well-defined, Hamilton continued

> *"It is easy to show that no choice simpler than the following,*
>
> $$\gamma_1 = -1, \quad \gamma_2 = 0,$$
>
> *would satisfy this essential condition; and for that reason we shall now select these two numbers, [...], for the two constants of multiplication,...*

Following Hamilton's argument, and Hamilton's design choices, we finally arrive at the following formula for the multiplication of two algebraic couples:—

$$(x, y) \otimes (u, v) = (xu - yv, xv + yu).$$

Letting $\mathbf{E} = (1, 0)$ and $\mathbf{I} = (0, 1)$, and defining $t(x, y) = (tx, ty)$ for all algebraic couples (x, y) and real numbers t, we find that

$$(x, y) = x\mathbf{E} + y\mathbf{I},$$

and

$$(x, y) \otimes (u, v) = (xy - uv)\mathbf{E} + (xv + yu)\mathbf{I},$$

where

$$\mathbf{E} \otimes \mathbf{E} = \mathbf{E}, \quad \mathbf{E} \otimes \mathbf{I} = \mathbf{I},$$

$$\mathbf{I} \otimes \mathbf{E} = \mathbf{I}, \quad \mathbf{I} \otimes \mathbf{I} = -\mathbf{E}.$$

To recover the familiar notation used for complex numbers, it suffices to *identify* the real number x with the algebraic couple $x\mathbf{E}$, represent the algebraic couple $\mathbf{I}$ with the letter i, or with the expression $\sqrt{-1}$, and we find that each algebraic couple (x, y) represents and is represented by a "complex number" of the form $x + yi$, where x and y are real numbers, and $i^2 = -1$. Moreover

$$(x + yi)(u + vi) = (xu - yv) + (xv + yu)i$$

for all real numbers x, y, u and v.

The formula

$$(x + yi)(u + vi) = (xu - yv) + (xv + yu)i$$

can be taken as the *definition* of multiplication within the system of complex numbers. An important consequence follows on setting $u = x$ and $v = -y$. We find that

$$(x + yi)(x - yi) = x^2 + y^2.$$

It follows that, in the system of complex numbers

$$\frac{1}{x + yi} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}\, i,$$

The usual Commutative, Associative and Distributive Laws are satisfied within the system of complex numbers.

Complex numbers $x + yi$ and $u + vi$ are equal if and only if $x = u$ and $y = v$.
The definitions of addition, subtraction, multiplication and division for complex numbers are as follows:

$$\begin{aligned}
(x + yi) + (u + vi) &= (x + u) + (y + v)i, \\
(x + yi) - (u + vi) &= (x - u) + (y - v)i, \\
(x + yi) \times (u + vi) &= (xu - yv) + (xv + yu)i, \\
\frac{x + yi}{u + vi} &= \frac{xu + yv}{u^2 + v^2} + \frac{yu - xv}{u^2 + v^2} i.
\end{aligned}$$

(Note that division by $u + vi$ is defined if and only if $u + vi \neq 0$.)

These definitions are translations of the definitions of addition, subtraction, multiplication and division in Hamilton's algebra of algebraic couples:

$$\begin{aligned}(x,y)+(u,v) &= (x+u, y+v),\\ (x,y)-(u,v) &= (x-u, y-v),\\ (x,y)\otimes(u,v) &= (xu-yv, xv+yu),\\ (x,y)\div(u,v) &= \left(\frac{xu+yv}{u^2+v^2}, \frac{yu-xv}{u^2+v^2}\right).\end{aligned}$$

(Note that division by the algebraic couple (u,v) is defined if and only if $(u,v)\neq(0,0)$.)

Proposition 101.1

Let multiplication of algebraic couples be defined such that

$$(x, y) \otimes (u, v) = (xu - yv, xv + yu)$$

for all algebraic couples (x, y) *and* (u, v). *Then*

$$\mathbf{Z} \otimes \mathbf{W} = \mathbf{W} \otimes \mathbf{Z}$$

for all algebraic couples $\mathbf{Z}$ *and* $\mathbf{W}$. *Thus multiplication of algebraic couples satisfies the Commutative Law.*

Proof

This result follows directly from the definition of multiplication of algebraic couples. ∎

Proposition 101.2

Let multiplication of algebraic couples be defined such that

$$(x, y) \otimes (u, v) = (xu - yv, xv + yu)$$

for all algebraic couples (x, y) *and* (u, v). *Then*

$$(\mathbf{Z}_1 \otimes \mathbf{Z}_2) \otimes \mathbf{Z}_3 = \mathbf{Z}_1 \otimes (\mathbf{Z}_2 \otimes \mathbf{Z}_3)$$

for all algebraic couples $\mathbf{Z}_1$, $\mathbf{Z}_2$ *and* $\mathbf{Z}_3$. *Thus multiplication of algebraic couples satisfies the Associative Law.*

Proof

Let $\mathbf{Z}_1 = (x_1, y_1)$, $\mathbf{Z}_2 = (x_2, y_2)$ and $\mathbf{Z}_3 = (x_3, y_3)$. Then

$$
\begin{aligned}
(\mathbf{Z}_1 \otimes \mathbf{Z}_2) \otimes \mathbf{Z}_3 &= ((x_1, y_1) \otimes (x_2, y_2)) \otimes (x_3, y_3) \\
&= (x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) \otimes (x_3, y_3) \\
&= (x_1x_2x_3 - y_1y_2x_3 - x_1y_2y_3 + y_1x_2y_3, \\
&\qquad x_1y_2y_3 + y_1x_2y_3 + x_1x_2y_3 - y_1y_2y_3) \\
\mathbf{Z}_1 \otimes (\mathbf{Z}_2 \otimes \mathbf{Z}_3) &= (x_1, y_1) \otimes ((x_2, y_2) \otimes (x_3, y_3)) \\
&= (x_1, y_1) \otimes ((x_2x_3 - y_2y_3, x_2y_3 + y_2x_3) \\
&\quad (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3, \\
&\qquad x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3 - y_1y_2y_3) \\
&= (\mathbf{Z}_1 \otimes \mathbf{Z}_2) \otimes \mathbf{Z}_3
\end{aligned}
$$

The result follows. ∎

Remark

Translated into more traditional notation, the proof of Proposition 101.2 amounts to showing that if z_1, z_2 and z_3 are complex numbers, and if

$$z_1 = x_1 + y_1 i, \quad z_2 = x_2 + y_2 i, \quad z_3 = x_3 + y_3 i,$$

where x_1, y_1, x_2, y_2, x_3 and y_3 are real numbers and $i^2 = -1$, then

$$\begin{aligned} (z_1 z_2) z_3 &= x_1 x_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 + y_1 y_2 x_3 \\ &\qquad + (y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 - y_1 y_2 y_3) i \\ &= z_1 (z_2 z_3). \end{aligned}$$

Proposition 101.3

Let addition and multiplication of algebraic couples be defined such that

$$(x, y) + (u, v) = (x + u, y + v)$$

and

$$(x, y) \otimes (u, v) = (xu - yv, xv + yu)$$

for all algebraic couples (x, y) *and* (u, v)*. Then*

$$(\mathbf{Z}_1 + \mathbf{Z}_2) \otimes \mathbf{W} = (\mathbf{Z}_1 \otimes \mathbf{W}) + (\mathbf{Z}_2 \otimes \mathbf{W})$$

and

$$\mathbf{W} \otimes (\mathbf{Z}_1 + \mathbf{Z}_2) \otimes \mathbf{W} = \mathbf{W} \otimes \mathbf{Z}_1 + \mathbf{W} \otimes \mathbf{Z}_2$$

for all algebraic couples $\mathbf{Z}_1$, $\mathbf{Z}_2$ *and* $\mathbf{W}$*. Thus multiplication of algebraic couples satisfies the Distributive Law.*

Proof

Let $\mathbf{Z}_1 = (x_1, y_1)$, $\mathbf{Z}_2 = (x_2, y_2)$ and $\mathbf{W} = (u, v)$. Then

$$\begin{aligned}
(\mathbf{Z}_1 + \mathbf{Z}_2) \otimes \mathbf{W} &= ((x_1, y_1) + (x_2, y_2)) \otimes (u, v) \\
&= (x_1 + x_2, y_1 + y_2) \otimes (u, v) \\
&= (x_1 u + x_2 u - y_1 v + y_2 v, \\
&\qquad + x_1 v + x_2 v + y_1 u + y_2 u) \\
&= (x_1 u - y_1 v, x_1 + y_1 u) \\
&\qquad + (x_2 u - y_2 v, x_2 + y_2 u) \\
&= (\mathbf{Z}_1 \otimes \mathbf{W}) + (\mathbf{Z}_2 \otimes \mathbf{W}).
\end{aligned}$$

The identity

$$\mathbf{W} \otimes (\mathbf{Z}_1 + \mathbf{Z}_2) \otimes \mathbf{W} = \mathbf{W} \otimes \mathbf{Z}_1 + \mathbf{W} \otimes \mathbf{Z}_2$$

then follows either from a calculation similar to the above, or alternatively as a consequence of the fact that multiplication of algebraic couples is commutative. The result follows. ∎

The results of the previous propositions guarantee that

$$zw = wz, \quad (z_1z_2)z_3 = z_1(z_2z_3),$$

$$(z_1 + z_2)w = z_1w + z_2w, \quad w(z_1 + z_2) = wz_1 + wz_2$$

for all complex numbers z, z_1, z_2, z_3 and w. Many standard properties of the complex number system follow from application of these fundamental identities. These include the relevant forms of the Laws of Indices which we now discuss.

Given a complex number z and a natural number p, the pth power z^p of z is defined recursively so that $z^{p+1} = z^p z$ for all natural numbers z.

Lemma 101.4

Let z be a complex number and let p and q be natural numbers. Then $z^{p+q} = z^p z^q$.

Proof

The identity $z^{p+q} = z^p z^q$ can be proved by induction on q. The recursive definition of z^{p+1} ensures that, for fixed p, $z^{p+q} = z^p z^q$ when $q = 1$. Suppose that $z^{m+k} = z^m z^k$ for some natural number k. Then

$$z^{m+k+1} = z^{m+k} a = (z^m z^k) z$$

But multiplication of complex numbers is associative (see Proposition 101.2 and the remarks following the proof of this proposition). It follows that

$$z^{m+k+1} = z^m (z^k z) = z^m z^{k+1}.$$

Thus if the identity $z^{m+n} = z^m z^n$ holds when $q = k$ for some natural number k, then it also holds for $q = k + 1$. It follows from the Principle of Mathematical Induction that the identity $z^{p+q} = z^p z^q$ holds for all real numbers a and for all natural numbers p and q, as required. ∎

Given a non-zero complex number z and a positive integer p, the complex number z^{-p} is defined so that

$$z^{-p} = \frac{1}{z^p}.$$

One can then proceed, as in the case of powers of non-zero real numbers, to prove that $z^{m+n} = z^m z^n$ and $z^{mn} = (z^m)^n$ for all non-zero complex numbers z and for all integers m and n. Also the fact that multiplication of complex numbers is commutative can be used to prove that $(zw)^n = z^n w^n$ for all non-zero complex numbers z and w and for all integers n.